1. French Complex Systems Summer School

Theory and Practice

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Random perturbations of dynamical systems

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Abstract

These lectures will provide an introduction to the mathematics of random perturbations. We will start by discussing some examples arising in climate modelling, namely simple conceptual climate models where noise is used to model fluctuations on short time scales such as given by the weather. Typically, these models are multistable and evolve on several well-separated time scales. We shall see that many interesting questions in noisy dynamical systems can be viewed as diffusion exit from a domain or as noise-induced passage through a boundary.

We will than proceed to reviewing the basic mathematical tools for the study of noisy dynamical systems: Ito calculus, stochastic differential equations and the classical Wentzell–Freidlin theory for diffusion exit from a domain. Less well-known but useful tools include results on the distribution of the first-passage time of Brownian motion to a (curved) boundary and so-called small-ball probabilities.

Finally, we will turn to the multitude of interesting phenomena arising in slowly driven systems with noise such as reduction of bifurcation delay, stochastic resonance, noise-induced synchronisation, the effect of noise on the size of hysteresis cycles. Using a constructive method developed by Berglund and the lecturer, we will describe the typical behaviour of a slowly-driven random system by specifying space-time sets in which the system's sample paths are typically concentrated. At the same time, we obtain precise bounds on the probability of atypical paths. We shall conclude by extending this method to general slow-fast systems and applying it to a conceptual model for the thermohaline circulation in the North-Atlantic.

Topics

I Motivation: Climate models

- ▷ Three examples of conceptual (i.e., simple !) climate models

II Review

- ▷ Brownian motion, stochastic integration, stochastic differential equations

III The paradym

- ▷ The overdamped motion of a Brownian particle in a potential
- ▷ Time scales

IV Diffusion exit from a domain

- ▷ Exponential asymptotics: Wentzell–Freidlin theory
- ▷ Refined results for gradient dynamics
- ▷ New phenomena for non-gradient systems: Cycling
- ▷ The density of the time of first passage through an unstable periodic orbit

V Small-ball probabilities for Brownian motion

VI First-passage of Brownian motion to a (curved) boundary

VII The simplest class of slow–fast systems: Slowly driven systems

- ▷ Concentration of sample paths near the bottom of a well
- ▷ Stochastic resonance
- ▷ Hysteresis cycles
- ▷ Bifurcation delay

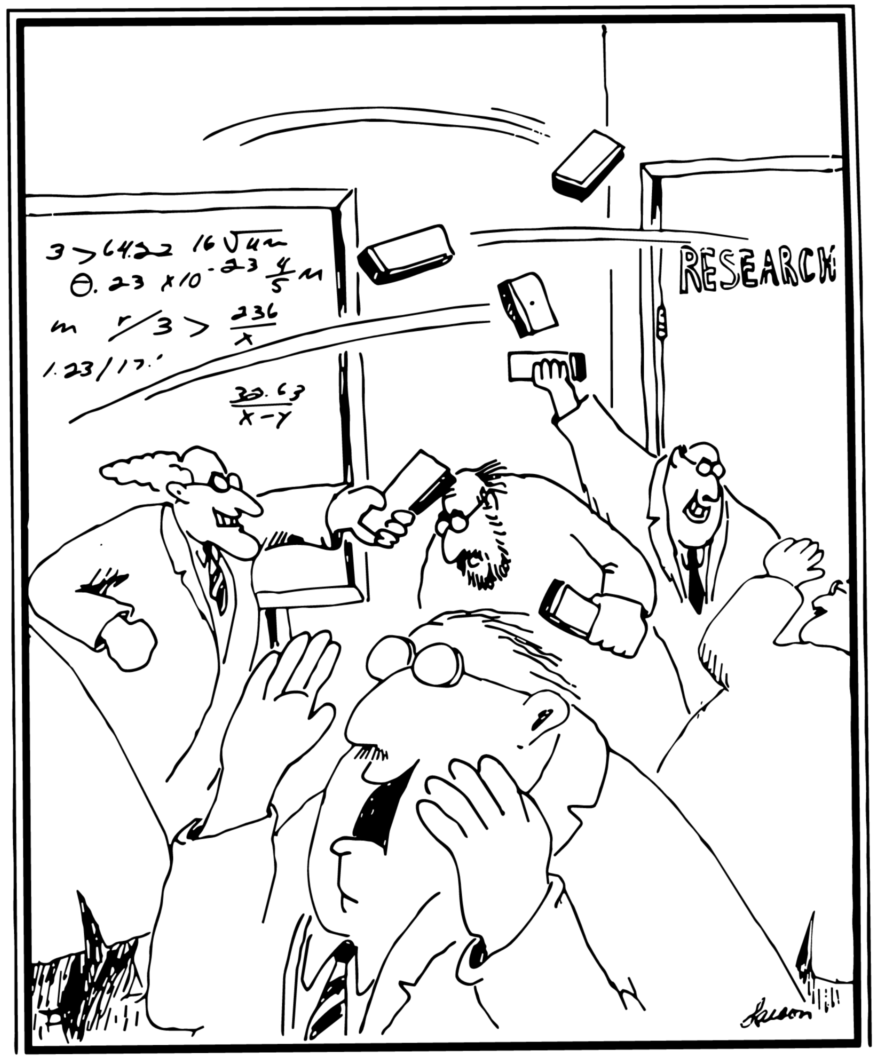
VIII Random perturbations of general slow–fast systems

- ▷ Controlling the random fluctuations of the fast variables
- ▷ Reduced dynamics

The results on random perturbations of slow–fast systems were obtained in joint work with Nils Berglund (Université d'Orléans; previously CPT–CNRS, Marseille)

Slides available at
`http://www.math.uni-bielefeld.de/~gentz/files/Paris_August07.pdf`

This course will focus on (the mathematics of) random perturbations . . .



"Eraser fight!"

PART I

Motivation: Climate models

- ▷ Different classes of climate models
- ▷ Examples of conceptual climate models
  - I Ice Ages: An energy-balance model
  - II Dansgaard–Oeschger events
  - III North-Atlantic thermohaline circulation: Stommel's box model
- ▷ Examples I & II: Stochastic resonance
- ▷ Example III: Relaxation oscillations, excitability, stochastic resonance, hysteresis
- ▷ Random perturbations of general slow-fast systems

Motivation: Climate models

Task: Describe the evolution of the Earth's climate over time spans of several millennia

Seems impossible?

Numerous models have been developed

Goal: Capture the dynamics of the more relevant quantities (such as atmosphere and ocean temperatures averaged over long time intervals and large volumes)

Types of climate models

One distinguishes

General Circulation Models (GCMs): Discretised versions of PDEs governing the atmospheric and oceanic dynamics (including the effect of land masses, ice sheets, etc.)

Earth Models of Intermediate Complexity (EMICs): Focus on certain parts of the climate system, using a more coarse-grained description of the rest of the system

Simple conceptual models (such as box models): Variables are quantities averaged over large volumes. Dynamics based on global conservation laws

Climate models

- ▷ GCMs and EMICs can only be analysed numerically
- ▷ Simple conceptual models are usually chosen such that they are accessible to analytic methods
- ▷ They can provide some insight into the basic mechanisms governing the climate system
- ▷ Even the most refined GCMs have limited resolution, with high-frequency and short-wavelength modes being neglected
- ▷ How to include the effect of unresolved degrees of freedom?

Climate models

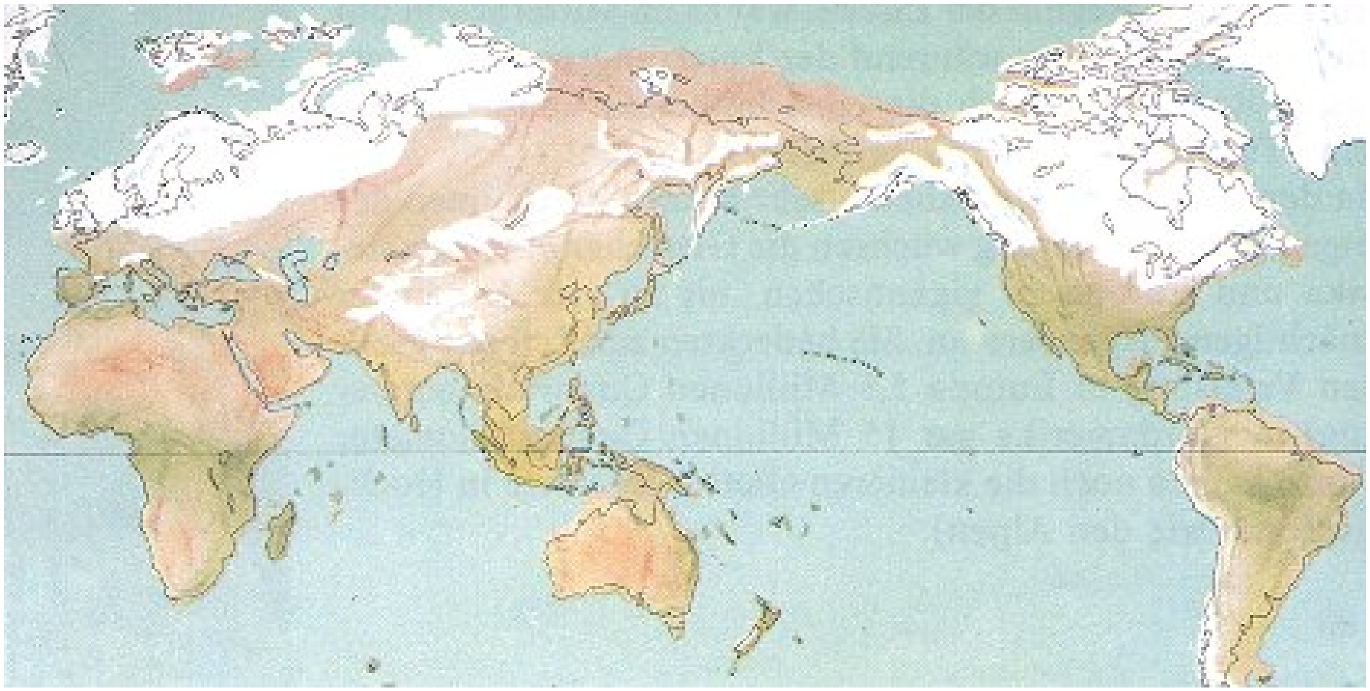
Parametrisation assumes that the unresolved degrees of freedom can be expressed as a function of the resolved ones
(like fast variables enslaved by the slow ones on a stable slow manifold of a slow–fast system)
The parametrisation is chosen on more or less empirical grounds

Averaging means that the equations for the resolved degrees of freedom are averaged over the unresolved ones, using (if possible) an invariant measure of the unresolved system in the averaging process

Modelling unresolved degrees of freedom by a noise term [Hasselmann 1976 (for climate models)]
Approach not yet rigorously justified (partial results by [Khasminskii 1966], [Kifer 1999–], [Bakhtin & Kifer 2004], [Just *et al* 2003])
Deviations from the averaged equations often have Gaussian fluctuations (CLT)
Approach provides a plausible model for rapid transition phenomena observed in the climate system

Examples for conceptual climate models

- ▷ Ice Ages
- ▷ Dansgard–Oeschger events
- ▷ Thermohaline circulation of the North-Atlantic (Gulf stream)



Riss Ice Age, 110.000 years ago

Example I: Ice Ages

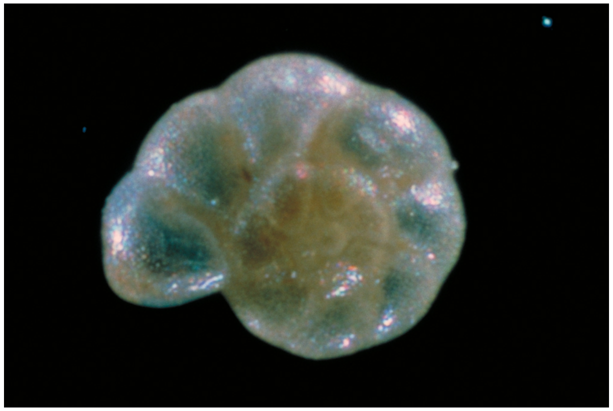
- ▷ During the last 2 million years: more than 20 glacier advances
- ▷ During the last 750.000 years: 8 glacier advances
- ▷ Period: 92.000–100.000 years

How do we know?

Several ways to estimate the amount of ice on Earth

Investigate sediments

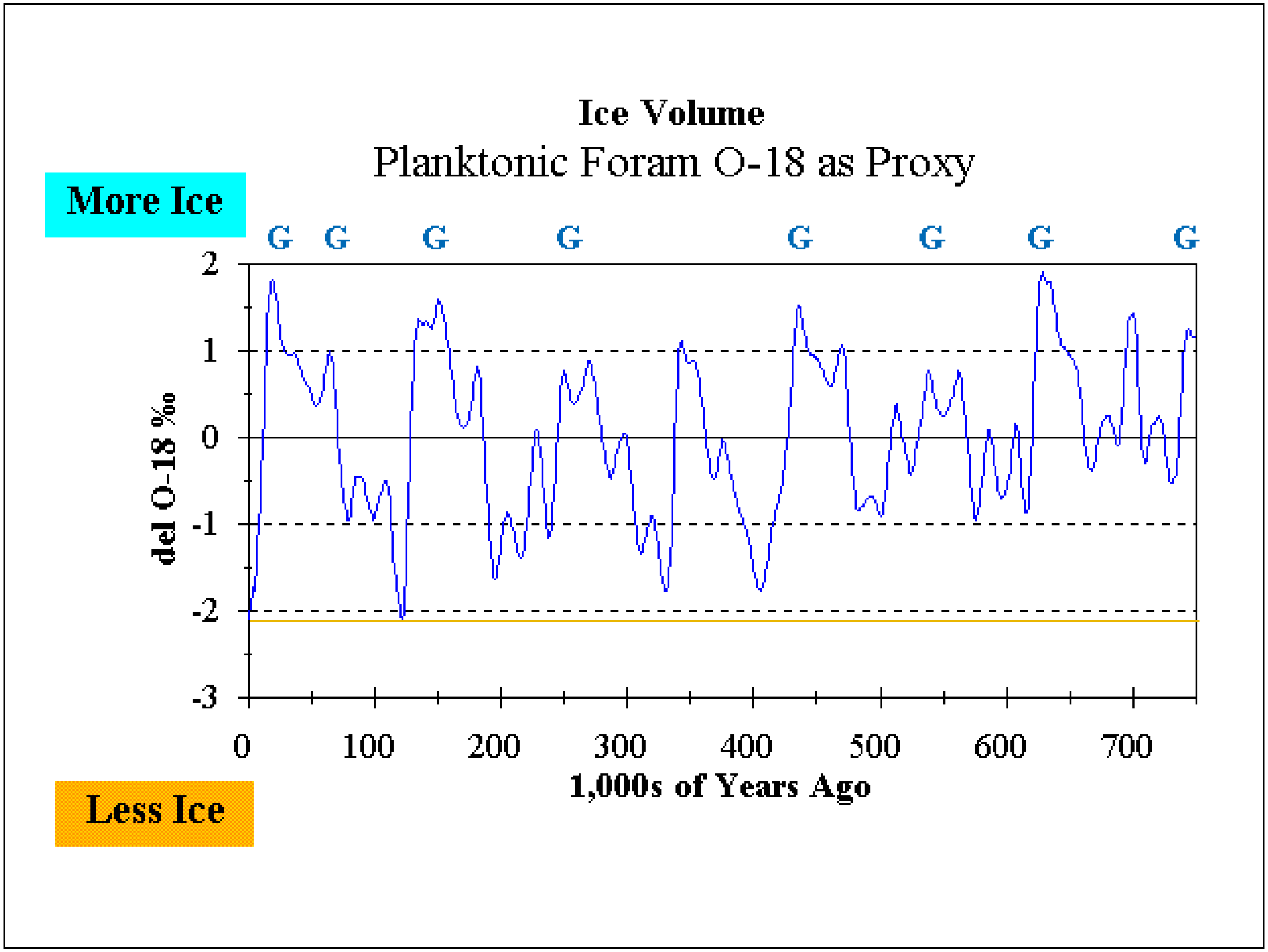
- ▷ Type of plankton:
  Indicator for water temperature
- ▷ Oxygen isotopes:
  Allows conclusions about ice volume



Plankton: Helenina anderseni

(Diameter 1/20–1/10 mm)

