W I A S

Weierstraß-Institut für Angewandte Analysis und Stochastik

Period of Concentration: Stochastic Climate Models

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Residence-time distributions as a measure for stochastic resonance

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Outline

- ▷ A brief introduction to stochastic resonance
  - – Example: Dansgaard–Oeschger events
- ▷ First-passage-time distributions as a qualitative measure for SR
- ▷ Diffusion exit from a domain
  - – Exponential asymptotics: Wentzell–Freidlin theory
- ▷ Noise-induced passage through an unstable periodic orbit
- ▷ The first-passage time density
  - – Universality
  - – Plots of the density: Cycling and synchronisation
- ▷ The residence-time density
  - – Definition and computation
  - – Plots of the density

Joint work with Nils Berglund (CPT–CNRS, Marseille)

What is stochastic resonance (SR)?

SR = mechanism to amplify weak signals in presence of noise

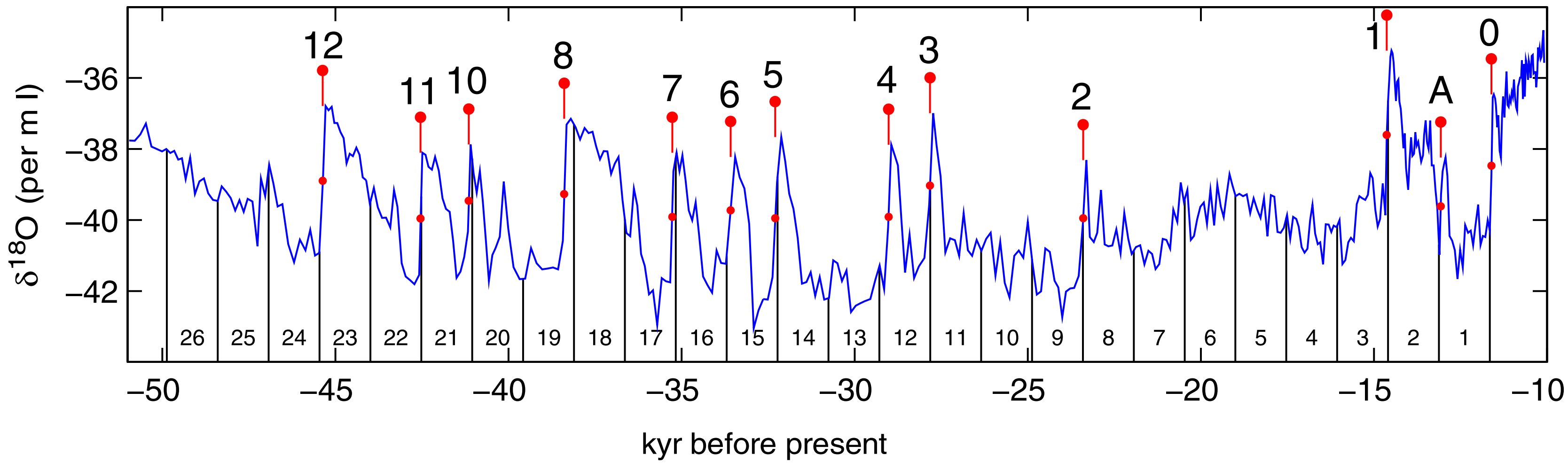
Requirements

- ▷ (background) noise
- ▷ weak input
- ▷ characteristic barrier or threshold (nonlinear system)

Examples

- ▷ periodic occurrence of ice ages (?)
- ▷ Dansgaard–Oeschger events (?)
- ▷ bidirectional ring lasers
- ▷ visual and auditory perception
- ▷ receptor cells in crayfish
- ▷ . . .

Example: Dansgaard–Oeschger events

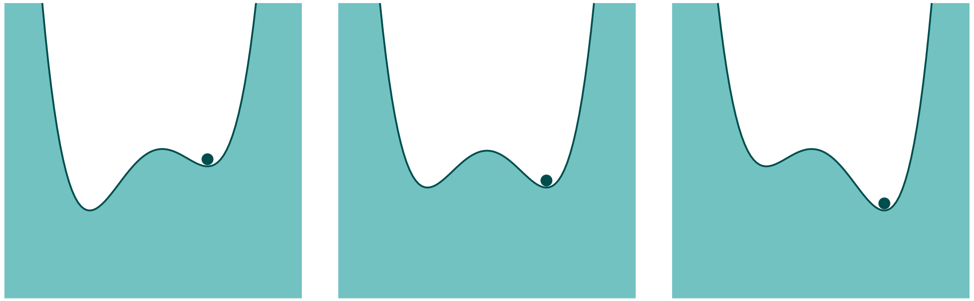


GISP2 climate record for the second half of the last glacial

[from: Rahmstorf, *Timing of abrupt climate change: A precise clock*, Geophys. Res. Lett. 30 (2003)]

- ▷ Abrupt, large-amplitude shifts in global climate during last glacial
- ▷ Cold stadials; warm Dansgaard–Oeschger interstadials
- ▷ Rapid warming; slower return to cold stadial
- ▷ 1 470-year cycle?
- ▷ Occasionally a cycle is skipped

The paradigm



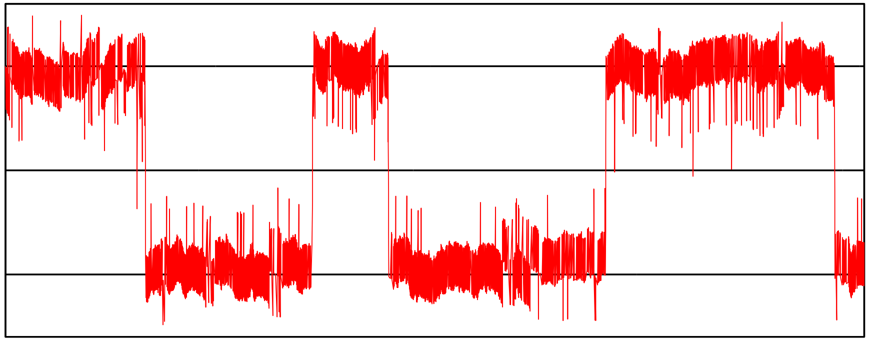
Overdamped motion of a Brownian particle . . .

