Dynamics of Non-densely Defined Stochastic Evolution Equations

seit 1558

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Bielefeld, 6th November 2015

1. Motivation

SDE:

$$\begin{cases} dU(t) = (AU(t) + F(U(t)))dt + dW(t), \ t \in [0, T] \\ U(0) = U_0. \end{cases} \tag{1.1}$$

RDE:

$$\begin{cases} \frac{dv(t)}{dt} = Av(t) + F(\omega, v(t)), \ t \in [0, T] \\ v(0) = v_0. \end{cases} \tag{1.2}$$

Dynamics

1. random attractors;
2. invariant manifolds.

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Dynamics

1. random attractors;
2. invariant manifolds.

Here: A is a non-densely defined linear operator: NO C_0-semigroup!

Example: Deterministic case

Age-structured models in population dynamics [P. Magal and S. Ruan (2009)]

$$\begin{cases} \partial_t v(t,a) + \partial_a v(t,a) = -\mu v(t,a), \;\; t>0, \;\; a>0, \\ v(t,0) = f\left(\int\limits_0^\infty \beta(a) v(t,a) da\right) \\ v(0,\cdot) = v_0(\cdot) \in L^1(0,\infty). \end{cases}$$

Ricker type birth function: $f(x) = xe^{-bx}$, $x \in \mathbb{R}$ and $b > 0$.

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Ricker type birth function: $f(x) = xe^{-bx}$, $x \in \mathbb{R}$ and $b > 0$.

Set $X = \mathbb{R} \times L^1(0,\infty)$ and $u(t,\cdot) = \begin{pmatrix} 0 \\ v(t,\cdot) \end{pmatrix}$.

$A \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} -v(0) \\ -v' - \mu v \end{pmatrix}$ with $D(A) = \{0\} \times W^{1,1}(0,\infty)$.

$F : \{0\} \times L^1(0,\infty) \to X$, $F\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} f\left(\int\limits_0^\infty \beta(a) v(a) da\right) \\ 0 \end{pmatrix}$.

Abstract Cauchy-Problem

One obtains

$$du = Au + F(u), \quad u(0) = u_0 \in \overline{D(A)}. \tag{1.3}$$

Note that $\overline{D(A)} = \{0\} \times L^1(0, \infty) \neq X$.

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Preliminaries

Definition

Let $\theta : \mathbb{R} \times \Omega \to \Omega$ be a family of $\mathbb{P}$-preserving transformations having following properties:

1. the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable;
2. $\theta_0 = Id_\Omega$;
3. $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s, \in \mathbb{R}$.

Then the quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system.

Definition

A linear random dynamical system is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \ (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

1. φ is $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable;
2. $\varphi(0, \omega, \cdot) = Id_X$ for all $\omega \in \Omega$;
3. the cocycle property:
   $\varphi(t + s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x)),$ for all $x \in X, s, t \in \mathbb{R}^+, \omega \in \Omega$;
4. for each $\omega \in \Omega$ and $t \in \mathbb{R}^+$, $[X \ni x \mapsto \varphi(t, \omega, x) \in X] \in \mathcal{L}(X)$.

RDE

Let X be a separable Banach space and $X_0 := \overline{D(A)}$;

$$u'(t) = Au(t) + F(\theta_t\omega, u(t)), \ u(0) = u_0 \in X_0. \tag{2.1}$$

Definition

A family of linear bounded operators $(S(t))_{t\geq 0}$ is called an integrated semigroup if

1. $S(0) = 0$;
2. $t \mapsto S(t)$ is strongly continuous;
3. $S(s)S(t) = \int_0^s (S(r+t) - S(r))dr, \ t, s \geq 0.$

Examples:

1. $S(t) = \int_0^t T(s)ds$ where $(T(t))_{t\geq 0}$ is a C_0-semigroup;
2. $Au = i\Delta u$ generates an integrated semigroup in $L^p(\mathbb{R}^n)$ for $p \neq 2$.

Definition

A continuous map $u \in C([0,T];X)$ is an integrated solution of (2.1) if

1. $\int_0^t u(s)ds \in D(A),\ t \in [0,T]$;
2. $u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t F(\theta_s\omega, u(s,\omega,u_0))ds,\ t \in [0,T]$.

Assumptions:

$$\text{(a)}\ \left\|(\lambda I - A)^{-k}\right\|_{\mathcal{L}(X_0)} \leq \frac{M}{(\lambda-\omega_A)^k},\ \text{for all } \lambda > \omega_A \text{ and all } k \geq 1;$$

$$\text{(b)}\ \lim_{\lambda\to\infty}(\lambda I - A)^{-1}x = 0,\ \text{for all } x \in X.$$

- $A_0 = A$ on $D(A_0) = \{x \in D(A) : Ax \in X_0\}$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X_0;
- A generates an integrated semigroup $(S(t))_{t\geq 0}$ on X.