Ice Ages



G: Glacier advance in the Middle West of the US

Ice Ages

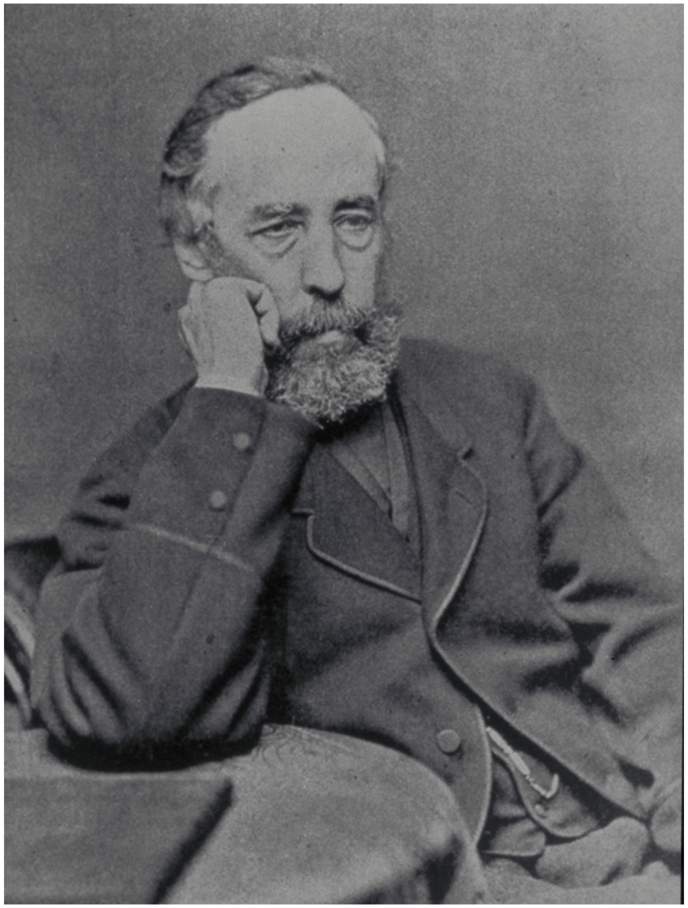
Various proxies indicate that during the last 700 000 years, the Earth's climate has repeatedly experienced dramatic transitions between "warm" phases (with average temperatures comparable to today's values), and Ice Ages (with temperatures about ten degrees lower)

Transitions occured with a striking, though not perfect, regularity

Average period of about 92 000 years

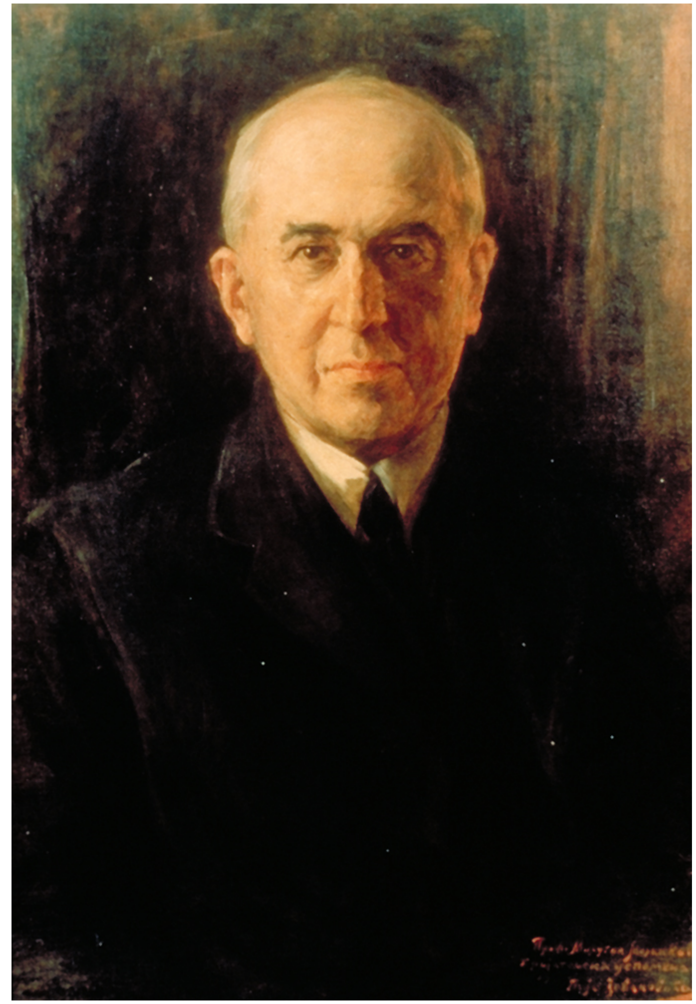
How to explain this regularity?

Milankovitch factors



James Croll

(1821–1890)



Milutin Milankovitch

(1879–1958)

Milankovitch factors

Idea: Regularity of transitions between warm and cold phases might be related to (quasi-)periodic variations of the Earth's orbital parameters [Croll 1864]
Milankovitch (≈1920): Theoretical considerations and calculations

Changes in the eccentricity of the Earth's orbit
(→ Distance Earth–Sun)

Periods: 90.000–100.000 years and 400.000 years
Large excentricity ⟶ large seasonal contrast on one hemisphere
Effect: 0,1–0,2 % variation in insolation

Changes in the tilt of the Earth's axis (22,1°–24,5°)

Period: 41.000 years
more tilt ⟶ enhanced seasonal contrast

The precession of the equinoxes (⟶ Dates of equinox)

Periods: 19 000 years and 23.000 years
⟶ seasonal contrast

Energy-balance model

Simplest model for the variation of the average climate is an energy-balance model

Sole dynamic variable: Mean temperature T of the atmosphere

Its time evolution is described by

$$c\,\frac{\mathrm{d}T}{\mathrm{d}s} = R_{\mathrm{in}}(s) - R_{\mathrm{out}}(T, s)$$

where

- s denotes time
- c is the heat capacity

Energy-balance model

$$c\,\frac{\mathrm{d}T}{\mathrm{d}s} = R_{\text{in}}(s) - R_{\text{out}}(T,s)$$

- $R_{\text{in}}(s)$ is the incoming solar radiation, modelled by the periodic function

$$R_{\text{in}}(s) = Q\Big(1 + K\cos\omega s\Big)$$

- Constant Q is called *solar constant*
- Amplitude K of the modulation is small (of order 5×10^{-4})
- Period $2\pi/\omega = 92\,000$ years
- $R_{\text{out}}(T,s)$ is the outgoing radiation, decomposing into directly reflected radiation and thermal emission:

$$R_{\text{out}}(T,s) = \alpha(T)R_{\text{in}}(s) + E(T)$$

- $\alpha(T)$ is called the Earth's albedo
- $E(T)$ is called emissivity

Energy-balance model

Approximate emissivity $E(T)$ by the Stefan–Boltzmann law of black-body radiation: $E(T) \sim T^4$

$E(T)$ varies little in the range of interest: Replace by constant E_0

Richness of the model lies in modelling the albedo's temperature-dependence (which is influenced by factors such as size of ice sheets and vegetation coverage)

The evolution equation can be rewritten as

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{E_0}{c}\Big[\gamma(T)(1 + K\cos\omega s) + K\cos\omega s\Big]$$

where

$$\gamma(T) = Q(1 - \alpha(T))/E_0 - 1$$

Energy-balance model

For two stable climate regimes to coexist, $\gamma(T)$ should have three roots, the middle root corresponding to an unstable state

Following [Benzi, Parisi, Sutera & Vulpiani 1983], we model $\gamma(T)$ by the cubic polynomial

$$\gamma(T) = \beta\left(1 - \frac{T}{T_1}\right)\left(1 - \frac{T}{T_2}\right)\left(1 - \frac{T}{T_3}\right)$$

where

- $T_1 = 278.6$ K and $T_3 = 288.6$ K are the representative temperatures of the two stable climate regimes
- $T_2 = 283.3$ K represents an intermediate, unstable regime
- β determines the relaxation time τ of the system in the "temperate climate" state, taken to be 8 years, by

$$\frac{1}{\tau} = (\text{curvature at } T_3) \simeq -\frac{E_0}{c}\gamma'(T_3)$$

Energy-balance model

Introduce

- ▷ slow time $t = \omega s$
- ▷ "dimensionless temperature" $x = (T - T_2)/\Delta T$ with $\Delta T = (T_3 - T_1)/2 = 5$ K

Rescaled equation of motion

$$\varepsilon \frac{\mathrm{d}x}{\mathrm{d}t} = -x(x - X_1)(x - X_3)(1 + K \cos t) + A \cos t$$

with $X_1 = (T_1 - T_2)/\Delta T \simeq -0.94$ and $X_3 = (T_3 - T_2)/\Delta T \simeq 1.06$

Adiabatic parameter $\varepsilon = \omega\tau \dfrac{2(T_3 - T_2)}{\Delta T} \simeq 1.16 \times 10^{-3}$

Effective driving amplitude $A = \dfrac{K}{\beta} \dfrac{T_1 T_2 T_3}{(\Delta T)^3} \simeq 0.12$

(according to the value $E_0/c = 8.77 \times 10^{-3}/4000$ Ks^{-1} given in [Benzi, Parisi, Sutera & Vulpiani 1983])

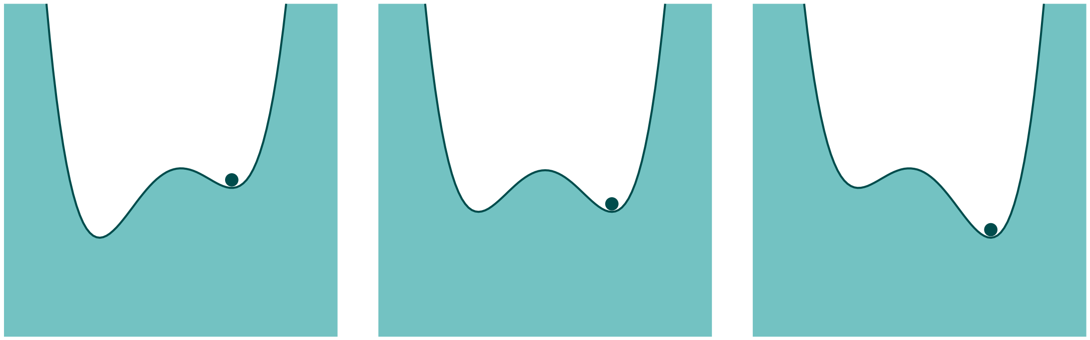
Energy-balance model

For simplicity, replace X_1 by -1, X_3 by 1, and neglect the term $K\cos 2\pi t$

This yields the equation

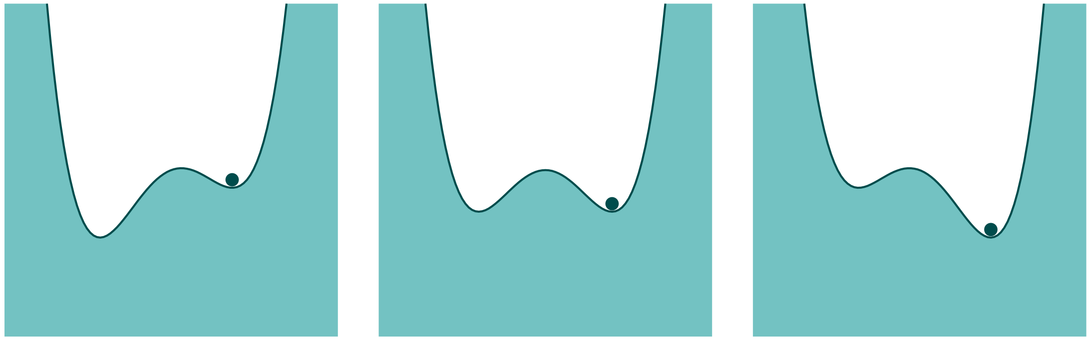
$$\varepsilon\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^3 + A\cos t$$

The right-hand side derives from a double-well potential, and therefore has two stable equilibria and one unstable equilibrium, for all $A < A_c = 2/3\sqrt{3} \simeq 0.38$



Overdamped particle in a periodically forced double-well potential

Energy-balance model



Overdamped particle in a periodically forced double-well potential

In our simple climate model, the two potential wells represent Ice Age and temperate climate

The periodic forcing is subthreshold and thus not sufficient to allow for transitions between the stable equilibria

Model too simple? The slow variations of insolation can only explain the rather drastic changes between climate regimes if some powerful feedbacks are involved, for example a mutual enhancement of ice cover and the Earth's albedo

Energy-balance model

New idea in [Benzi, Sutera & Vulpiani 1981] and [Nicolis & Nicolis 1981]: Incorporate the effect of short-timescale atmospheric fluctuations, by adding a noise term, as suggested by [Hasselmann 1976]

This yields the SDE

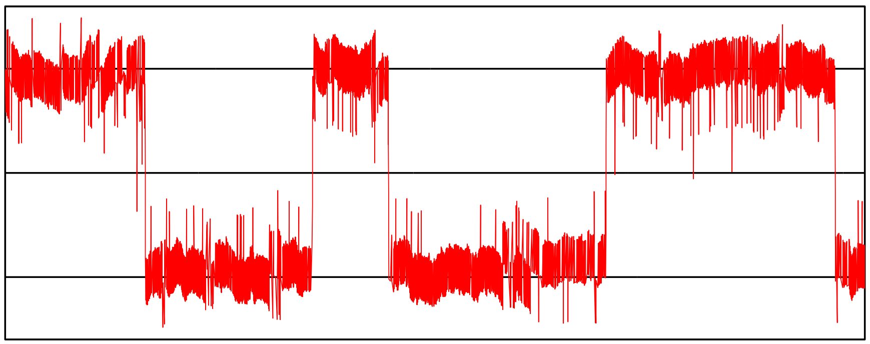
$$\dot{x}_t = \frac{1}{\varepsilon}\Big[x_t - x_t^3 + A\cos t\Big] + \widetilde{\sigma}(\varepsilon)\,\dot{W}_t$$

(considered on the slow timescale, $\widetilde{\sigma} = \sigma/\sqrt{\varepsilon}$)

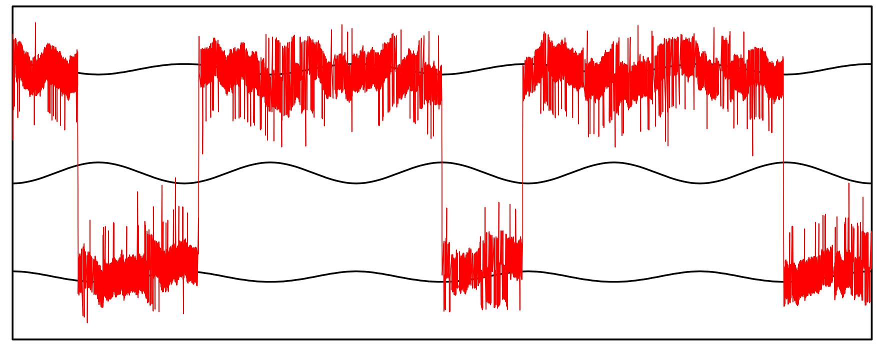
For adequate parameter values, typical solutions are likely to cross the potential barrier twice per period, producing the observed sharp transitions between climate regimes. This is a manifestation of stochastic resonance (SR).

Whether SR is indeed the right explanation for the appearance of Ice Ages is controversial, and hard to decide.