$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{= -\frac{\partial}{\partial x} V(x_t, \varepsilon t)} \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

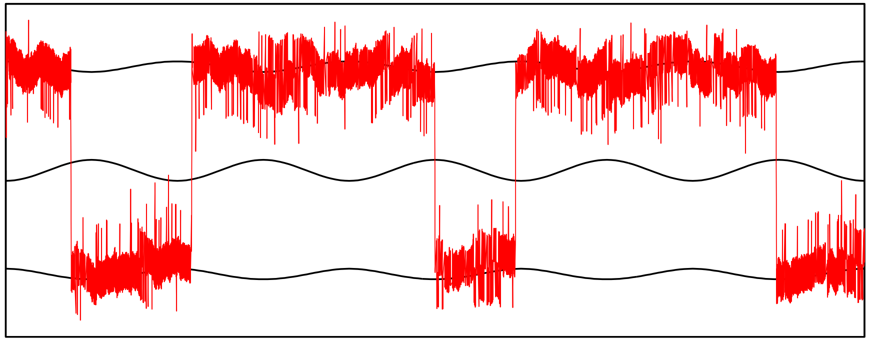
. . . in a periodically modulated double-well potential

$$V(x,s) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\cos(s)x \; , \qquad A < A_\mathrm{c}$$

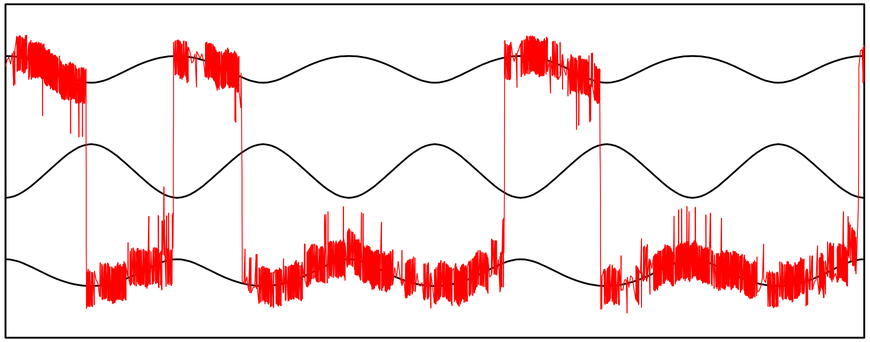
Sample paths



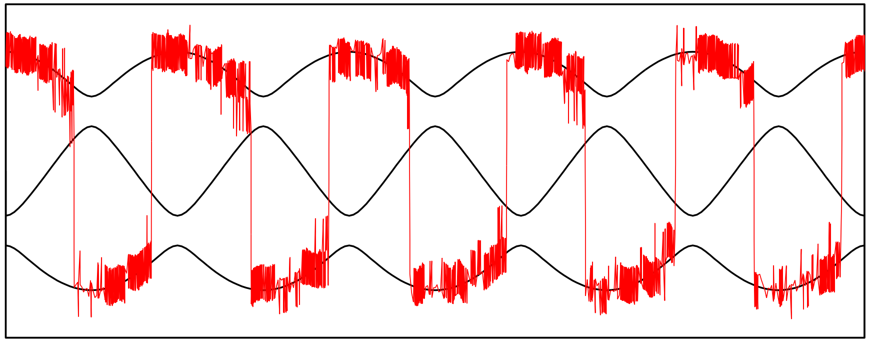
$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$



$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$



$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

Different parameter regimes

Synchronisation I

- For matching time scales: $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp \mathrm{e}^{2H/\sigma^2}$
- Quasistatic approach: Transitions twice per period with high probability
  (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- Requires exponentially long forcing periods

Synchronisation II

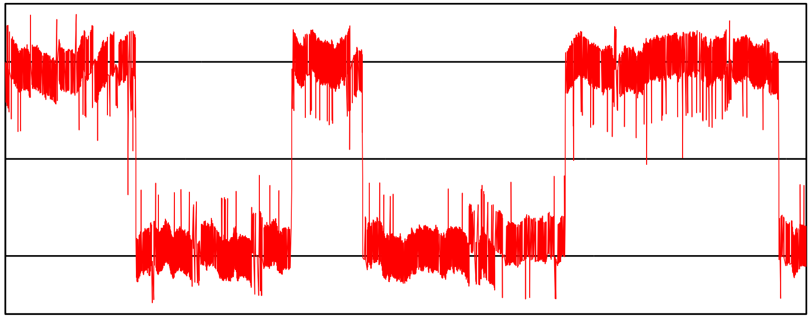
- For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$
  and close-to-critical forcing amplitude: $A \approx A_{\text{c}}$
- Transitions twice per period with high probability
- Subtle dynamical effects: Effective barrier heights [Berglund & G '02]