Variation of constants

$$(\lambda I - A)^{-1} : X \to X_0 \text{ and } \lim_{\lambda\to\infty} \lambda(\lambda I - A)^{-1}x = x, \text{ for } x \in X_0.$$

Equation on X_0:

$$(\lambda I - A)^{-1}du(t) = A_0(\lambda I - A)^{-1}u(t)dt + (\lambda I - A)^{-1}F(\theta_t\omega, u(t))dt,$$

$$(\lambda I - A)^{-1}u(t) = T(t)(\lambda I - A)^{-1}u_0 + \int_0^t T(t-s)(\lambda I - A)^{-1}F(\theta_s\omega, u(s))ds.$$

Theorem

Equation (2.1) possesses a unique global integrated solution

$$u(t, \omega, u_0) = T(t)u_0 + \lim_{\lambda\to\infty} \int_0^t T(t-s)\lambda(\lambda I - A)^{-1}F(\theta_s\omega, u(s, \omega, u_0))ds. \tag{2.2}$$

Special case

Consider

$$du(t) = (Au(t) + f(u(t)))dt + \sigma dW(t),\ u(0) = u_0 \in \overline{D(A)},\ \sigma \in D(A).\ (2.3)$$

- Ornstein-Uhlenbeck process: $dz = zdt + dW$, $z(\omega) = -\int_{-\infty}^{0} e^s \omega(s)ds$

$$(t,\omega) \mapsto z(\theta_t\omega):\quad z(\theta_t\omega) = -\int_{-\infty}^{0} e^s\omega(t+s)ds + \omega(t),\ t \in \mathbb{R}.$$

- Transformation: $x(t) = u(t) - z(\theta_t\omega)$.

Equation (2.3) becomes: $x'(t) = Ax(t) + F(\theta_t\omega, x(t))$,

$$F(\theta_t\omega, x(t)) = f(x(t) + z(\theta_t\omega)) + Az(\theta_t\omega) + z(\theta_t\omega).$$

The parabolic case

Assumptions: A_0 generates an analytic semigroup, $B \in \gamma(H; X_0)$ and W is an H-cylindrical Wiener process.

$$dU(t) = (AU(t) + F(U(t)))dt + BdW(t)$$
$$v(t) = U(t) - Z(\theta_t \omega)$$
$$dv(t) = Av(t)dt + F(v(t) + Z(\theta_t \omega))dt.$$

Infinite dimensional noise: $L^p(\mathbb{R})$-valued Brownian motion: formally

$$W(t) = \sum_{k=1}^{\infty} g_k(x) w_k(t) = \sum_{k=1}^{\infty} W_H(t) e_k B e_k.$$

$(g_k)_{k\geq 1} \in L^p(\mathbb{R}, l_2)$ define $Bh := \sum_{k\geq 1} [h, e_k] g_k, \; h \in l_2$ and $(e_k)_{k\geq 1}$ ONB in l_2.

$$\mathbb{E}\Big\| \sum_{k\geq 1} \gamma_k B e_k \Big\|^2_{L^p(\mathbb{R})} \lesssim_p \mathbb{E}\Big\| \sum_{k\geq 1} \gamma_k B e_k \Big\|^p_{L^p(\mathbb{R})} = \int_{\mathbb{R}} \mathbb{E}\Big| \sum_{k\geq 1} \gamma_k g_k(x) \Big|^p dx$$

$$\leq \int_{\mathbb{R}} \left(\sum_{k\geq 1} |g_k(x)|^2 \right)^{\frac{p}{2}} dx < \infty.$$

Applications: Parabolic SPDE-s with nonlinear boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} = \mu u - \frac{\partial^2 u}{\partial x^2} + M(u(t,\cdot))(x) + dW(t), \ \mu > 0, t > 0, x > 0 \\ -\frac{\partial u(t,0)}{\partial x} = G(u(t,\cdot)) \\ u(0,\cdot) = u_0 \in L^p((0,\infty);\mathbb{R}). \end{cases} \tag{3.1}$$

Set $X := \mathbb{R} \times L^p((0,\infty);\mathbb{R})$, $A\begin{pmatrix}0\\u\end{pmatrix} := \begin{pmatrix}u'(0)\\ \mu u - u''\end{pmatrix}$, $F\left(\begin{pmatrix}0\\u\end{pmatrix}\right) := \begin{pmatrix}G(u)\\M(u)\end{pmatrix}$.