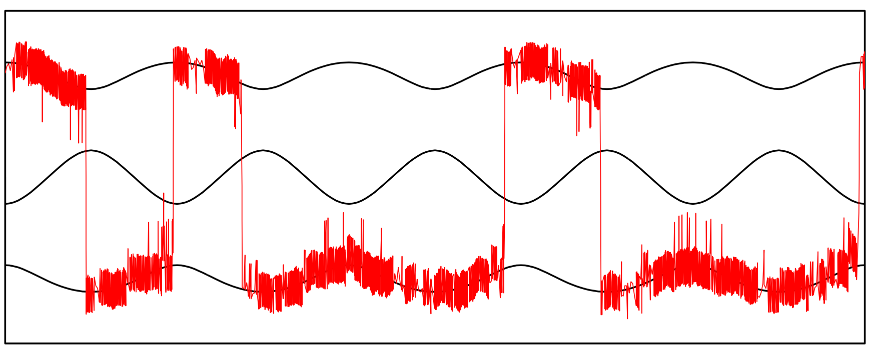
Sample paths



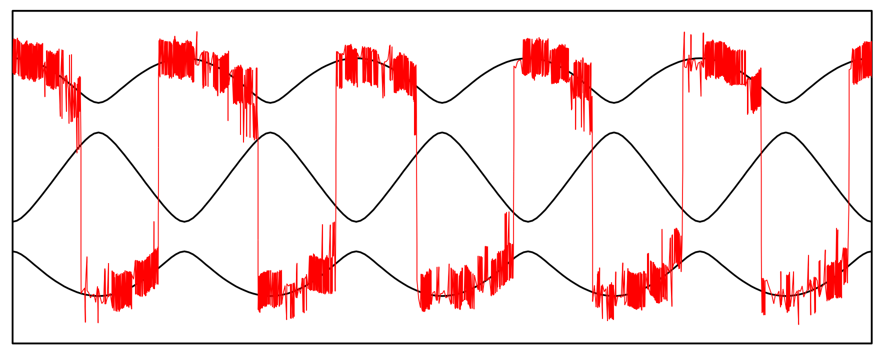
$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$



$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$

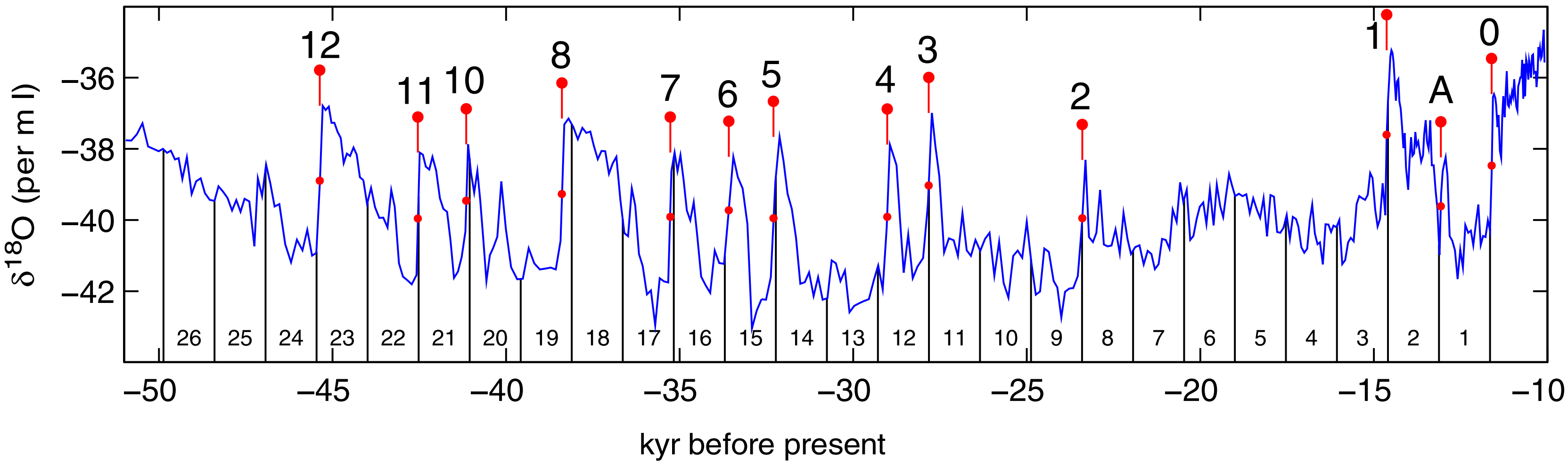


$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

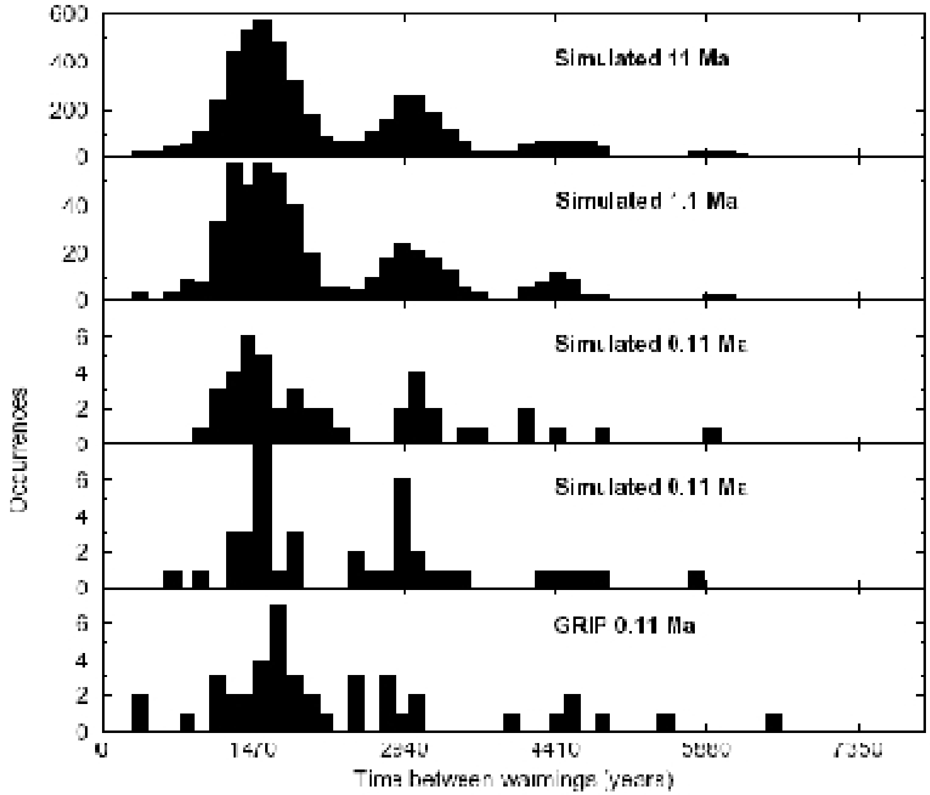
Example II: Dansgaard–Oeschger events



GISP2 climate record for the second half of the last glacial
[Rahmstorf, *Timing of abrupt climate change: A precise clock*, Geophys. Res. Lett. 30 (2003)]

- Abrupt, large-amplitude shifts in global climate during last glacial
- Cold stadials; warm Dansgaard–Oeschger interstadials
- Rapid warming; slower return to cold stadial
- 1 470-year cycle?
- Occasionally a cycle is skipped

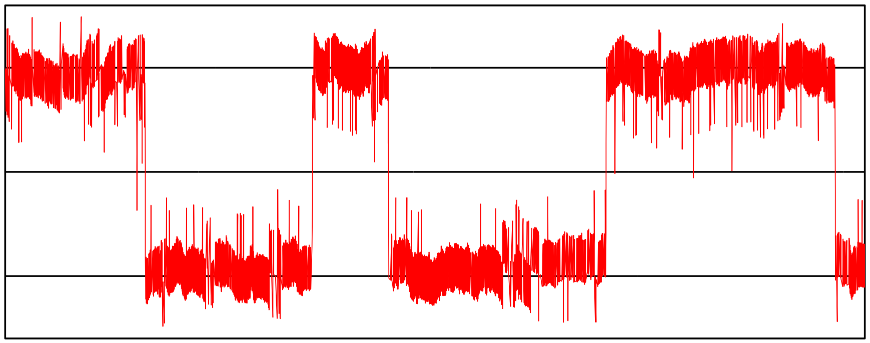
Interspike times for Dansgaard–Oeschger events



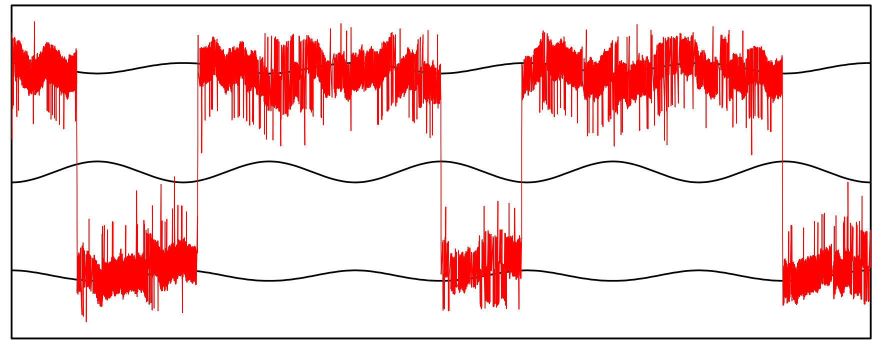
Histogram for "waiting times" between transitions

[from: Alley, Anandakrishnan & Jung, *Stochastic resonance in the North Atlantic*, Paleoceanography 16 (2001)]

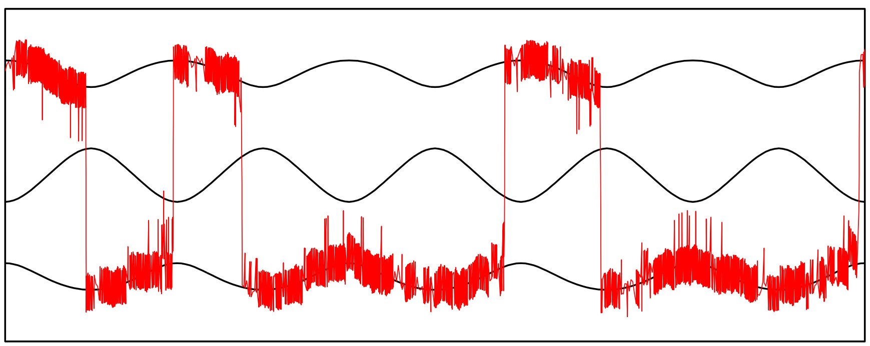
Sample paths



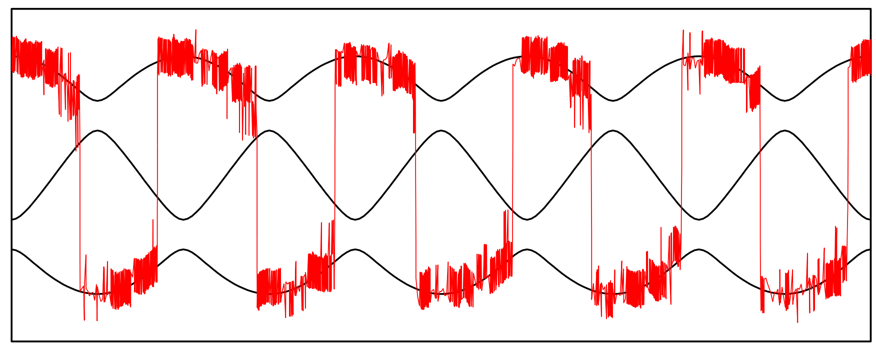
$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$



$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$



$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

Stochastic resonance

What is stochastic resonance (SR)?

SR = mechanism to amplify weak signals in presence of noise

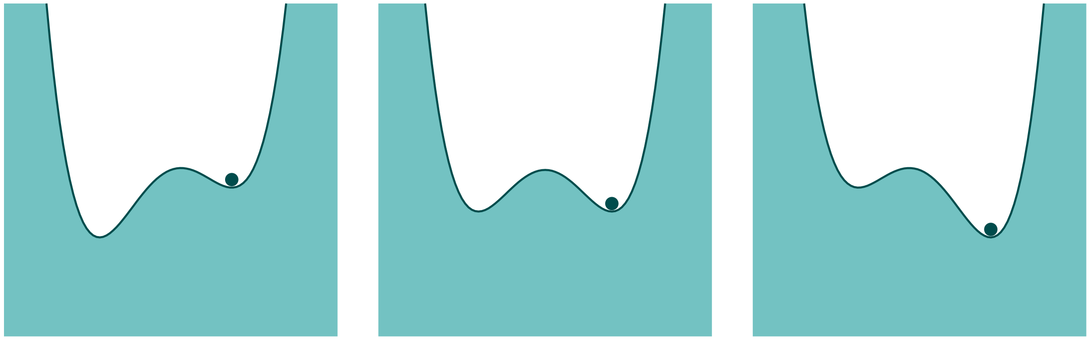
Requirements

- ▷ (background) noise
- ▷ weak input
- ▷ characteristic barrier or threshold (nonlinear system)

Examples

- ▷ periodic occurrence of ice ages (?)
- ▷ Dansgaard–Oeschger events (?)
- ▷ bidirectional ring lasers
- ▷ visual and auditory perception
- ▷ receptor cells in crayfish
- ▷ . . .

Stochastic resonance: The paradigm model



Overdamped motion of a Brownian particle ...

$$\mathrm{d}x_s = \underbrace{\Big[-x_s^3 + x_s + A\cos(\varepsilon s)\Big]}_{= -\frac{\partial}{\partial x} V(x_t, \varepsilon s)} \mathrm{d}s + \sigma \, \mathrm{d}W_s$$

... in a periodically modulated double-well potential

$$V(x,t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\cos(t)x \quad \text{with} \quad A < A_{\mathrm{c}}$$

SR: Different parameter regimes

Synchronisation I

- ▷ Matching time scales $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp \mathrm{e}^{2H/\sigma^2}$
- ▷ Quasistatic approach: Transitions twice per period likely
  (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- ▷ Requires exponentially long forcing periods

Synchronisation II

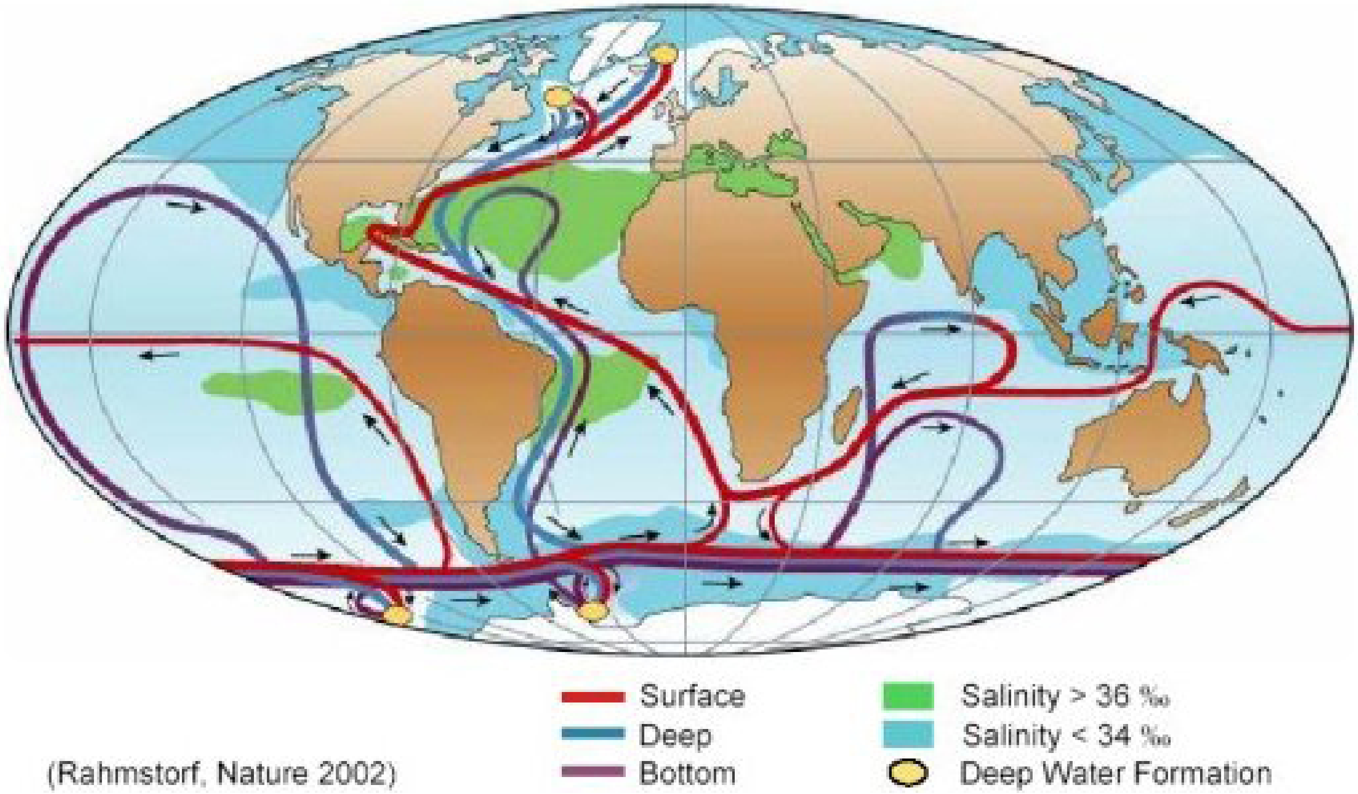
- ▷ Intermediate forcing periods $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$
  and close-to-critical forcing amplitude $A \approx A_{\text{c}}$
- ▷ Transitions twice per period with high probability
- ▷ Subtle dynamical effects: Effective barrier heights [Berglund & G '02]

SR outside synchronisation regimes

- ▷ Only occasional transitions
- ▷ But transition times localised within forcing periods

Unified description / understanding of transition between regimes ?