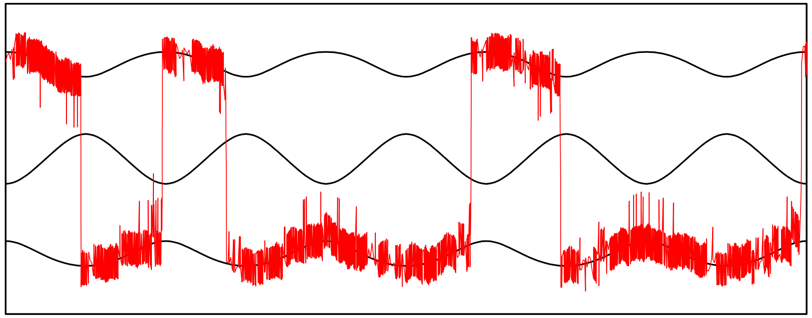
SR outside synchronisation regimes

- Only occasional transitions
- But transition times localised within forcing periods

Unified description / understanding of transition between regimes ?

Qualitative measures for SR





How to measure combined effect of periodic and random perturbations?

Spectral-theoretic approach

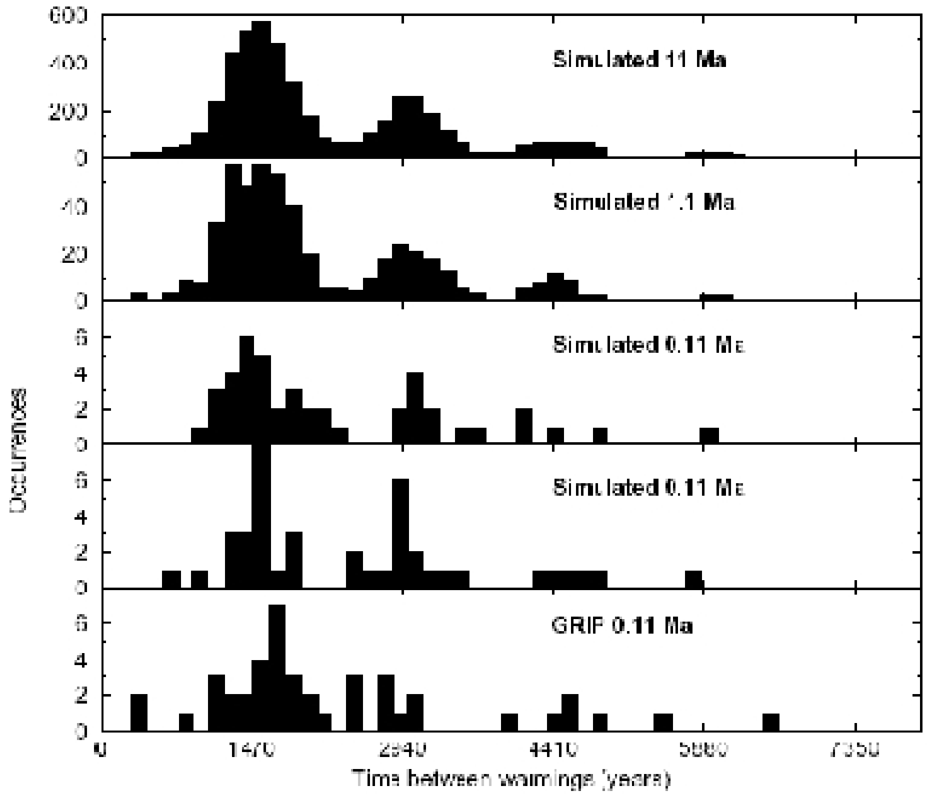
- ▷ Power spectrum
- ▷ Spectral power amplification
- ▷ Signal-to-noise ratio

Probabilistic approach

- ▷ Distribution of interspike times
- ▷ Distribution of first-passage times
- ▷ Distribution of residence times

Look for periodic component in density of these distributions

Interspike times for Dansgaard–Oeschger events



Histogram for “waiting times”

[from: Alley, Anandakrishnan, Jung, *Stochastic resonance in the North Atlantic*, Paleoceanography 16 (2001)]