$$A_0\begin{pmatrix}0\\u\end{pmatrix} = \begin{pmatrix}0\\ \mu u - u''\end{pmatrix} \text{ with } D(A_0) = \{0\} \times \{u \in W^{2,p}(\mathbb{R}) : u'(0) = 0\}.$$

1. A_0 generates an analytic C_0-semigroup;
2. there exists $p^* \geq 1$ such that

$$\limsup_{\lambda\to\infty} \lambda^{\frac{1}{p^*}} ||(\lambda I - A)^{-1}|| < \infty.$$

Fractional power: $(-A)^{-\beta}$ for $\beta > 1 - \frac{1}{p^*}$ [Magal et.al. (2010)].

$$v(t) = T(t)v_0 + \int_0^t (\lambda I - A_0)^{\beta} T(t-s)(\lambda I - A)^{-\beta} F(v(s) + Z(\theta_s\omega))ds.$$

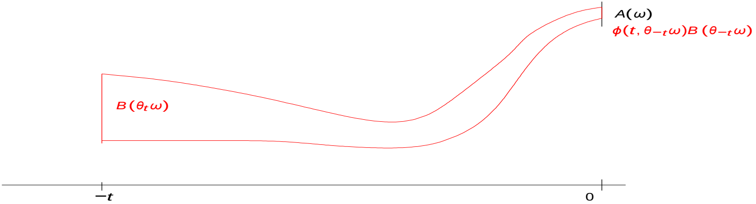
Random Dynamics

Definition

Let $\mathcal{D}$ be the collection of the tempered random subsets of X and consider $\{A(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{A(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for ϕ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and $\omega \in \Omega$, there exists a $t_B(\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq A(\omega), \text{ for all } t \geq t_B(\omega).$$

For each ω fix:



Singular Gronwall Lemma

Lemma (Henry, (1993))

Let f be a nonnegative locally integrable function on $[0,T)$ *with*

$$f(t) \leq a(t) + L\int_0^t (t-s)^{-\beta} f(s)ds \text{ on } [0,T).$$

Then it holds on $[0,T)$

$$f(t) \leq a(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(L\Gamma(1-\beta))^n}{\Gamma(n(1-\beta))}(t-s)^{n(1-\beta)-1} a(s)ds.$$

Apply to:

$$||v(t)|| \leq e^{-\mu t}||v_0|| + L\int_0^t e^{-\mu(t-s)}(t-s)^{-\beta}(||v(s)|| + ||Z(\theta_s\omega)||)ds.$$

Random attractor

Definition

A random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a random $\mathcal{D}$-attractor if for all $\omega \in \Omega$:

a) $\mathcal{A}(\omega)$ is compact and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;

b) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant:

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \text{ for all } t \geq 0;$$

c) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} d(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where d is the Hausdorff semimetric, $d(Y, Z) = \sup_{y\in Y} \inf_{z\in Z} ||y - z||_X$.

Problem: Compactness on unbounded domains

Remark

- *closed absorbing set;*
- *RDS* ϕ *is called* $\mathcal{D}$*-pullback asymptotically compact if for all* $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, u_n)\}_{n=1}^{\infty}$ *has a convergent subsequence, for* $t_n \to \infty$ *and* $u_n \in B(\theta_{-t_n}\omega)$ *with* $\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$.

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Purpose: show ϕ is pullback asymptotically compact.

1. $$\{\phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))\}_{n=1}^{\infty} \text{ bounded in } L^p(\mathbb{R});$$
$$\phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)) \rightharpoonup \xi$$
2. $$||\phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega))||_{W^{2\alpha,p}(\mathbb{R})} \leq k\rho(\omega),$$
since $D((-A_0)^{\alpha}) = W^{2\alpha,p}(\mathbb{R})$.
3. There exist $R^* = R^*(\omega, \varepsilon)$ and $T = T(B, \omega)$ for all $t_n \geq T$:
$$\int_{|x|\geq R^*} |\phi(t_n, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)) - \xi|^p dx < \varepsilon.$$

Outlook

1. Multiplicative noise;
2. Random invariant manifolds: (Lyapunov-Perron method);
3. Oseledets splitting;
4. Delay equations

$$\begin{cases} du(t) = Au(t)dt + F(u_t)dt + dW(t), & \text{for } t \geq 0. \\ u(t) = u_0(t), & \text{for } t \in [-r, 0]. \end{cases}$$

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Thank you for your attention!