Example III: North-Atlantic thermohaline circulation

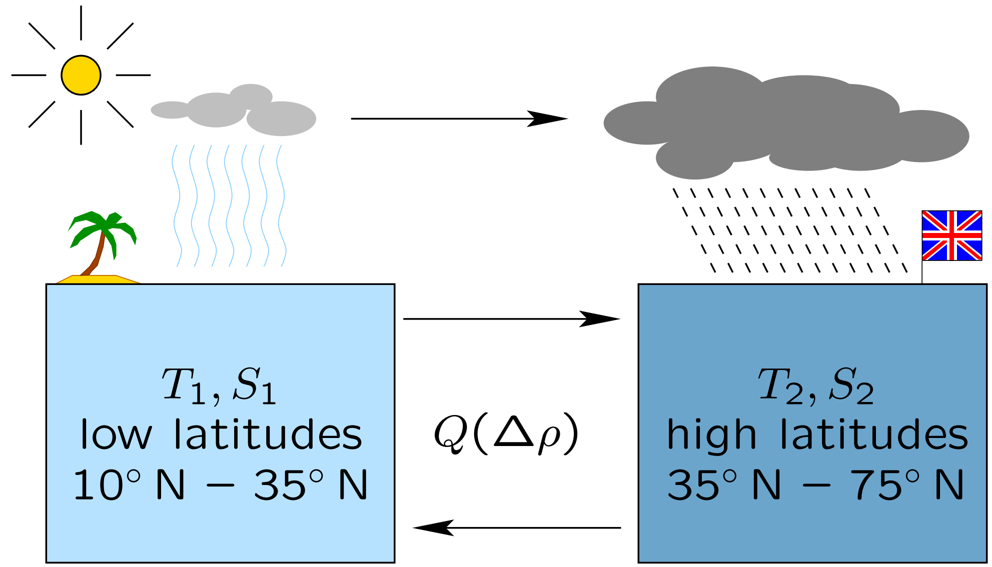


(Rahmstorf, Nature 2002)

- ▷ "Realistic" models (GCMs, EMICs): Numerical analysis
- ▷ Simple conceptual models: Analytical results
- ▷ In particular: Box models

North-Atlantic THC: Stommel's Box Model ('61)

T_i: Temperatures
S_i: Salinities
F: Freshwater flux
$Q(\Delta\rho)$: Mass exchange
$\Delta\rho = \alpha_S \Delta S - \alpha_T \Delta T$
$\Delta T = T_1 - T_2$
$\Delta S = S_1 - S_2$



$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T \\ \frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F - Q(\Delta\rho)\Delta S \end{cases}$$

Model for Q [Cessi '94]: $Q(\Delta\rho) = \frac{1}{\tau_d} + \frac{q}{V}(\Delta\rho)^2$

Stommel's box model as a slow–fast system

Separation of time scales: $\tau_r \ll \tau_d$

Rescaling: $x = \Delta_T/\theta$, $y = (\alpha_S/\alpha_T)(\Delta S/\theta)$, $s = \tau_d t$

$$\begin{cases} \varepsilon\dot{x} = -(x-1) - \varepsilon x[1+\eta^2(x-y)^2] \\ \dot{y} = \mu - y[1+\eta^2(x-y)^2] \end{cases}$$

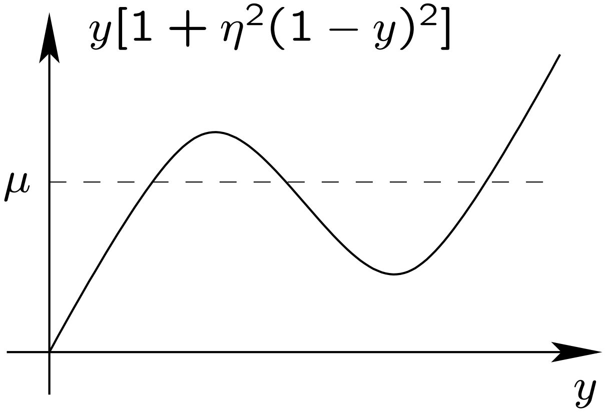
$\varepsilon = \tau_r/\tau_d \ll 1$

Slow manifold ($\varepsilon\dot{x} = 0$):

$$x = x^\star(y) = 1 + \mathcal{O}(\varepsilon)$$

Reduced equation on slow manifold:

$$\dot{y} = \mu - y[1+\eta^2(1-y)^2 + \mathcal{O}(\varepsilon)]$$



1 or 2 stable equilibria, depending on freshwater flux μ (and η)

Stommel's box model with Ornstein–Uhlenbeck noise

$$
\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}\Big[-(x_t - 1) - \varepsilon x_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^1_t \\
\mathrm{d}\xi^1_t &= -\frac{\gamma_1}{\varepsilon}\xi^1_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W^1_t \\
\mathrm{d}y_t &= \Big[\mu - y_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^2_t \\
\mathrm{d}\xi^2_t &= -\gamma_2\xi^2_t\,\mathrm{d}t + \sigma'\,\mathrm{d}W^2_t
\end{aligned}
$$

- ▷ Variance of $x_t - 1 \simeq \sigma^2/(2(1+\gamma_1))$
- ▷ Reduced system for (y_t, ξ^2_t) is bistable (for suitable choice of μ)

How to choose μ, i. e., how to model the freshwater flux?

Modelling the freshwater flux

$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta T = -\frac{1}{\tau_r}(\Delta T - \theta) - Q(\Delta\rho)\Delta T$$
$$\frac{\mathrm{d}}{\mathrm{d}s}\Delta S = \frac{S_0}{H}F(s) - Q(\Delta\rho)\Delta S$$

- ▷ Feedback: F or $\dot{F}$ depending on ΔT and ΔS
  $\Rightarrow$ relaxation oscillations, excitability
- ▷ External periodic forcing
  $\Rightarrow$ stochastic resonance, hysteresis
- ▷ Internal periodic forcing of ocean–atmosphere system
  $\Rightarrow$ stochastic resonance, hysteresis

Case I: Feedback (with Gaussian white noise)

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\Big[-(x_t - 1) - \varepsilon x_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^0$$

$$\mathrm{d}y_t = \Big[\mu_t - y_t Q(x_t - y_t)\Big]\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_t^1$$

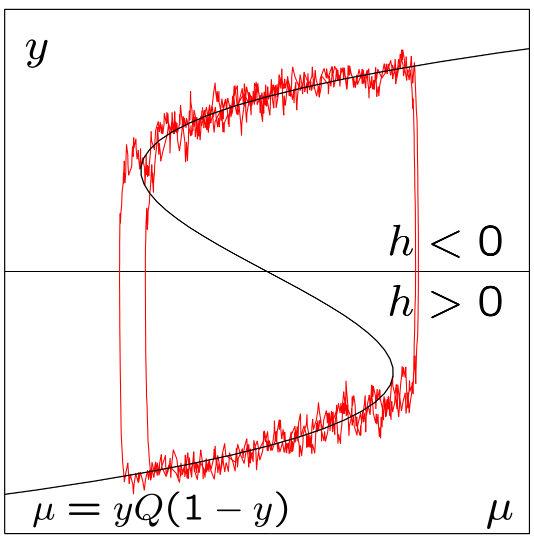
$$\mathrm{d}\mu_t = \tilde{\varepsilon} h(x_t, y_t, \mu_t)\,\mathrm{d}t + \sqrt{\tilde{\varepsilon}}\sigma_2\,\mathrm{d}W_t^2$$ (slow change in freshwater flux)

Reduced equation (after time change $t \mapsto \tilde{\varepsilon} t$)

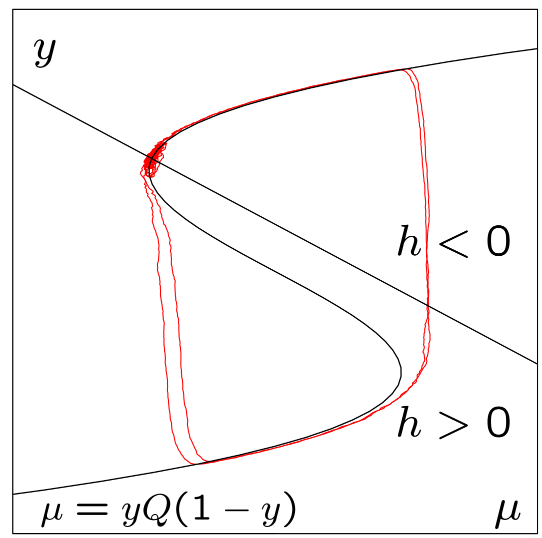
$$\mathrm{d}y_t = \frac{1}{\tilde{\varepsilon}}\Big[\mu_t - y_t Q(1 - y_t)\Big]\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\tilde{\varepsilon}}}\,\mathrm{d}W_t^1$$

$$\mathrm{d}\mu_t = h(1, y_t, \mu_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^2$$

Relaxation oscillations

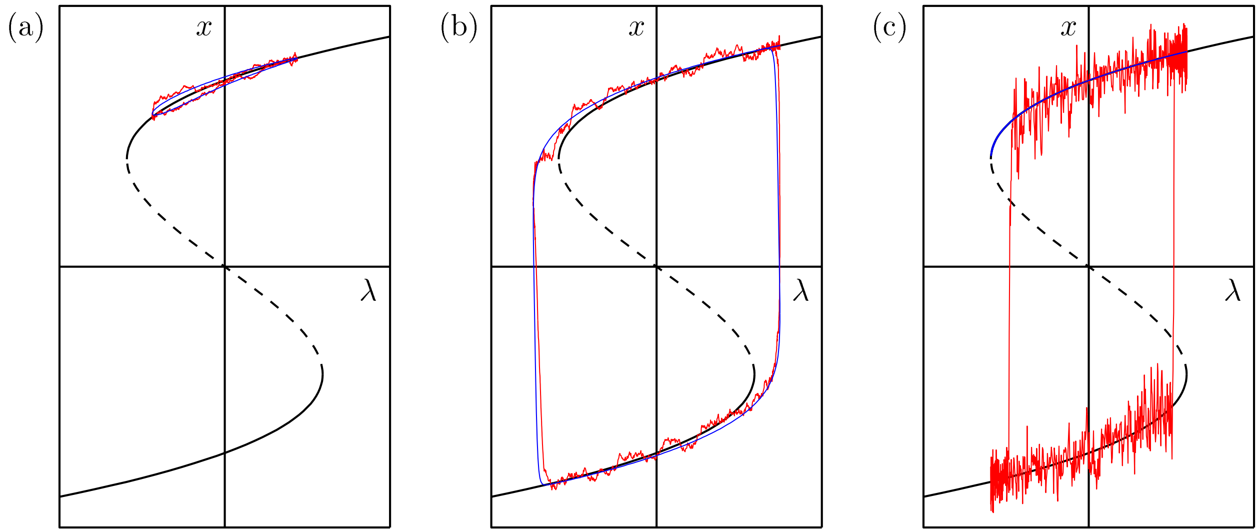


Excitability



Case II: Periodic forcing

Assume periodic freshwater flux $\mu(t)$ (centred w.r.t. bifurcation diagram)



Theorem [Berglund & G '02]

- ▷ Small amplitude, small noise: Transitions unlikely during one cycle (However: Concentration of transition times within each period)
- ▷ Large amplitude, small noise: Hysteresis cycles
  Area = static area + $\mathcal{O}(\varepsilon^{2/3})$ (as in deterministic case)
- ▷ Large noise: Stoch. resonance / noise-induced synchronization
  Area = static area − $\mathcal{O}(\sigma^{4/3})$ (reduced due to noise)

General slow–fast systems

Stommel's box model with noise

$$\begin{aligned}
\mathrm{d}x_t &= \frac{1}{\varepsilon}\Big[-(x_t-1) - \varepsilon x_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^1_t \\
\mathrm{d}\xi^1_t &= -\frac{\gamma_1}{\varepsilon}\xi^1_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W^1_t \\
\mathrm{d}y_t &= \Big[\mu - y_t Q(x_t-y_t)\Big]\,\mathrm{d}t + \mathrm{d}\xi^2_t \\
\mathrm{d}\xi^2_t &= -\gamma_2\xi^2_t\,\mathrm{d}t + \sigma'\,\mathrm{d}W^2_t
\end{aligned}$$

is a special case of a randomly perturbed slow–fast system

$$\begin{cases}
\mathrm{d}x_t = \frac{1}{\varepsilon}f(x_t,y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}F(x_t,y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\
\mathrm{d}y_t = \quad g(x_t,y_t)\,\mathrm{d}t + \sigma'\,G(x_t,y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)}
\end{cases}$$

General slow–fast systems

For deterministic slow–fast systems

$$\begin{cases} \varepsilon\dot{x} = f(x,y) & \text{(fast variables } \in \mathbb{R}^n) \\ \dot{y} = g(x,y) & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

geometric singular perturbation theory permits to study the reduced dynamics on a slow or centre manifold (under suitable assumptions)

Our goals:

- ▷ Analog for the case of random perturbations
- ▷ Effect of random perturbations near bifurcation points of the deterministic system

We will focus on simple cases, in particular slowly driven systems

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Data, figures and photographs:

- ▷ `http://www.ncdc.noaa.gov/paleo/slides`
- ▷ `http://www.museum.state.il.us/exhibits/ice_ages`
- ▷ `http://arcss.colorado.edu/data/gisp_grip` (ice-core date)
- ▷ `http://www.ncdc.noaa.gov/paleo/icecore/greenland/greenland.html` (ice-core date)

And last not least:

- ▷ `http://www.phdcomics.com/comics.php`

I'm inviting you now to follow me onto a journey into probability theory.

In case you're bored – I recommend . . .

Seminar BINGO!

To play, simply print out this bingo sheet and attend a departmental seminar.

Mark over each square that occurs throughout the course of the lecture.

The first one to form a straight line (or all four corners) must yell out BINGO!! to win!

SEMINAR BINGO

B	I	N	G	O
Speaker bashes previous work	Repeated use of "um…"	Speaker sucks up to host professor	Host Professor falls asleep	Speaker wastes 5 minutes explaining outline
Laptop malfunction	Work ties in to Cancer/HIV or War on Terror	"…et al."	You're the only one in your lab that bothered to show up	Blatant typo
Entire slide filled with equations	"The data *clearly* shows…"	FREE Speaker runs out of time	Use of Powerpoint template with blue background	References Advisor (past or present)
There's a Grad Student wearing same clothes as yesterday	Bitter Post-doc asks question	"That's an interesting question"	"Beyond the scope of this work"	Master's student bobs head fighting sleep
Speaker forgets to thank collaborators	Cell phone goes off	You've no idea what's going on	"Future work will…"	Results conveniently show improvement

PART II

Review

- ▷ Brownian motion
- ▷ Stopping times
- ▷ Stochastic integration (Itô integrals)
- ▷ Stochastic differential equations
- ▷ Diffusion processes and Fokker–Planck equation

Stochastic processes

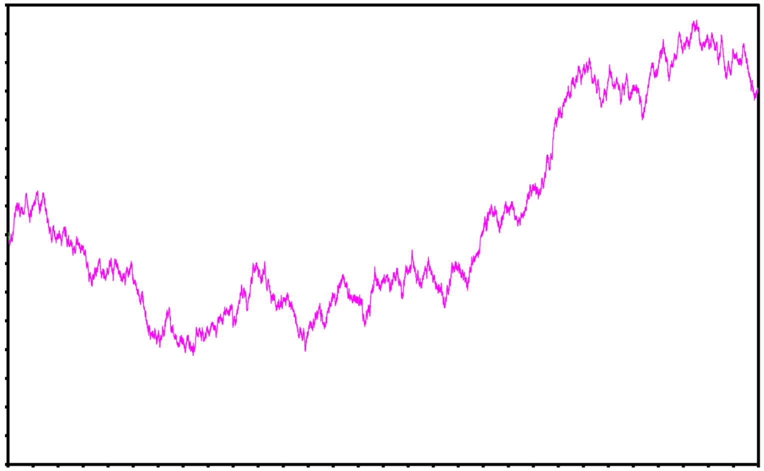
A stochastic process is a collection $\{X_t(\omega)\}_{t\geq 0}$ of random (chance) variables $\omega \mapsto X_t(\omega)$, indexed by time.

ω denotes the dependence on chance

More precisely:
ω denotes the realisation of chance / randomness / noise

View stochastic process as a random function of time: $t \mapsto X_t(\omega)$ (for fixed ω)

We call $t \mapsto X_t(\omega)$ a sample path.



Brownian motion

Physics' literature:

Gaussian white noise $\dot{W}_t(\omega)$ is a Gaussian stationary stochastic process with autocorrelation function

$$C(s) := \mathbb{E}(\dot{W}_t \dot{W}_{t+s}) = \delta(s)$$

- ▷ $\mathbb{E}$ denotes expectation (weighted average over all realizations of the noise)
- ▷ $\delta(s)$ denotes the Dirac delta function

- ▷ $\dot{W}_t$ is completely uncorrelated

Brownian motion (BM): $W_t = \int_0^t \dot{W}_s \, ds$

(In the sense that Gaussian white noise is the generalized mean-square derivative of Brownian motion.)