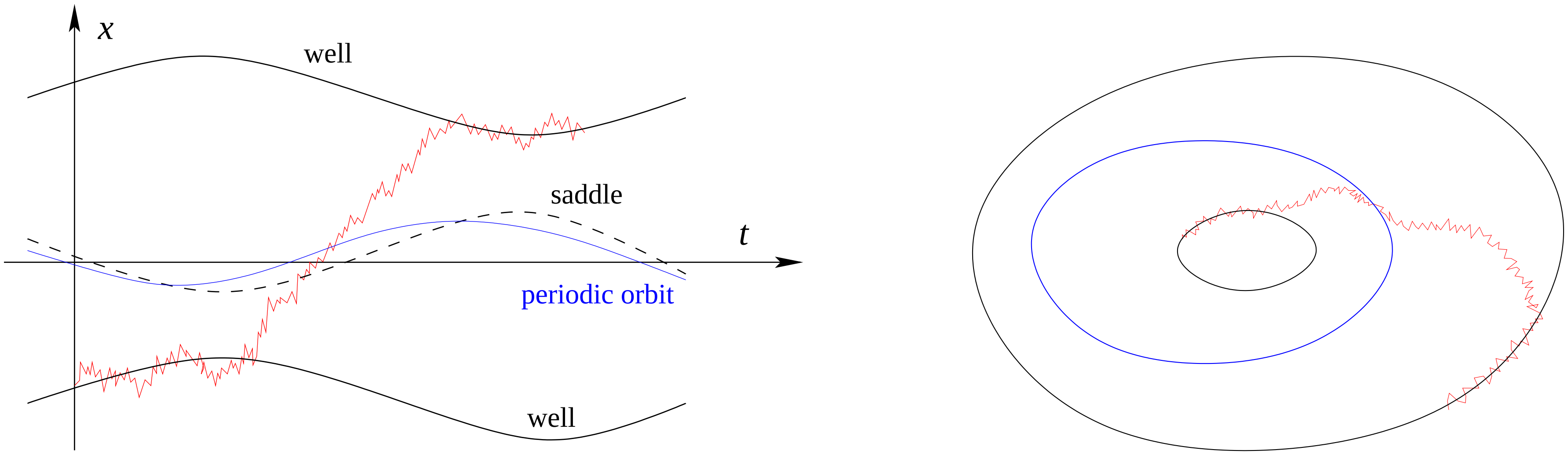
Interwell transitions

Deterministic motion in a periodically modulated double-well potential

- ▷ 2 stable periodic orbits tracking bottoms of wells
- ▷ 1 unstable periodic orbit tracking saddle
- ▷ Unstable periodic orbit separates basins of attraction

Brownian particle in a periodically modulated double-well potential

- ▷ Interwell transitions characterised by crossing of unstable orbit



Exit problem

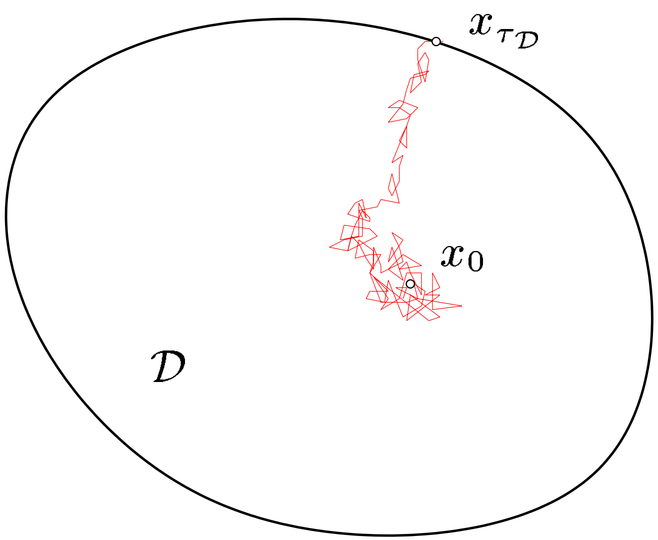
Deterministic ODE $\dot{x}_t^{\mathrm{det}} = f(x_t^{\mathrm{det}})$ $x_0 \in \mathbb{R}^d$

Small random perturbation $\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$ (same initial cond. x_0)

Bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- *first-exit time* $\tau = \tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
- *first-exit location* $x_\tau \in \partial\mathcal{D}$

Distribution of τ and x_τ ?



Interesting case

$\mathcal{D}$ positively invariant under deterministic flow

Approaches

- Mean first-exit times and locations via PDEs
- Exponential asymptotics via Wentzell–Freidlin theory

Exponential asymptotics: Wentzell–Freidlin theory I

Assumptions (for this slide)

- $\mathcal{D}$ positively invariant
- unique, asympt. stable equilibrium point at $0 \in \mathcal{D}$
- $\partial\mathcal{D} \subset$ basin of attraction of 0

Concepts

- Rate function / action functional :

  $I_{[0,t]}(\varphi) = \frac{1}{2}\int_0^t \|\dot{\varphi}_s - f(\varphi_s)\|^2 \, \mathrm{d}s$ for $\varphi \in H_1$, $\qquad I_{[0,t]}(\varphi) = +\infty$ otherwise

  Probability $\sim \exp\{-I(\varphi)\}$ to observe sample paths close to φ (LDP)

- Quasipotential: Cost to go against the flow from 0 to z

  $$V(0,z) = \inf_{t>0} \inf\{I_{[0,t]}(\varphi) \colon \varphi \in \mathcal{C}([0,t],\mathbb{R}^d),\ \varphi_0 = 0,\ \varphi_t = z\}$$

- Minimum of quasipotential on boundary $\partial\mathcal{D}$:

  $$\overline{V} := \min_{z\in\partial\mathcal{D}} V(0,z)$$

Exponential asymptotics: Wentzell–Freidlin theory II

Theorem [Wentzell & Freidlin $\geqslant$ '70]

For arbitrary initial condition in $\mathcal{D}$

- ▷ Mean first-exit time

  $\mathbb{E}\tau \sim e^{\overline{V}/\sigma^2}$ as $\sigma \to 0$

- ▷ Concentration of first-exit times

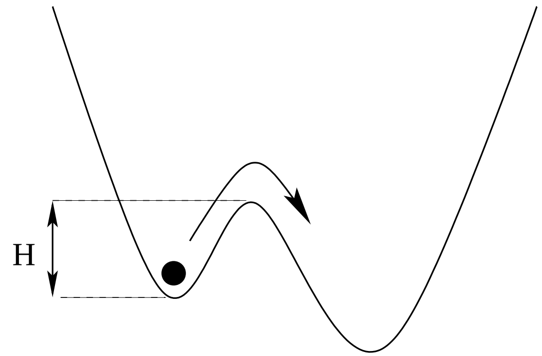
  $\mathbb{P}\Big\{ e^{(\overline{V}-\delta)/\sigma^2} \leqslant \tau \leqslant e^{(\overline{V}+\delta)/\sigma^2} \Big\} \to 1$ as $\sigma \to 0$ (for arbitrary $\delta > 0$)