Sample-path view on Brownian motion
(in the spirit of this course)

BM can be constructed as a scaling limit of a symmetric random walk

$$W_t(\omega) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i(\omega)$$

- $X_i(\omega)$ are independent, identically distributed (i.i.d.) random variables (r.v.'s)
- $\mathbb{E}X_i = 0$, $\mathrm{Var}(X_i) = 1$

Special case:
Nearest-neighbour random walk ($X_i = \pm 1$ with probability 1/2)

The limit is to be understood as convergence in distribution.

Definition of Brownian motion

A one-dimensional standard Brownian motion (or Wiener process) is a stochastic process $\{W_t\}_{t \geq 0}$, satisfying

1. $W_0 = 0$

2. *Independent increments:*
   $W_t - W_s$ is independent of $\{W_u\}_{0 \leq u \leq s}$ (for all $t > s \geq 0$)

3. *Gaussian increments:*
   $W_t - W_s \sim \mathcal{N}(0, t-s)$ (for all $t > s \geq 0$)

   That is:
   $W_t - W_s$ has (probability) density $x \mapsto \dfrac{1}{\sqrt{2\pi(t-s)}}\, \mathrm{e}^{-x^2/2(t-s)}$

   (the famous bell-shape curve!)

Properties of Brownian motion

- ▷ Continuity of sample paths
  We may assume that the sample paths $t \mapsto W_t(\omega)$ of BM are continuous for almost all ω. (Kolmogorov's continuity theorem)
- ▷ Non-differentiability of sample paths
  The sample paths are nowhere differentiable for almost all ω.
- ▷ Markov property
  BM is a Markov process

$$\mathbb{P}\Big\{W_{t+s} \in A \,\Big|\, W_u,\ u \leq t\Big\} = \mathbb{P}\Big\{W_{t+s} \in A \,\Big|\, W_t\Big\}$$

- ▷ Gaussian transition probabilities

$$\mathbb{P}\Big\{W_{t+s} \in A \,\Big|\, W_t = x\Big\} = \mathbb{P}^{t,x}\Big\{W_{t+s} \in A\Big\} = \int_A \frac{\mathrm{e}^{-(y-x)^2/2s}}{\sqrt{2\pi s}}\,\mathrm{d}y$$

- ▷ Fokker–Planck equation (FPE)
  The transition densities $p(t,x)$ satiesfy the FPE / forward Kolmogorov equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} p = \frac{1}{2}\triangle p \qquad \text{(in the } d\text{-dim. case)}$$

Properties of Brownian motion

- ▷ Gaussian process

  $\{W_t\}_{t\geq 0}$ is a Gaussian process (i.e., all its finite-dimensional marginals are Gaussian random variables) with

  – mean zero

  – $\text{Cov}\{W_t, W_s\} := \mathbb{E}(W_t W_s) = t \wedge s$

  Conversely, any mean-zero Gaussian process with this covariance structure is a standard Brownian motion.

- ▷ Scaling property

  $\{cW_{t/c^2}\}_{t\geq 0}$ is a standard Brownian motion (for any $c > 0$)

A k-dimensional standard Brownian motion is a vector

$$W_t = (W_t^{(1)}, \ldots, W_t^{(k)})$$

of k independent one-dimensional standard Brownian motions

Stopping times

A random variable $\tau : \Omega \to [0, \infty]$ is called a stopping time (with respect to the BM $\{W_t\}_t$) if

$$\{\tau \leq t\} = \{\omega \in \Omega \colon \tau(\omega) \leq t\}$$

can be decided from the knowledge of W_s for $s \leq t$ alone.
(No need to "look into the future".)

Formally, we request $\{\tau \leq t\} \in \mathcal{F}_t = \sigma\{W_s,\ 0 \leq s \leq t\}$ for all $t > 0$.

Example: First-exit time from a set

$$\tau_A = \inf\{t > 0 \colon W_t \notin A\} \in [0, \infty]$$

Note: The time

$$\widetilde{\tau}_A = \sup\{t > 0 \colon W_t \in A\} \in [0, \infty]$$

of the last visit to A is in general no stopping time.

André's reflection principle

Consider a Brownian motion $\{W_t\}_t$, starting in $-b < 0$.
(Shift to whole sample path vertically by $-b$.)

First-passage time $\tau_0 = \inf\{t > 0\colon W_t \geq 0\}$ at level $x = 0$

$$\mathbb{P}^{0,-b}\{\tau_0 < t\} = \mathbb{P}^{0,-b}\{\tau_0 < t, W_t \geq 0\} + \mathbb{P}^{0,-b}\{\tau_0 < t, W_t < 0\}$$

Now, for $\tau_0 < t$, $W_t = W_t - W_{\tau_0}$ depends (by the *strong* Markov property) only on W_{τ_0} but not on the rest of the past of the sample path.

We can restart W_t at time τ_0 in $W_{\tau_0} = 0$.

By symmetry of the distribution of the Brownian sample path, starting in 0 at time τ_0,

$$\dots = 2\mathbb{P}^{0,-b}\{\tau_0 < t, W_t \geq 0\} = 2\mathbb{P}^{0,-b}\{W_t \geq 0\} = \int_b^\infty \frac{\mathrm{e}^{-y^2/2t}}{\sqrt{2\pi t}}\,\mathrm{d}y$$

Depends only on the endpoint at time t !

Stochastic integrals (Itô integrals)

Goal: Give a meaning to stochastic differential equations (SDE's)

$$\dot{x}_t = f(x_t, t) + F(x_t, t)\dot{W}_t$$

Consider the discrete-time version

$$x_{t_{k+1}} - x_{t_k} = f(x_{t_k}, t_k)\Delta t_k + F(x_{t_k}, t_k)\Delta W_k, \qquad k \in \{0, \dots, K-1\}$$

with

- ▷ a partition $0 = t_0 < t_1 < \cdots < t_K = T$
- ▷ $\Delta t_k = t_{k+1} - t_k$
- ▷ Gaussian increments $\Delta W_k = W_{t_{k+1}} - W_{t_k}$

Observe that

$$\sum_{k=0}^{K-1} f(x_{t_k}, t_k)\Delta t_k \to \int_0^t f(x_s, s)\,\mathrm{d}s \quad \text{as the partition is chosen finer and finer}$$

Stochastic integrals (Itô integrals)

This suggests to interpret the SDE as an integral equation

$$x_t = x_0 + \int_0^t f(x_s, s)\,\mathrm{d}s + \int_0^t F(x_s, s)\,\mathrm{d}W_s$$

provided the second integral can be defined as

$$\int_0^t F(x_s, s)\,\mathrm{d}W_s = \lim_{\Delta t_k \to 0} \sum_{k=0}^{K-1} F(x_{t_k}, t_k)\Delta W_k$$

in some suitable sense

Thus we want to define (stochastic) integrals of the type

$$\int_0^t h(s, \omega)\,\mathrm{d}W_s(\omega)$$

for suitable integrands $h(s, \omega)$

A heuristic approach to stochastic integrals

Assume for the moment:
$s \mapsto h(s, \omega)$ continuous and of bounded variation for (almost) all ω

Were the paths of the Brownian motion $s \mapsto W_s(\omega)$ also of finite variation, we could apply integration by parts:

$$\begin{aligned}\int_0^t h(s,\omega)\,\mathrm{d}W_s(\omega) &= h(t)W_t(\omega) - h(0)W_0(\omega) - \int_0^t W_s(\omega)h(\mathrm{d}s,\omega) \\ &= h(t)W_t(\omega) - \int_0^t W_s(\omega)h(\mathrm{d}s,\omega)\end{aligned}$$

The integral on the right-hand side is defined as a Stieltjes integral for each fixed ω.
We can use this equation to define $\int_0^t h(s,\omega)\,\mathrm{d}W_s(\omega)$ ω-wise

Unfortunately, the paths of BM are almost surely *not* of finite variation, and we can not expect $s \mapsto h(s,\omega) = F(x_s(\omega), s)$ to be of finite variation either. Thus the class of possible integrands is not large enough for our purpose!

Elementary functions

Let $\mathcal{F}_t = \sigma\{W_s, s \leq t\}$ be the σ-algebra generated by the Brownian motion up to time t. We think of $\mathcal{F}_t$ as the past of the BM up to time t

We start by defining the stochastic integral for a class of particularly simple functions:

$h : [0, T] \times \Omega \to \mathbb{R}$ is called elementary if there exists a partition $0 = t_0 < t_1 < \dots t_K = T$ such that

- ▷ $h(t, \omega) = \sum_{k=0}^{K-1} h_k(\omega) 1_{(t_k, t_{k+1}]}(t)$
- ▷ $\omega \mapsto h_k(\omega)$ is $\mathcal{F}_{t_k}$-measurable for all k

For such elementary integrands h, define

$$\int_0^t h(s, \omega)\, \mathrm{d}W_s(\omega) = \sum_{k=0}^{K-1} h_k(\omega)[W_{t_{k+1}}(\omega) - W_{t_k}(\omega)]$$

Stochastic integrals: L_2-theory

To extend this definition, we use the following isometry

Itô isometry

Let h be elementary with $h_k \in L^2(\Omega)$ for all k. Then,

$$\mathbb{E}\left\{\left(\int_0^t h(s)\,\mathrm{d}W_s\right)^2\right\} = \int_0^t \mathbb{E}\{h(s)^2\}\,\mathrm{d}s$$

Importance of the Itô isometry

The map $h \mapsto \int_0^T h(s)\,\mathrm{d}W_s$ which maps (elementary) h to the stochastic integral of h is an isometry between $L_2([0,T] \times \Omega)$ and $L_2(\Omega)$

Stochastic integrals: L_2-theory

Class of possible integrands $h : [0, T] \times \Omega \to \mathbb{R}$:

- ▷ $(t, \omega) \mapsto h(t, \omega)$ jointly measurable
- ▷ $\omega \mapsto h(t, \omega)$ $\mathcal{F}_t$-measurable for any fixed t (Not looking into future!)
- ▷ $\int_0^T \mathbb{E}\{h(t)^2\} \, \mathrm{d}t < \infty$.

Such h can be approximated by elementary functions $e^{(n)}$

$$\int_0^T \mathbb{E}\{(h(s) - e^{(n)}(s))^2\} \, \mathrm{d}s \to 0, \qquad \text{as } n \to \infty$$

By Itô isometry

$$\int_0^t h(s) \, \mathrm{d}W_s = L_2\text{-}\lim_{n\to\infty} \int_0^t e^{(n)}(s) \, \mathrm{d}W_s$$

is well-defined (its value does not depend on the choice of the sequence of elementary functions)

Stratonovich integral

By our definition of elementary functions, h is approximated by (random) step functions, where the value of such a step function at all times $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$ is $\mathcal{F}_{t_k^{(n)}}$-measurable.

If h is a bounded function and continuous in t for (almost) all ω, the elementary functions $e^{(n)}$ can be chosen by setting $e^{(n)}(t) = h(t_k^{(n)})$ for all $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$.

If we were to choose $e^{(n)}(t) = h(t^\star)$ on $[t_k^{(n)}, t_{k+1}^{(n)}]$ for some different $t^\star \in [t_k^{(n)}, t_{k+1}^{(n)}]$, the definition of the stochastic integral would yield a different value. For instance, choosing $t^\star$ as the midpoint the interval would yield the so-called Stratonovich integral.

Properties of the Itô integral

For $[a,b] \subset [0,T]$, define

$$\int_a^b h(s)\,\mathrm{d}W_s = \int_0^T 1_{[a,b]}(s)h(s)\,\mathrm{d}W_s$$

- Splitting
  $$\int_s^t h(s)\,\mathrm{d}W_s = \int_s^u h(s)\,\mathrm{d}W_s + \int_u^t h(s)\,\mathrm{d}W_s \text{ for } 0 \le s \le u \le t \le T$$
- Linearity
  $$\int_0^t (ch_1(s) + h_2(s))\,\mathrm{d}W_s = c\int_0^t h_1(s)\,\mathrm{d}W_s + \int_0^t h_2(s)\,\mathrm{d}W_s$$
- Expectation
  $$\mathbb{E}\left\{\int_0^t h(s)\,\mathrm{d}W_s\right\} = 0;$$
- Covariance / Itô isometry
  $$\mathbb{E}\left\{\left(\int_0^t h_1(s)\,\mathrm{d}W_s\right)\left(\int_0^t h_2(s)\,\mathrm{d}W_s\right)\right\} = \int_0^t \mathbb{E}\{h_1(s)h_2(s)\}\,\mathrm{d}s$$

Itô integrals as stochastic processes

Consider $X_t = \int_0^t h(s)\,\mathrm{d}W_s$ as a function of t

- X_t is $\mathcal{F}_t$-measurable (not looking into the future)
- X_t is an $\mathcal{F}_t$-martingale: $\mathbb{E}\{X_t|\mathcal{F}_s\} = X_s$ for $0 \leq s \leq t \leq T$
- We may assume that $t \mapsto X_t(\omega)$ is continuous for allmost all ω

Extending the definition

The definition of the Itô integral can be extended to integrands h satisfying the same measurability assumptions as before but a weaker integrability assumption. It is sufficient to assume that

$$\mathbb{P}\left\{\int_0^t h(s,\omega)^2 \,\mathrm{d}s < \infty \quad \text{for all } t \geq 0\right\} = 1.$$

The stochastic integral is then defined as the limit in probability of integrals of elementary functions.

Keep in mind that for such h, those of the above properties of the stochastic integral which involve expectations may fail.

Examples

(a) Calculate $\int_0^t W_s \, \mathrm{d}W_s$ directly from the definition by approximating W_s by elementary functions. (Homework!)
Note that the result

$$\int_0^t W_s \, \mathrm{d}W_s = \frac{1}{2} W_t^2 - \frac{1}{2} t$$

contains an unexpected term $-t/2$, which shows that Itô integrals can not be calculated like ordinary integrals.

(The stochastic integral is a martingale, and the Itô correction $-t$ is the quadratic variation of W_t which makes $W_t^2 - t$ a martingale.)

Below we will state Itô's formula which replaces the chain rule for Riemann integrals. Useful for calculating Itô integrals.

(b) Case of deterministic integrands (h not depending on ω):
$\int_0^t h(s) \, \mathrm{d}W_s$ is Gaussian with mean zero and variance $\int_0^t h(s)^2 \, \mathrm{d}s$

Itô's formula

Assume

- $\triangleright$ h and f satisfy the standard measurability assumptions
- $\triangleright$ $\mathbb{P}\left\{\int_0^t h(s,\omega)^2 \, \mathrm{d}s < \infty \quad \text{for all } t \geq 0\right\} = 1$
- $\triangleright$ $\mathbb{P}\left\{\int_0^t |f(s,\omega)| \, \mathrm{d}s < \infty \quad \text{for all } t \geq 0\right\} = 1$

Itô process

$$X_t = X_0 + \int_0^t f(s) \, \mathrm{d}s + \int_0^t h(s) \, \mathrm{d}W_s$$

Let $g : \mathbb{R} \times [0,T] \to \mathbb{R}$ be continuous with cont. partial derivatives

$$g_t = \frac{\partial}{\partial t} g(x,t), \qquad g_x = \frac{\partial}{\partial x} g(x,t), \qquad g_{xx} = \frac{\partial^2}{\partial x^2} g(x,t)$$

Itô's formula

Then $Y_t = g(X_t, t)$ is again an Itô process, given by

$$Y_t = g(X_0, 0) + \int_0^t \left[g_t(X_s, s) + g_x(X_s, s) f(s) + \frac{1}{2} g_{xx}(X_s, s) h(s)^2 \right] \mathrm{d}s \\ + \int_0^t g_x(X_s, s) h(s) \, \mathrm{d}W_s$$

Using the shorthand

$$\mathrm{d}X_t = f \, \mathrm{d}t + h \, \mathrm{d}W_t$$

Itô's formula can be written as

$$\mathrm{d}Y_t = g_t \, \mathrm{d}t + g_x \, \mathrm{d}X_t + \frac{1}{2} g_{xx} (\mathrm{d}X_t)^2$$

where $(\mathrm{d}X_t)^2$ is calculated according to the scheme

$$(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W_t) = (\mathrm{d}W_t)(\mathrm{d}t) = 0, \qquad (\mathrm{d}W_t)^2 = \mathrm{d}t$$

Examples

(a) Using Itô's formula, we can calculate $\int_0^t s \, \mathrm{d}W_s$:
Set $g(x,t) = t \cdot x$ and $Y_t = g(W_t, t)$.
Then $\mathrm{d}Y_t = W_t \, \mathrm{d}t + t \, \mathrm{d}W_t + \frac{1}{2} 0 \, \mathrm{d}t$, and, therefore,

$$\int_0^t s \, \mathrm{d}W_s = Y_t - Y_0 - \int_0^t W_s \, \mathrm{d}s = tW_t - \int_0^t W_s \, \mathrm{d}s.$$

Note that this is an integration-by-parts formula.
Similarly, by setting $g(x,t) = h(t) \cdot x$, the integration-by-parts formula from Slide 51 can be established for suitable h.