- ▷ Concentration of exit locations near minima of quasipotential

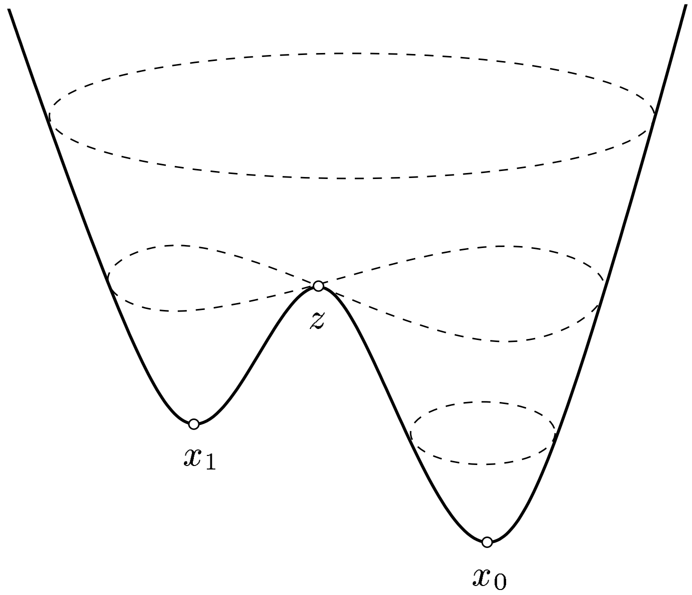
Gradient case (reversible diffusion)

Drift coefficient deriving from potential: $f = -\nabla V$

- ▷ Cost for leaving potential well: $\overline{V} = 2H$
- ▷ Attained for paths going against the flow: $\dot{\varphi}_t = -f(\varphi_t)$



Refined results in the gradient case



Simplest case: V double-well potential

First-hitting time τ^{hit} of deeper well

- $\mathbb{E}_{x_1} \tau^{\text{hit}} = c(\sigma)\ \mathrm{e}^{2\,[V(z)-V(x_1)]\,/\,\sigma^2}$

- $\displaystyle\lim_{\sigma\to 0} c(\sigma) = \frac{2\pi}{\lambda_1(z)} \sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x_1)}}$ exists !

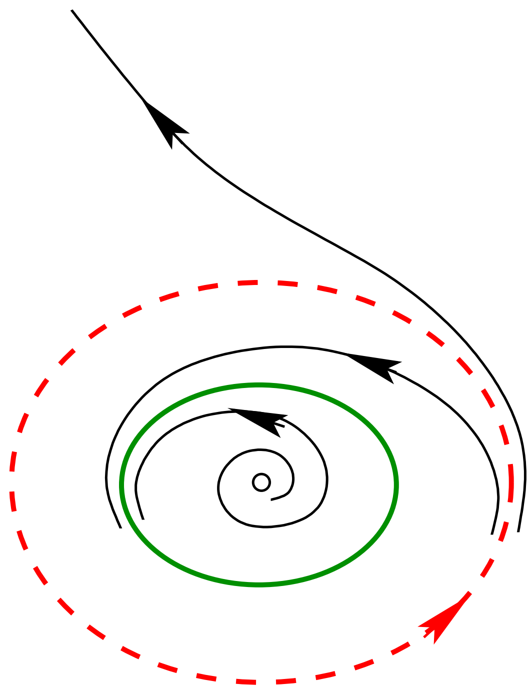
  $\lambda_1(z)$ unique negative e.v. of $\nabla^2 V(z)$

  (Physics' literature: [Eyring '35], [Kramers '40]; [Bovier, Gayrard, Eckhoff, Klein '02])

- Subexponential asymptotics known !
  Related to geometry at well and saddle / small eigenvalues of the generator
  ([Bovier *et al* '02], [Helffer, Klein, Nier '04])

- $\tau^{\text{hit}} \approx$ exp. distributed: $\displaystyle\lim_{\sigma\to 0} \mathbb{P}\{\tau^{\text{hit}} > t\,\mathbb{E}\,\tau^{\text{hit}}\} = \mathrm{e}^{-t}$ ([Day '82], [Bovier *et al* '02])

New phenomena for drift not deriving from a potential?



Simplest situation of interest

Nontrivial invariant set which is a single periodic orbit

Assume from now on

$d = 2$, $\partial\mathcal{D} =$ *unstable* periodic orbit

- ▷ $\mathbb{E}\tau \sim \mathrm{e}^{\overline{V}/\sigma^2}$ still holds [Day '90]
- ▷ Quasipotential $V(0,z) \equiv \overline{V}$ is constant on $\partial\mathcal{D}$:
  Exit equally likely anywhere on $\partial\mathcal{D}$ (on exp. scale)
- ▷ Phenomenon of cycling [Day '92]:
  Distribution of x_τ on $\partial\mathcal{D}$ generally does *not* converge as $\sigma \to 0$.
  Density is *translated* along $\partial\mathcal{D}$ proportionally to $|\log\sigma|$.
- ▷ In *stationary regime*: (obtained by reinjecting particle)
  Rate of escape $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{x_t \in \mathcal{D}\}$ has $|\log\sigma|$-periodic prefactor [Maier & Stein '96]