(b) Choosing $g(x,t) = x^2$ and $Y_t = g(t, W_t)$, Itô's formula gives a much easier way to calculate $\int_0^t W_s \, \mathrm{d}W_s$. (Homework!)

(c) Let $X_t = W_t - t/2$. Use Itô's formula to show that $Y_t = \mathrm{e}^{X_t}$ satisfies

$$\mathrm{d}Y_t = Y_t \, \mathrm{d}W_t$$

Y_t is called the Doléans exponential of W_t.

The multidimensional case

Extension to $\mathbb{R}^n$ is easy:

- ▷ $W_t = (W_t^{(1)}, \ldots, W_t^{(k)})$ k-dimensional standard BM
- ▷ $h(s,\omega) = (h_{ij}(s,\omega))_{i \le n, j \le k}$ a matrix-valued function, taking values in the set of $(n \times k)$-matrices
- ▷ Assume, each h_{ij} allows for stochastic integration in $\mathbb{R}$

Define the ith component of the n-dim. stochastic integral by

$$\sum_{j=1}^{k} \int_0^t h_{ij}(s)\,\mathrm{d}W_s^{(j)}$$

The above mentioned properties of stochastic integrals carry over in the natural way. In particular, the covariance of stochastic integrals can be calculated as

$$\mathbb{E}\Big\{\Big(\int_0^t f(s)\,\mathrm{d}W_s\Big)\Big(\int_0^t g(s)\,\mathrm{d}W_s\Big)^{\mathsf{T}}\Big\} = \int_0^t \mathbb{E}\{f(s)g(s)^{\mathsf{T}}\}\,\mathrm{d}s$$

Itô's formula: The multidimensional case

As the multidimensional integral can be defined componentwise, it is sufficient to consider $Y_t = g(X_t, t)$ for multidimensional X_t and one-dimensional Y_t.

- ▷ $h : [0, \infty) \times \Omega \to \mathbb{R}^{n \times k}$
- ▷ $f : [0, \infty) \times \Omega \to \mathbb{R}^{n}$
- ▷ $g : \mathbb{R}^{n} \times [0, T] \to \mathbb{R}$
- ▷ Assumptions as before . . .

Let $\quad \mathrm{d}X_t = f(t)\,\mathrm{d}t + h(t)\,\mathrm{d}W_t \quad$ and $\quad Y_t = g(X_t, t)$

Then

$$\mathrm{d}Y_t = g_t(X_t, t)\,\mathrm{d}t + \sum_{i=1}^{n} g_{x_i}(X_t, t)\,\mathrm{d}X_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} g_{x_i x_j}(X_t, t)(\mathrm{d}X_t^{(i)})(\mathrm{d}X_t^{(j)})$$

using the scheme

$(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}W_t^{(\mu)}) = (\mathrm{d}W_t^{(\mu)})(\mathrm{d}t) = 0$ and $(\mathrm{d}W_t^{(\mu)})(\mathrm{d}W_t^{(\nu)}) = \delta_{\mu\nu}\,\mathrm{d}t$

Application of the multidimensional version of Itô's formula

Integration-by-parts formula

Let $\mathrm{d}X_t^{(i)} = f_i \, \mathrm{d}t + h_i \, \mathrm{d}W_t$ for $i = 1, 2$

The multidimensional version of Itô's formula shows

$$X_t^{(1)} X_t^{(2)} = X_0^{(1)} X_0^{(2)} + \int_0^t X_s^{(1)} \, \mathrm{d}X_s^{(2)} + \int_0^t X_s^{(2)} \, \mathrm{d}X_s^{(1)} + \int_0^t h_1(s) h_2(s) \, \mathrm{d}s$$

Stochastic differential equations

Goal: Give a meaning to SDE's of the form

$$\mathrm{d}x_t = f(x_t, t)\,\mathrm{d}t + F(x_t, t)\,\mathrm{d}W_t$$

$\{x_t\}_{t\in[0,T]}$ is called a strong solution with initial condition x_0 if

- ▷ For all t: x_t is $\{W_s; s \leq t\}$-measurable
  (depends only on the past of the BM up to time t)
- ▷ Integrability condition:

$$\mathbb{P}\biggl\{\int_0^T \|f(x_s, s)\|\,\mathrm{d}s < \infty\biggr\} = 1\,, \quad \mathbb{P}\biggl\{\int_0^T \|F(x_s, s)\|^2\,\mathrm{d}s < \infty\biggr\} = 1$$

- ▷ For all t:

$$x_t = x_0 + \int_0^t f(x_s, s)\,\mathrm{d}s + \int_0^t F(x_s, s)\,\mathrm{d}W_s \quad \text{holds for almost all } \omega$$

If the initial condition x_0 is random, we assume that it does not depend on the BM !

Existence and uniqueness

Assume

- ▷ Lipschitz condition (local Lipschitz condition suffices)

$$\|f(x,t) - f(y,t)\| + \|F(x,t) - F(y,t)\| \leq K\|x - y\|$$

- ▷ Bounded-growth condition

$$\|f(x,t)\| + \|F(x,t)\| \leq K(1 + \|x\|)$$

(Can be relaxed, f.e. to $xf(x,t)+F(x,t)^2 \leq K^2(1+x^2)$ in the one-dim. case)

Then: The SDE has a (pathwise) unique almost surely continuous solution x_t

Uniqueness means:
For any two almost surely continuous solutions x_t and y_t

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|x_t - y_t\| > 0\right\} = 0$$

Existence and uniqueness: Remarks

- ▷ As in the deterministic case: Uniqueness requires only the Lipschitz condition
- ▷ As in the deterministic case: The bounded-growth condition excludes explosions of the solution
- ▷ Conditions can be relaxed in many ways
- ▷ Proof by a stochastic version of Picard–Lindelöf iterations
- ▷ The solution x_t satisfies the strong Markov property, meaning that we can restart the process not only at fixed times s in x_s but even at any stopping time τ in x_τ

Example: Linear SDE's

- ▷ We frequently approximate solutions of SDE's locally by linearizing
- ▷ Linear SDE's can be solved easily

One-dimensional linear SDE

$$\mathrm{d}x_t = \Big[a(t)x_t + b(t)\Big]\,\mathrm{d}t + F(t)\,\mathrm{d}W_t$$

Admits a strong solution

$$x_t = x_0\,\mathrm{e}^{\alpha(t,t_0)} + \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)}\,b(s)\,\mathrm{d}s + \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)}\,F(s)\,\mathrm{d}W_s$$

where

$$\alpha(t,s) = \int_s^t a(u)\,\mathrm{d}u$$

(Use Itô's formula to solve the equation! Hint: $y_t = \mathrm{e}^{-\alpha(t,t_0)}\,x_t$)

Example: Linear SDE's

▷ If the initial condition x_0 is either deterministic of Gaussian, then

$$x_t = x_0 \, \mathrm{e}^{\alpha(t,t_0)} + \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)} \, b(s) \; \mathrm{d}s + \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)} \, F(s) \; \mathrm{d}W_s$$

is a Gaussian process

▷ For arbitrary initial conditions (independent of the BM):

$$\mathbb{E}\{x_t\} = \mathbb{E}\{x_0\} \, \mathrm{e}^{\alpha(t)} + \int_0^t b(s) \, \mathrm{e}^{\alpha(t,s)} \, \mathrm{d}s,$$

$$\mathrm{Var}\,\{x_t\} = \mathrm{Var}\,\{x_0\} \, \mathrm{e}^{2\alpha(t)} + \int_0^t F(s)^2 \, \mathrm{e}^{2\alpha(t,s)} \, \mathrm{d}s,$$

If $a(t) \leq -a_0$, the effect of the initial condition is suppressed exponentially fast in t

Example: Ornstein–Uhlenbeck process

Consider the particular case

$$a(t) \equiv -\gamma \;, \qquad b(t) \equiv 0 \;, \qquad F(t) \equiv 1$$

leading to the SDE

$$\mathrm{d}x_t = -\gamma x_t \,\mathrm{d}t + \mathrm{d}W_t$$

Its solution

$$x_t = x_0 \,\mathrm{e}^{-\gamma(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-\gamma(t-s)} \,\mathrm{d}W_s$$

is known as Ornstein–Uhlenbeck process, modelling the velocity of a Brownian particle. In this context, $-\gamma x_t$ is the damping or frictional force

As soon as $t \gg 1/2\gamma$, x_t relaxes quickly towards its equilibrium distribution which is Gaussian with mean zero and variance

$$\lim_{t\to\infty} \mathrm{Var}\{x_t\} = \lim_{t\to\infty} \int_{t_0}^{t} \mathrm{e}^{-2\gamma(t-s)} \,\mathrm{d}s = \lim_{t\to\infty} \frac{1}{2\gamma}\Big[1 - \mathrm{e}^{-2\gamma t}\Big] = \frac{1}{2\gamma}$$

Diffusion processes and Fokker–Planck equation

Diffusion process

$$\mathrm{d}x_t = f(x_t, t)\,\mathrm{d}t + F(x_t, t)\,\mathrm{d}W_t$$

The solution x_t is an (inhomogenous) Markov process, and the densities of the transition properties satisfy Kolmogorov's forward or Fokker–Planck equation

$$\frac{\partial}{\partial t}\rho(y,t) = L\rho(y,t)$$

- $L\varphi = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\Big(f_i(y,t)\varphi\Big) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j}\Big(d_{ij}(y,t)\varphi\Big)$
- $d_{ij}(x,t)$ are the matrix elements of $D(x,t) := F(x,t)F(x,t)^{\mathsf{T}}$
- $\rho : (y,t) \mapsto p(y,t|x,s)$ is the (time-dependent) density of the transition probability, when starting in x at time s

Note: If x_t admits an invariant density ρ_0, then $L\rho_0 = 0$

Gradient systems and Fokker–Planck equation

Consider an (autonomous) SDE of the form

$$\mathrm{d}x_t = -\nabla U(x)\,\mathrm{d}x + \sigma\,\mathrm{d}W_t$$

Then

$$L = \Delta U + \nabla U \cdot \nabla + \frac{\sigma^2}{2}\Delta$$

If the potential grows sufficiently quickly at infinity, the stochastic process admits an invariant density

$$\rho_0(x) = \frac{1}{\mathcal{N}}\,\mathrm{e}^{-2U(x)/\sigma^2}$$

(Homework: Compute L and verify that $L\rho_0 = 0$.)

For the Ornstein–Uhlenbeck process, $U(x)$ is quadratic, and thus the invariant density is indeed Gaussian.

References for PART II

The covered material is pretty standard, and you can choose your favourite text book. Standard references are for instance

- ▷ R. Durrett, *Brownian motion and martingales in analysis*, Wadswort (1984)
- ▷ I. Karatzas, and S. E. Shreve, *Brownian motion and stochastic calculus*, Springer (1991)
- ▷ Ph. E. Protter, *Stochastic integration and differential equations*, Springer (2003)
- ▷ B. K. Øksendal, *Stochastic differential equations*, Springer (2000)

For those who can read French, I'd like to recommend also the lecture notes by Jean-François Le Gall, available at

- ▷ `http://www.dma.ens.fr/~legall`

PART III

The paradym

- ▷ The overdamped motion of a Brownian particle in a potential
- ▷ Time scales
- ▷ Metastability
- ▷ Slowly driven systems

The motion of a particle in a double-well potential

Two-parameter family of ODEs

$$\frac{\mathrm{d}x_s}{\mathrm{d}s} = \mu x_s - x_s^3 + \lambda$$

describes the overdamped motion of a particle in the potential

$$U(x) = -\frac{1}{2}\mu x^2 + \frac{1}{4}x^4 - \lambda x$$

- ▷ $\mu^3 > (27/4)\lambda^2$: Two wells, one saddle
- ▷ $\mu^3 < (27/4)\lambda^2$: One well
- ▷ $\mu^3 = (27/4)\lambda^2$ and $\lambda \neq 0$: Saddle–node bifurcation between the saddle and one of the wells
- ▷ $(x, \lambda, \mu) = (0, 0, 0)$: Pitchfork bifurcation point

Notation

$x^\star_\pm$ for (the position of) the well bottoms and $x^\star_0$ for the saddle

The motion of a Brownian particle in a double-well potential

For a Brownian particle:

$$\mathrm{d}x_s = \left[\mu x_s - x_s^3 + \lambda\right] \mathrm{d}s + \sigma \, \mathrm{d}W_s$$

x_s has an invariant density

$$p_0(x) = \frac{1}{N} \mathrm{e}^{-2U(x)/\sigma^2}$$

- ▷ For small σ, $p_0(x)$ is strongly concentrated near the minima of the potential
- ▷ If $U(x)$ has two wells of different depths, the invariant density favours the deeper well

The invariant density does not contain all the information needed to describe the motion!

Time scales

Assume : U double-well potential and x_0 concentrated at the bottom $x_+^\star$ of the right-hand well

How long does it take, until we may safely assume that x_t is well described by the invariant distribution?

- If the noise is sufficiently weak, paths are likely to stay in the right-hand well for a long time
- x_t will first approach a Gaussian in a time of order

$$T_{\text{relax}} = \frac{1}{c} = \frac{1}{\text{curvature at the bottom } x_+^\star \text{ of the well}}$$

- With overwhelming probability, paths will remain inside the *same* well, for all times significantly shorter than Kramers' time $T_{\text{Kramers}} = \mathrm{e}^{2H/\sigma^2}$, where $H = U(x_0^\star) - U(x_+^\star) =$ barrier height
- Only on longer time scales, the density of x_t will approach the bimodal stationary density p_0

Time scales

Dynamics is thus very different on the different time scales

- $t \ll T_{\text{relax}}$
- $T_{\text{relax}} \ll t \ll T_{\text{Kramers}}$
- $t \gg T_{\text{Kramers}}$

Method of choice to study the SDE depends on the time scale we are interested in

Hierarchical description

- On a coarse-grained level, the dynamics is described by a two-state Markovian jump process, with transition rates $\mathrm{e}^{-2H_\pm/\sigma^2}$
- Dynamics between transitions (inside a well) can be approximated by ignoring the other well

  Approximate local dynamics of the deviation $x_t - x^\star_\pm$ by the linearisation (OU process)

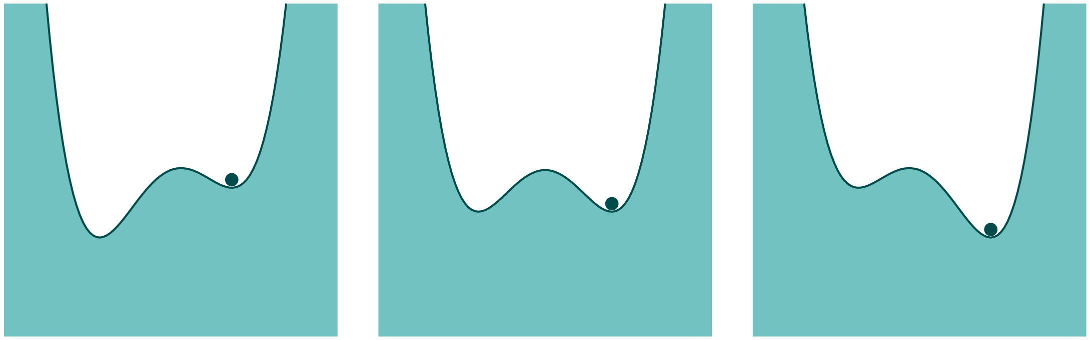
$$\mathrm{d}y_s = -\omega_\pm^2 y_s \,\mathrm{d}s + \sigma \,\mathrm{d}W_s$$

Metastability

The fact, that the double-well structure of the potential is not visible on time scales shorter than T_{Kramers} is a manifestation of metastability: The distribution concentrated near $x_+^\star$ seems to be invariant

The relevant time scales for metastability are related to the small eigenvalues of the generator of the diffusion

Slowly driven systems



Let us now turn to situations in which the potential $U(x) = U(x, \varepsilon s)$ depends slowly on time:

$$\mathrm{d}x_s = -\frac{\partial U}{\partial x}(x_s, \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

In slow time $t = \varepsilon s$

$$\mathrm{d}x_t = -\frac{1}{\varepsilon}\frac{\partial U}{\partial x}(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

($\mathrm{d}t = \varepsilon\,\mathrm{d}s$, $\mathrm{d}W_t = \sqrt{\varepsilon}\,\mathrm{d}W_s$ as $W_{\varepsilon s}$ and $\sqrt{\varepsilon}W_s$ have the same distribution)

Note that the probability density of x_t still obeys a Fokker–Planck equation, but there will be no stationary solution in general

Slowly driven systems

- ▷ Depths $H_\pm = H_\pm(t)$ of the well may now depend on time, and may even vanish if one of the bifurcation curves is crossed
- ▷ "Instantaneous" Kramers timescales $e^{2H_\pm(t)/\sigma^2}$ no longer fixed
- ▷ If the forcing timescale ε^{-1}, at which the potential changes shape, is longer than the maximal Kramers time of the system, one can expect the dynamics to be a slow modulation of the dynamics for frozen potential
- ▷ Otherwise, the interplay between the timescales of modulation and of noise-induced transitions becomes nontrivial

ε introduces additional timescale via the forcing speed $T_{\text{forcing}} = 1/\varepsilon$

Slowly driven systems

Questions

- ▷ How long do sample paths remain concentrated near stable equilibrium branches, that is, near the bottom of slowly moving potential wells?
- ▷ How fast do sample paths depart from unstable equilibrium branches, that is, from slowly moving saddles?
- ▷ What happens near bifurcation points, when the number of equilibrium branches changes?
- ▷ What can be said about the dynamics far from equilibrium branches?