Back to SR

$$dx_t = -\frac{\partial}{\partial x} V(x_t, \varepsilon t)\, dt + \sigma\, dW_t$$

where $V(x,s)$ is a periodically modulated double-well potential

$$V(x,s) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\cos(s)x\ , \qquad A < A_{\mathrm{c}}$$

- ▷ Time t as auxiliary variable $\rightarrow$ 2-dimensional system
- ▷ Deterministic system: 3 periodic orbits tracking bottoms of wells and saddle
- ▷ 2 stable, 1 unstable
- ▷ Unstable periodic orbit separates basins of attraction
- ▷ Choose $\mathcal{D}$ as interior of unstable periodic orbit
- ▷ $\partial\mathcal{D}$ is *unstable* periodic orbit

Degenerate case: No noise acting on auxiliary variable

Density of the first-passage time at an unstable periodic orbit

Taking number of revolutions into account

Idea

Density of first-passage time at unstable orbit

$$p(t) = c(t,\sigma)\ \mathrm{e}^{-\overline{V}/\sigma^2} \times \text{ transient term } \times \text{ geometric decay per period}$$

Identify $c(t,\sigma)$ as periodic component in first-passage density

Notations

- ▷ Value of quasipotential on unstable orbit: $\overline{V}$
- ▷ Period of unstable orbit: $T = 2\pi/\varepsilon$
- ▷ Curvature at unstable orbit: $a(t) = -\dfrac{\partial^2}{\partial x^2} V(x^{\text{unst}}(t), t)$
- ▷ Lyapunov exponent of unstable orbit: $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(t)\ \mathrm{d}t$

Universality in first-passage-time distributions

Theorem ([Berglund & G '04], [Berglund & G '05], work in progress)

Using a (model dependent) "natural" parametrisation of the boundary:

For any $\Delta \geqslant \sqrt{\sigma}$ and all $t \geqslant t_0$

$$\mathbb{P}\{\tau \in [t, t+\Delta]\} = \int_t^{t+\Delta} p(s, t_0)\, \mathrm{d}s \left[1 + \mathcal{O}(\sqrt{\sigma})\right]$$

where

- $p(t, t_0) = \dfrac{1}{\mathcal{N}} Q_{\lambda T}\big(t - |\log\sigma|\big) \dfrac{1}{\lambda T_{\mathsf{K}}(\sigma)} \mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\mathsf{K}}(\sigma)} f_{\text{trans}}(t, t_0)$ is the "density"
- $Q_{\lambda T}(y)$ is a *universal* λT-periodic function
- $T_{\mathsf{K}}(\sigma)$ is the analogue of Kramers' time: $T_{\mathsf{K}}(\sigma) = \dfrac{C}{\sigma} e^{\overline{V}/\sigma^2}$
- f_{trans} grows from 0 to 1 in time $t - t_0$ of order $|\log\sigma|$

The different regimes

$$p(t,t_0) = \frac{1}{\mathcal{N}}\, Q_{\lambda T}\big(t - |\log\sigma|\big)\frac{1}{\lambda T_{\mathsf{K}}(\sigma)}\,\mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\mathsf{K}}(\sigma)}\, f_{\mathsf{trans}}(t,t_0)$$

Transient regime

f_{trans} is increasing from 0 to 1; exponentially close to 1 after time $t - t_0 > 2|\log\sigma|$

Metastable regime

$$Q_{\lambda T}(y) = 2\lambda T \sum_{k=-\infty}^{\infty} P(y - k\lambda T) \qquad \text{with peaks} \qquad P(z) = \frac{1}{2}\,\mathrm{e}^{-2z}\exp\Big\{-\frac{1}{2}\,\mathrm{e}^{-2z}\Big\}$$

kth summand: Path spends

- k periods near stable periodic orbit
- the remaining $[(t-t_0)/T] - k$ periods near unstable periodic orbit

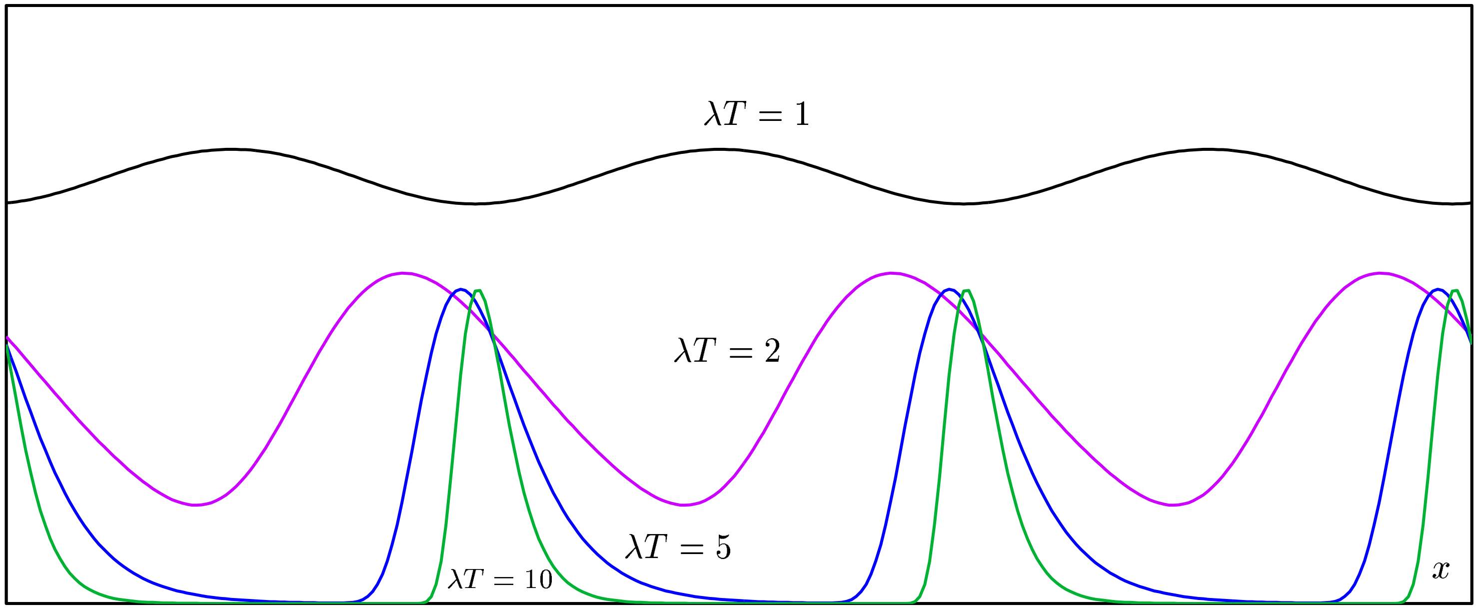
Periodic dependence on $|\log\sigma|$: Peaks rotate as σ decreases

Asymptotic regime

Significant decay only for $t - t_0 \gg T_{\mathsf{K}}(\sigma)$

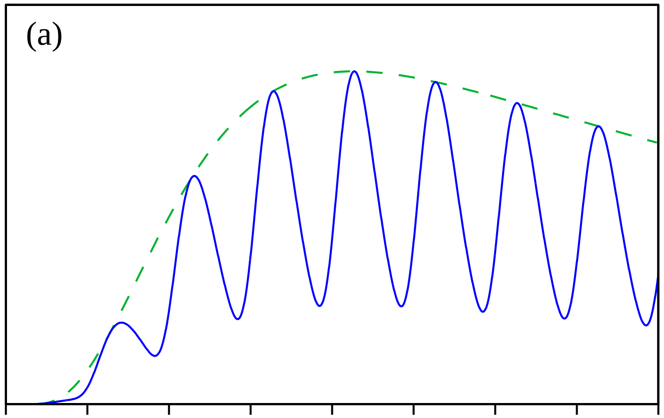
The universal profile

$$y \mapsto Q_{\lambda T}(\lambda T y)/2\lambda T$$

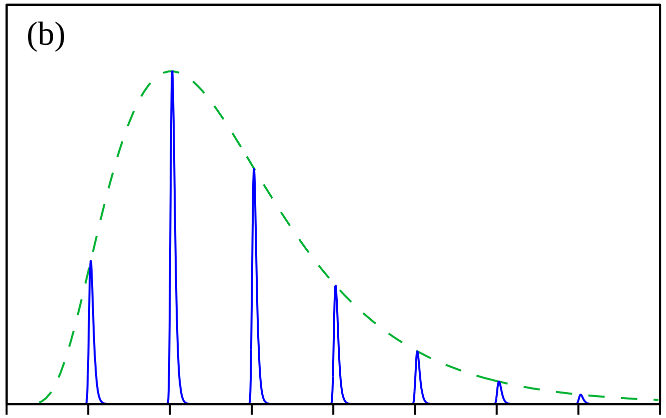


- ▷ Profile determines concentration of first-passage times within a period
- ▷ The larger λT, the more pronounced the peaks
- ▷ For smaller values of λT, the peaks overlap more

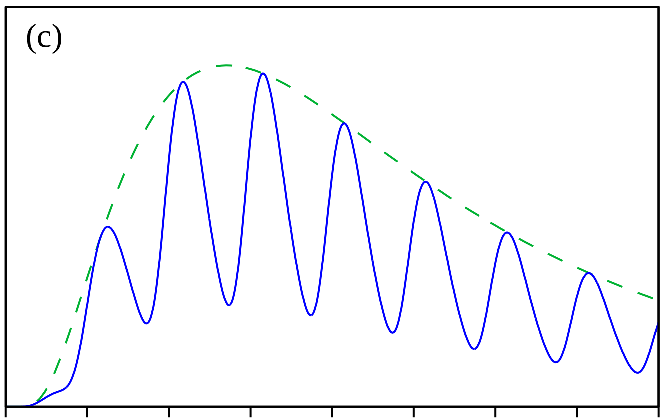
Density of the first-passage time $\overline{V} = 0.5,\ \lambda = 1$