PART IV

Diffusion exit from a domain

- ▷ Large deviations for Brownian motion
- ▷ Large deviations for diffusion processes
- ▷ Diffusion exit from a domain
- ▷ Relation to PDEs
- ▷ The concept of a quasipotential
- ▷ Asymptotic behaviour of first-exit times and locations (small-noise asymptotics)
- ▷ Refined results for gradient systems
- ▷ Refined results for non-gradient systems: Passage through an unstable periodic orbit
- ▷ Cycling

Introduction: Small random perturbations

Consider a small random perturbation

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,g(x_t^\varepsilon)\,\mathrm{d}W_t, \qquad x_0^\varepsilon = x_0$$

of ODE

$$\dot{x}_t = b(x_t) \qquad\qquad \text{(with same initial cond.)}$$

We expect $x_t^\varepsilon \approx x_t$ for small ε

Depends on

- ▷ deterministic dynamics
- ▷ noise intensity ε
- ▷ time scale

Introduction: Small random perturbations

Indeed, for b Lipschitz continuous and $g = \mathrm{Id}$

$$\|x_t^\varepsilon - x_t\| \leq L \int_0^t \|x_s^\varepsilon - x_s\| \, \mathrm{d}s + \sqrt{\varepsilon} \, \|W_t\|$$

Gronwall's lemma shows

$$\sup_{0 \leq s \leq t} \|x_s^\varepsilon - x_s\| \leq \sqrt{\varepsilon} \sup_{0 \leq s \leq t} \|W_s\| \, \mathrm{e}^{Lt}$$

Remains to estimate $\sup_{0 \leq s \leq t} \|W_s\|$

- $d = 1$: Use reflection principle

$$\mathbb{P}\Big\{ \sup_{0 \leq s \leq t} |W_s| \geq r \Big\} \leq 2 \, \mathbb{P}\Big\{ \sup_{0 \leq s \leq t} W_s \geq r \Big\} \leq 4 \, \mathbb{P}\Big\{ W_t \geq r \Big\} \leq 2 \, \mathrm{e}^{-r^2/2t}$$

- $d > 1$: Reduce to $d = 1$ using independence

$$\mathbb{P}\Big\{ \sup_{0 \leq s \leq t} \|W_s\| \geq r \Big\} \leq 2d \, \mathrm{e}^{-r^2/2dt}$$

Introduction: Small random perturbations

For $\Gamma \subset \mathcal{C} = \mathcal{C}([0,T], \mathbb{R}^d)$ with $\Gamma \subset B((x_s)_s, \delta)^c$

$$\mathbb{P}\Big\{x^\varepsilon \in \Gamma\Big\} \leq \mathbb{P}\bigg\{\sup_{0 \leq s \leq t} \|x_s^\varepsilon - x_s\| \geq \delta\bigg\} \leq \mathbb{P}\bigg\{\sup_{0 \leq s \leq t} \|W_s\| \geq \frac{\delta}{\sqrt{\varepsilon}} \, \mathrm{e}^{-Lt}\bigg\}$$

and

$$\mathbb{P}\Big\{x^\varepsilon \in \Gamma\Big\} \leq 2d \exp\bigg\{-\frac{\delta^2 \, \mathrm{e}^{-2Lt}}{2\varepsilon d t}\bigg\} \to 0 \qquad \text{as } \varepsilon \to 0$$

- ▷ Event $\{x^\varepsilon \in \Gamma\}$ is atypical: Occurrence a large deviation
- ▷ Question: Rate of convergence as a function of Γ?
- ▷ Generally not possible, but exponential rate can be found

Aim: Find functional $I : \mathcal{C} \to [0, \infty]$ s.t.

$$\mathbb{P}\Big\{\|x^\varepsilon - \varphi\|_\infty < \delta\Big\} \approx \mathrm{e}^{-I(\varphi)/\varepsilon} \qquad \text{for } \varepsilon \to 0$$

- ▷ Provides local description

Large deviations for Brownian motion: The endpoint

Special case: Scaled Brownian motion, $d = 1$

$$\mathrm{d}W_t^\varepsilon = \sqrt{\varepsilon}\,\mathrm{d}W_t, \qquad \Longrightarrow \qquad W_t^\varepsilon = \sqrt{\varepsilon}\,W_t$$

- ▷ Consider endpoint instead of whole path

$$\mathbb{P}\{W_t^\varepsilon \in A\} = \int_A \frac{1}{\sqrt{2\pi\varepsilon t}} \exp\Bigl\{-x^2/2\varepsilon t\Bigr\}\,\mathrm{d}x$$

- ▷ Use Laplace method to evaluate integral

$$\varepsilon \log \mathbb{P}\{W_t^\varepsilon \in A\} \sim -\frac{1}{2} \inf_{x\in A} \frac{x^2}{t} \qquad \text{as} \quad \varepsilon \to 0$$

Caution

- ▷ $|A| = 1$: l.h.s. $= -\infty <$ r.h.s. $\in (-\infty, 0]$
- ▷ Limit does not necessarily exit

Large deviations for Brownian motion: The endpoint

Remedy: Use interior and closure $\implies$ Large deviation principle

$$-\frac{1}{2}\inf_{x\in A^\circ}\frac{x^2}{t} \leq \liminf_{\varepsilon\to 0}\,\varepsilon\log\mathbb{P}\{W_t^\varepsilon\in A\}$$
$$\leq \limsup_{\varepsilon\to 0}\,\varepsilon\log\mathbb{P}\{W_t^\varepsilon\in A\} \leq -\frac{1}{2}\inf_{x\in\overline{A}}\frac{x^2}{t}$$

Large deviations for Brownian motion: Schilder's theorem

Schilder's Theorem (1966)
Scaled BM satisfies a (full) large deviation principle (LDP) with good rate function

$$I(\varphi) = I_{[0,T],0}(\varphi) = \begin{cases} \frac{1}{2}\|\varphi\|^2_{H_1} = \frac{1}{2}\int_{[0,T]} \|\dot{\varphi}_s\|^2 \, \mathrm{d}s & \text{if } \varphi \in H_1,\ \varphi_0 = 0 \\ +\infty & \text{otherwise} \end{cases}$$

- $I : \mathcal{C}_0 := \{\varphi \in \mathcal{C} : \varphi_0 = 0\} \to [0, \infty]$ is lower semi-continuous
- Good rate function: I has compact level sets
- Upper and lower large-deviation bound:

$$-\inf_{\Gamma^\circ} I \le \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in \Gamma\} \le \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in \Gamma\} \le -\inf_{\overline{\Gamma}} I$$

- Infinite-dimensional version of Laplace method
- $W^\varepsilon \notin H^1 \implies I(W^\varepsilon) = +\infty$ (almost surely)
- $I(0) = 0$ reflects $W^\varepsilon \to 0 \quad (\varepsilon \to 0)$

Large deviations for Brownian motion: Examples

Example I: Endpoint again ... ($d = 1$) $\qquad \Gamma = \Big\{\varphi \in \mathcal{C}_0 \colon \varphi_t \in A\Big\}$

$$\inf_\Gamma I = \inf_{x \in A} \frac{1}{2} \int_0^t \left| \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{xs}{t}\right) \right|^2 \mathrm{d}s = \inf_{x \in A} \frac{x^2}{2t}$$

$$\inf_\Gamma I = \text{cost to force BM to be in } A \text{ at time } t$$

$$\Longrightarrow \quad \mathbb{P}\Big\{W_t^\varepsilon \in A\Big\} \sim \exp\Big\{-\inf_{x \in A} x^2/2t\varepsilon\Big\}$$

Note: Typical spreading of W_t^ε is $\sqrt{\varepsilon t}$

Example II: BM leaving a small ball $\qquad \Gamma = \Big\{\varphi \in \mathcal{C}_0 \colon \|\varphi\|_\infty \geq \delta\Big\}$

$$\inf_\Gamma I = \inf_{0 \leq t \leq T} \inf_{\varphi \in \mathcal{C}_0 \colon \|\varphi_t\| = \delta} I(\varphi) = \inf_{0 \leq t \leq T} \frac{\delta^2}{2t} = \frac{\delta^2}{2T}$$

$$\inf_\Gamma I = \text{cost to force BM to leave } B(0, \delta) \text{ before } T$$

$$\Longrightarrow \quad \mathbb{P}\Big\{\exists\, t \leq T,\ \|W_t^\varepsilon\| \geq \delta\Big\} \sim \exp\Big\{-\delta^2/2T\varepsilon\Big\}$$

Large deviations for Brownian motion: Examples

Example III: BM staying in a cone (similar . . . Homework!)

Large deviations for Brownian motion: Lower bound

To show: Lower bound for open sets

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in G\} \geq - \inf_G I \qquad \text{for all open } G \subset \mathcal{C}_0$$

Lemma (local variant of lower bound)

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in B(\varphi, \delta)\} \geq -I(\varphi)$$

for all $\forall\, \varphi \in \mathcal{C}_0$ s.t. $I(\varphi) < \infty$ and all $\delta > 0$

▷ Lemma $\Longrightarrow$ lower bound

Rewrite ($\widehat{W}_t = W_t - \varphi_t/\sqrt{\varepsilon}$)

$$\mathbb{P}\{W^\varepsilon \in B(\varphi, \delta)\} = \mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} = \mathbb{P}\{\widehat{W} \in B(0, \delta/\sqrt{\varepsilon})\}$$

▷ Proof of Lemma: via Cameron–Martin–Girsanov formula, allows to transform away the drift

Cameron–Martin–Girsanov formula (special case, $d = 1$)

$$\{W_t\}_t \quad \mathbb{P}\text{–BM} \quad \Longrightarrow \quad \{\widehat{W}_t\}_t \quad \mathbb{Q}\text{–BM}$$

where

$$\widehat{W}_t = W_t - \int_0^t h(s)\, \mathrm{d}s, \qquad h \in \mathcal{L}_2$$

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t h(s)\, \mathrm{d}W_s - \frac{1}{2}\int_0^t h(s)^2\, \mathrm{d}s\right\}$$

Proof of Cameron–Martin–Girsanov formula

First step

$$X_t = \exp\left\{\int_0^t h(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t h(s)^2\,\mathrm{d}s\right\} \qquad h \in \mathcal{L}_2$$

$$Y_t = \exp\left\{\int_0^t (\gamma + h(s))\,\mathrm{d}W_s - \frac{1}{2}\int_0^t (\gamma + h(s))^2\,\mathrm{d}s\right\} = X_t \exp\left\{\gamma \widehat{W}_t - \frac{1}{2}\gamma^2 t\right\}$$

are exponential martingales wrt. $\mathbb{P}$ (for any $\gamma > 0$)

Second step

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\{Z \exp\{\gamma(\widehat{W}_t - \widehat{W}_s)\}\} &= \mathbb{E}^{\mathbb{P}}\{Z\,X_t \exp\{\gamma(\widehat{W}_t - \widehat{W}_s)\}\} \\ &= \mathbb{E}^{\mathbb{P}}\left\{Z \exp\left\{-\gamma\widehat{W}_s + \frac{1}{2}\gamma^2 t\right\}\mathbb{E}^{\mathbb{P}}\{Y_t \mid \mathcal{F}_s\}\right\} \\ &= \mathbb{E}^{\mathbb{P}}\left\{Z\,X_s \exp\left\{\frac{1}{2}\gamma^2(t-s)\right\}\right\} = \mathbb{E}^{\mathbb{Q}}\{Z\} \exp\left\{\frac{1}{2}\gamma^2(t-s)\right\}\end{aligned}$$

- ▷ $\widehat{W}_t - \widehat{W}_s$ is $\mathbb{Q}$−independent of $\mathcal{F}_s \Longrightarrow$ increments are independent
- ▷ Increments are Gaussian

$\Longrightarrow$ $\widehat{W}_t$ is BM with respect to $\mathbb{Q}$

LDP for Brownian motion: Proof of the lower bound

$d = 1$, $\delta > 0$, $\varphi \in \mathcal{C}_0$ with $I(\varphi) < \infty$, $\widehat{W}_t = W_t - \varphi_t/\sqrt{\varepsilon}$

$$\mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} = \mathbb{P}\{\|\widehat{W}\|_\infty < \delta/\sqrt{\varepsilon}\,\}$$
$$= \int_{\widehat{W}\in B(0,\delta/\sqrt{\varepsilon})} \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot\varphi_s \,\mathrm{d}W_s + \frac{1}{2\varepsilon}\int_0^T \dot\varphi_s^2 \,\mathrm{d}s\biggr\}\,\mathrm{d}\mathbb{Q}$$

Estimate integral by Jensen's inequality

$$\dots = \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\}\mathbb{Q}\,\{\widehat{W} \in B(0,\delta/\sqrt{\varepsilon}\,)\}$$
$$\times\frac{1}{\mathbb{Q}\,\{\dots\}}\int_{\widehat{W}\in B(0,\delta/\sqrt{\varepsilon})} \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot\varphi_s \,\mathrm{d}\widehat{W}_s\biggr\}\,\mathrm{d}\mathbb{Q}$$
$$\geq \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\}\mathbb{P}\bigl\{W \in B(0,\delta/\sqrt{\varepsilon}\,)\bigr\} \times \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}\,\mathbb{P}\{\dots\}}\int_{W\in B(0,\delta/\sqrt{\varepsilon})}\int_0^T \dot\varphi_s \,\mathrm{d}W_s \,\mathrm{d}\mathbb{P}\biggr\}$$
$$= \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\}\mathbb{P}\bigl\{W \in B(0,\delta/\sqrt{\varepsilon}\,)\bigr\} \times 1$$

Finally note

$$\mathbb{P}\bigl\{W \in B(0,\delta/\sqrt{\varepsilon}\,)\bigr\} \nearrow 1 \quad (\,\varepsilon \searrow 0\,) \implies \liminf_{\varepsilon\to 0}\,\varepsilon \log \mathbb{P}\bigl\{\|W^\varepsilon - \varphi\|_\infty < \delta\bigr\} \geq -I(\varphi)$$

LDP for Brownian motion: Approximation by polygons (upper bound)

Approximate W^ε by the random polygon $W^{n,\varepsilon}$ joining the random points $(0, W_0^\varepsilon), (T/n, W_{T/n}^\varepsilon), \dots, (T, W_T^\varepsilon)$

To show: $W^{n,\varepsilon}$ is a good approximation to W^ε

$$\begin{aligned}\mathbb{P}\Big\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geq \delta\Big\} &\leq n\,\mathbb{P}\left\{\sup_{0\leq s\leq T/n} \|W_s^\varepsilon - W_s^{n,\varepsilon}\| \geq \delta\right\}\\ &\leq n\,\mathbb{P}\left\{\sup_{0\leq s\leq T/n} \|W_s^\varepsilon\| \geq \frac{\delta}{2}\right\}\\ &= n\,\mathbb{P}\left\{\sup_{0\leq s\leq T/n} \|W_s\| \geq \frac{\delta}{2\sqrt{\varepsilon}}\right\} \leq 2nd\,\exp\left\{-\frac{n\delta^2}{8\varepsilon dT}\right\}\end{aligned}$$

$\Longrightarrow$ Difference is negligible:

$$\limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geq \delta\Big\} = -\infty \qquad \text{for all} \quad \delta > 0$$