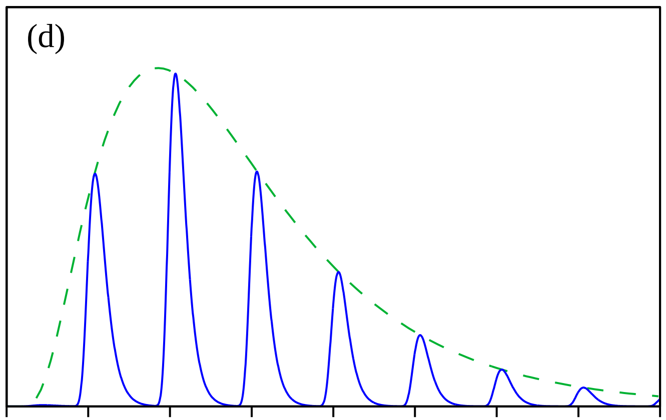
$\sigma = 0.4,\ T = 2$



$\sigma = 0.4,\ T = 20$



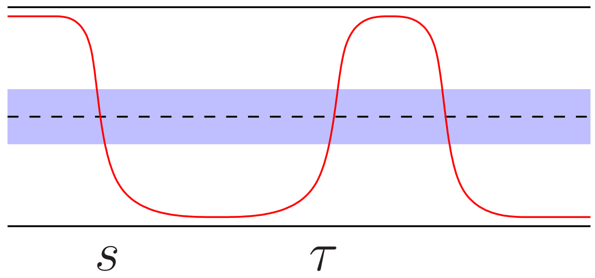
$\sigma = 0.5,\ T = 2$



$\sigma = 0.5,\ T = 5$

Definition of residence-time distributions

x_t crosses unstable periodic orbit $x^{\mathrm{per}}(t)$ at time s
τ: time of first crossing back after time s



- ▷ First-passage-time density:

$$p(t,s) = \frac{\partial}{\partial t} \mathbb{P}_{s,x^{\mathrm{per}}(s)}\{\tau < t\}$$

- ▷ Asymptotic transition-phase density: (stationary regime)

$$\psi(t) = \int_{-\infty}^{t} p(t|s)\psi(s - T/2)\,\mathrm{d}s = \psi(t+T)$$

- ▷ Residence-time distribution:

$$q(t) = \int_{0}^{T} p(s+t|s)\psi(s - T/2)\,\mathrm{d}s$$

Computation of residence-time distributions

Without forcing $(A = 0)$

$p(t,s) \sim$ exponential, $\psi(t)$ uniform $\Longrightarrow q(t) \sim$ exponential

With forcing $(A \gg \sigma^2)$

- First-passage-time density:

$$p(t,s) \simeq \frac{1}{\mathcal{N}} Q_{\lambda T}(t - |\log \sigma|) \frac{1}{\lambda T_{\mathsf{K}}} \mathrm{e}^{-(t-s)/\lambda T_{\mathsf{K}}} f_{\mathsf{trans}}(t,s)$$

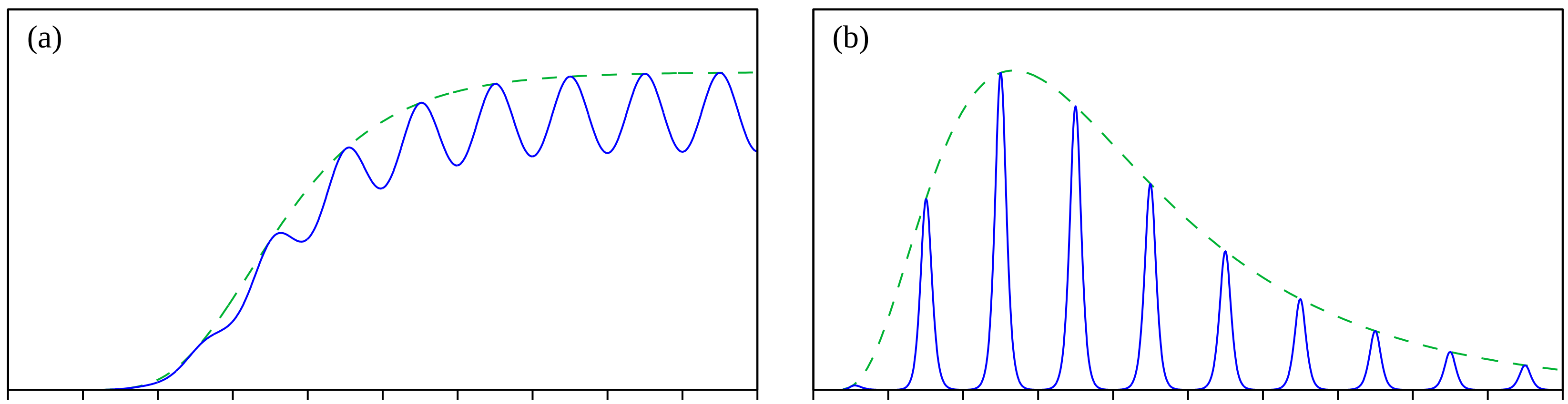
- Asymptotic transition-phase density:

$$\psi(s) \simeq \frac{1}{\lambda T} Q_{\lambda T}(s - |\log \sigma|) \left[1 + \mathcal{O}(T/T_{\mathsf{K}})\right]$$

- Residence-time distribution: (no cycling)

$$q(t) \simeq \tilde{f}_{\mathsf{trans}}(t) \frac{\mathrm{e}^{-t/\lambda T_{\mathsf{K}}}}{\lambda T_{\mathsf{K}}} \frac{\lambda T}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(t + \lambda T/2 - k\lambda T)}$$

Density of the residence-time distribution $\overline{V} = 0.5$, $\lambda = 1$



$\sigma = 0.2$, $T = 2$ $\sigma = 0.4$, $T = 10$

- Peaks symmetric
- No cycling
- σ fixed, λT increasing: Transition into synchronisation regime

- Picture as for Dansgaard–Oeschger events:
  Periodically perturbed *asymmetric* double-well potential

Concluding remarks . . .