LDP for Brownian motion: Proof of the upper bound

$F \subset \mathcal{C}_0$ closed, $\delta > 0$, $\ell_\delta = \inf_{F^{(\delta)}} I = \inf\Big\{I(\varphi) \colon \varphi \in F^{(\delta)}\Big\}$, $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{P}\Big\{W^\varepsilon \in F\Big\} &\leq \mathbb{P}\Big\{W^{n,\varepsilon} \in F^{(\delta)}\Big\} + \mathbb{P}\Big\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geq \delta\Big\} \\ &\leq \mathbb{P}\Big\{I(W^{n,\varepsilon}) \geq \ell_\delta\Big\} + \text{negligible term}\end{aligned}$$

$W^{n,\varepsilon}$ being a polygon yields

$$\begin{aligned}I(W^{n,\varepsilon}) &= \frac{1}{2}\int_0^T \|\dot{W}_s^{n,\varepsilon}\|^2 \,\mathrm{d}s = \frac{1}{2}\sum_{k=1}^{n} \frac{T}{n}\Big\|\frac{n}{T}\Big(W^{n,\varepsilon}_{kT/n} - W^{n,\varepsilon}_{(k-1)T/n}\Big)\Big\|^2 \\ &\overset{(\mathcal{D})}{=} \frac{\varepsilon}{2}\sum_{k=1}^{nd} \xi_i^2 \qquad\qquad (\xi_i \sim \mathcal{N}(0,1) \quad \text{i.i.d.})\end{aligned}$$

LDP for Brownian motion: Proof of the upper bound

By Chebychev's inequality, for $\gamma < 1/2$

$$\mathbb{P}\Big\{I(W^{n,\varepsilon}) \geq \ell_\delta\Big\} \leq \mathbb{P}\Big\{\sum_{k=1}^{nd} \xi_i^2 \geq \frac{2\ell_\delta}{\varepsilon}\Big\} \leq \exp\Big\{-\frac{2\gamma\ell_\delta}{\varepsilon}\Big\}\Big(\mathbb{E}\exp\Big\{\gamma\xi_1^2\Big\}\Big)^{nd}$$
$$= \exp\Big\{-\frac{2\gamma\ell_\delta}{\varepsilon}\Big\}\Big(1-2\gamma\Big)^{-nd/2}$$

$\gamma < 1/2$ being arbitrary and the lower semi-continuity of I show

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big\{W^\varepsilon \in F\Big\} \leq \limsup_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big\{I(W^{n,\varepsilon}) \geq \ell_\delta\Big\}$$
$$\leq -\ell_\delta = -\inf_{F^{(\delta)}} I \searrow -\inf_F I$$

Large deviations for solutions of SDEs: Special case

Special case: $g(x) \equiv$ identity matrix

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t\,, \qquad x_0^\varepsilon = x_0$$

Define $F : \mathcal{C}_0 \to \mathcal{C}$ by $\varphi \mapsto F(\varphi) = f$, f the unique solution in $\mathcal{C}$ to

$$f(t) = x_0 + \int_0^t b(f(s))\,\mathrm{d}s + \varphi(t)$$

- ▷ $F(W^\varepsilon) = x^\varepsilon$
- ▷ F is continuous (use Gronwall's lemma)

Large deviations for solutions of SDEs: Special case

Contraction principle (trivial version)

I is a good rate fct, governing LDP for W^ε

$\implies$ $J(f) = \inf\Big\{I(\varphi) \colon \varphi \in \mathcal{C}_0, F(\varphi) = f\Big\}$
is a good rate fct, governing LDP for $x^\varepsilon = F(W^\varepsilon)$

Identify J

$$J(f) = J_{[0,T],x_0}(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{[0,T]} \|\dot{f}_s - b(f_s)\|^2 \, \mathrm{d}s & \text{if } f \in H_1,\ f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

Large deviations for solutions of SDEs: General case

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,g(x_t^\varepsilon)\,\mathrm{d}W_t, \qquad x_0^\varepsilon = x_0$$

Assumptions

- ▷ b, g Lipschitz continuous
- ▷ bounded growth:
  $\|b(x)\| \leq M\,(1+\|x\|^2)^{1/2}$, $a(x) = g(x)g(x)^{\mathsf{T}} \leq M\,(1+\|x\|^2)\,\mathrm{Id}$
- ▷ ellipticity: $a(x) > 0$

Theorem (Wentzell–Freidlin)
x^ε satisfies a LDP with good rate function

$$J(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{[0,T]} \Big\| a(f_s)^{-1/2}\Big[\dot{f}_s - b(f_s)\Big]\Big\|^2\,\mathrm{d}s & \text{if } f \in H_1,\ f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

Large deviations for solutions of SDEs: General case

Remark

$a(x) = 0$: LDP remains valid with good rate function
but identification of J may fail

$$J(f) = \inf \Big\{ I(\varphi) \colon \varphi \in H_1, \\ f_t = x_0 + \int_0^t b(f_s)\, \mathrm{d}s + \int_0^t a(f_s)^{1/2} \dot{\varphi}_s\, \mathrm{d}s, t \in [0, T] \Big\}$$

LDP for SDEs: Sketch of the proof for the general case

- ▷ Difficulty: Cannot apply contraction principle directly
- ▷ Introduce Euler approximations

$$x_t^{n,\varepsilon} = x_0 + \int_0^t b(x_s^{n,\varepsilon})\, \mathrm{d}s + \sqrt{\varepsilon} \int_0^t g(x_{T_n(s)}^{n,\varepsilon})\, \mathrm{d}W_s, \qquad T_n(s) = \frac{[ns]}{n}$$

- ▷ Schilder's Theorem and contraction principle imply LDP for $x^{n,\varepsilon}$ with good rate function J^n

$$J^n(f) = \begin{cases} \dfrac{1}{2} \displaystyle\int_{[0,T]} \left\| a(f_{T_n(s)})^{-1/2} \left[\dot{f}_s - b(f_s) \right] \right\|^2 \mathrm{d}s & \text{if } f \in H_1,\ f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

- ▷ To show:
  (1) $x^{n,\varepsilon}$ is a good approximation to x^ε
  (not difficult but tedious, uses Itô's formula)
  (2) J^n approximates J: $\lim_{n\to\infty} \inf_\Gamma J^n = \inf_\Gamma J$ for all Γ

Large deviations for solutions of SDEs: Varadhan's Lemma

Assumptions

- $\phi : \mathbb{R}^d \to \mathbb{R}$ continuous
- Tail condition

$$\lim_{L\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \int_{\phi(x^\varepsilon)\geq L} \exp\{\phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P} = -\infty$$

Theorem (Varadhan's Lemma)

$$\lim_{\varepsilon\to 0} \varepsilon \log \int \exp\{\phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P} = \sup_{\varphi} \left[\phi(\varphi) - J(\varphi)\right]$$

Remarks

- The moment condition

$$\sup_{0<\varepsilon\leq 1} \left(\int \exp\{\alpha\, \phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P}\right)^{\varepsilon} < \infty \qquad \text{for some } \alpha \in (1,\infty)$$

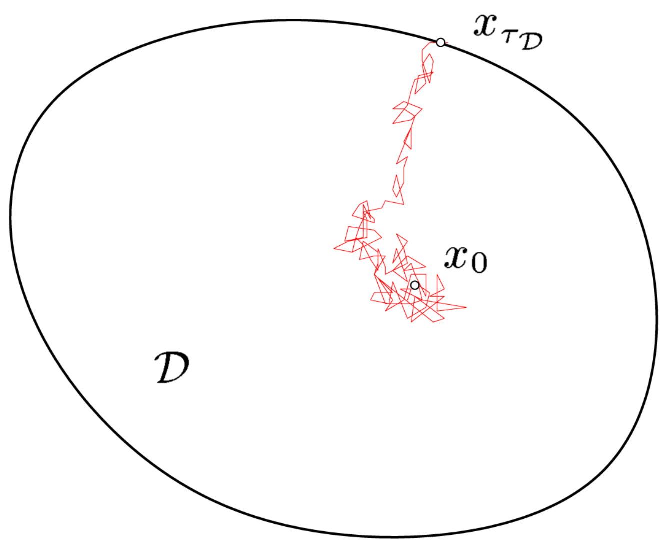
  implies tail condition
- Infinite-dimensional analogue of Laplace method
- Holds in great generality — as long as x^ε satisfies a LDP with a good rate function J

Diffusion exit from a domain: Introduction

Deterministic ODE	$\dot{x}_t^{\text{det}} = b(x_t^{\text{det}})$	$x_0 \in \mathbb{R}^d$
Small random perturbation	$\mathrm{d}x_t = b(x_t)\,\mathrm{d}t + \sqrt{\varepsilon} g(x_t)\,\mathrm{d}W_t$	

Bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- ▷ *first-exit time* $\tau = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
- ▷ *first-exit location* $x_\tau \in \partial\mathcal{D}$



Questions

- ▷ Does x_t^ε leave $\mathcal{D}$?
- ▷ If so: When and where?
- ▷ Expected time of first exit?
- ▷ Concentration of first-exit time and location?
- ▷ Distribution of τ and x_τ ?

Diffusion exit from a domain: Introduction

Towards answers

- ▷ If x_t leaves $\mathcal{D}$, so will x_t^ε. Use LDP to estimate deviation $x_t^\varepsilon - x_t$.
- ▷ Assume x_t does *not* leave $\mathcal{D}$
  ($\mathcal{D}$ positively invariant under deterministic flow)
  Study noise-induced exit

In the latter case:

- ▷ Mean first-exit times and locations via PDEs
- ▷ Exponential asymptotics via Wentzell–Freidlin theory

Diffusion exit from a domain: Relation to PDEs

Assumptions (from now on)

- b, g Lipschitz cont., bounded growth
- $a(x) = g(x)g(x)^{\mathsf{T}} \geq (1/M)\,\mathrm{Id}$ (uniform ellipticity)
- $\mathcal{D}$ bounded domain, smooth boundary

Infinitesimal generator $\mathcal{L}^\varepsilon$ of diffusion x^ε

$$\mathcal{L}^\varepsilon v(t,x) = \frac{\varepsilon}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} v(t,x) + \langle\, b(x), \nabla v(t,x) \,\rangle$$

Compare to FPE!

Diffusion exit from a domain: Relation to PDEs

Theorem

For $f : \partial\mathcal{D} \to \mathbb{R}$ continuous

▷ $\mathbb{E}_x\{\tau^\varepsilon\}$ is the unique solution of $\begin{cases} \mathcal{L}^\varepsilon u = -1 & \text{in } \mathcal{D} \\ u = 0 & \text{on } \partial\mathcal{D} \end{cases}$

▷ $\mathbb{E}_x\{f(x^\varepsilon_{\tau^\varepsilon})\}$ is the unique solution of $\begin{cases} \mathcal{L}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases}$

Remarks

▷ Information on first-exit times and exit locations can be obtained *exactly* from PDEs
▷ In principle . . .
▷ Smoothness assumption for $\partial\mathcal{D}$ can be relaxed to "exterior-ball condition"

Diffusion exit from a domain: An example

Motion of a Brownian particle in a single-well potential

$d = 1$, $b(0) = 0$, $x\,b(x) < 0$ for $x \neq 0$, $g(x) \equiv 1$

- ▷ Drift pushes particle towards bottom
- ▷ Probability of x^ε leaving $\mathcal{D} = (\alpha_1, \alpha_2) \ni 0$?

Solve the (one-dimensional) Dirichlet problem

$$\begin{cases} \mathcal{L}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases} \qquad \text{with} \qquad f(x) = \begin{cases} 1 & \text{for } x = \alpha_1 \\ 0 & \text{for } x = \alpha_2 \end{cases}$$

$$w(x) = \mathbb{P}_x\Big\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\Big\} = \mathbb{E}_x f(x^\varepsilon_{\tau^\varepsilon}) = \int_x^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\,\mathrm{d}y \Bigg/ \int_{\alpha_1}^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\,\mathrm{d}y$$

Diffusion exit from a domain: An example

$$w(x) = \mathbb{P}_x\Big\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\Big\} = \mathbb{E}_x f(x^\varepsilon_{\tau^\varepsilon}) = \int_x^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\, \mathrm{d}y \Bigg/ \int_{\alpha_1}^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\, \mathrm{d}y$$

What happens in the small-noise limit?

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = 1 \qquad \text{if } U(\alpha_1) < U(\alpha_2)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = 0 \qquad \text{if } U(\alpha_2) < U(\alpha_1)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = \frac{1}{|U'(\alpha_1)|} \Bigg/ \left(\frac{1}{|U'(\alpha_1)|} + \frac{1}{|U'(\alpha_2)|}\right) \qquad \text{if } U(\alpha_1) = U(\alpha_2)$$

Note that the information we obtained this way is more precise than results relying on the LDP can provide.

Diffusion exit from a domain: A first result

Corollary (to LDP for x^ε)

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_x\Bigl\{\tau^\varepsilon \leqslant t\Bigr\} = -\inf\Bigl\{V(x,y;s) \colon s \in [0,t],\ y \notin \mathcal{D}\Bigr\}$$

$$V(x,y;s) = \inf\Bigl\{J_{[0,s],x}(\varphi) \colon \varphi \in \mathcal{C}([0,s],\mathbb{R}^d),\ \varphi_0 = x,\ \varphi_s = y\Bigr\}$$

$$= \text{cost of forcing a path to connect } x \text{ and } y \text{ in time } s$$

Remarks

- Upper and lower LDP bounds coincide $\implies$ limit exists
- Calculation of asymptotical behaviour reduces to a variational problem
- $V(x,y;s)$ is solution to a Hamilton–Jacobi equation
- extremals solution to an Euler equation

The concept of a quasipotential

Assumptions (for the next slides)

- $\mathcal{D}$ positively invariant
- unique, asymptotically stable equilibrium point at $0 \in \mathcal{D}$
- $\partial\mathcal{D} \subset$ basin of attraction of 0

Quasipotential

- Quasipotential *with respect to* 0:
  Cost to go against the flow from 0 to z
  $$V(0,z) = \inf_{t>0} \inf\{I_{[0,t]}(\varphi) \colon \varphi \in \mathcal{C}([0,t], \mathbb{R}^d),\ \varphi_0 = 0,\ \varphi_t = z\}$$
- Minimum of quasipotential on boundary $\partial\mathcal{D}$
  $$\overline{V} := \min_{z \in \partial\mathcal{D}} V(0,z)$$

Wentzell–Freidlin theory

Theorem [Wentzell & Freidlin $\geqslant$ '70] (under above assumptions)

For arbitrary initial condition in $\mathcal{D}$

- ▷ Mean first-exit time
  $\mathbb{E}\tau \sim e^{\overline{V}/\sigma^2}$ as $\sigma \to 0$
- ▷ Concentration of first-exit times
  $\mathbb{P}\Big\{ e^{(\overline{V}-\delta)/\sigma^2} \leqslant \tau \leqslant e^{(\overline{V}+\delta)/\sigma^2} \Big\} \to 1$ as $\sigma \to 0$ (for arbitrary $\delta > 0$)
- ▷ Concentration of exit locations near minima of quasipotential

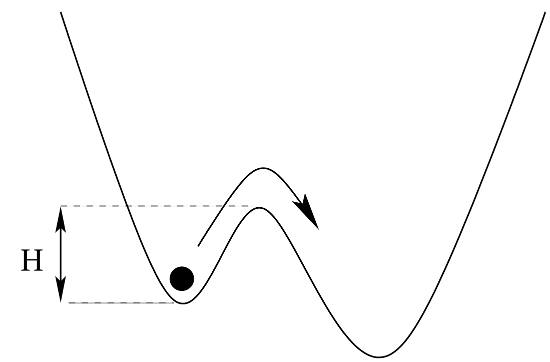
Gradient case (reversible diffusion)

Drift coefficient deriving from potential:

$f = -\nabla V$, $g = \mathrm{Id}$

$\mathcal{D}$ containing saddle $\Longrightarrow$ $\overline{\mathcal{D}}$ no longer invariant

- ▷ Cost for leaving potential well: $\overline{V} = 2H$
- ▷ Attained for paths going against the flow:
  $\dot{\varphi}_t = -f(\varphi_t)$



Wentzell–Freidlin theory: Idea of the proof

First step

x^ε cannot remain in $\mathcal{D}$ arbitrarily long without hitting a small neighbourhood $B(0,\mu)$ of 0:

$$\forall\, \mu \qquad \lim_{t\to\infty} \limsup_{\varepsilon\to0} \varepsilon \log \sup_{x\in\mathcal{D}} \mathbb{P}_x\Big\{ x_s^\varepsilon \in \mathcal{D} \setminus B(0,\mu) \text{ for all } s \leq t \Big\} = -\infty$$

$\Longrightarrow$ Restrict to initial conditions in $B(0,\mu)$

Second step

Lower bound on probability to leave $\mathcal{D}$:

$$\forall\, \eta > 0 \ \exists\, \mu_0 \ \forall\, \mu < \mu_0 \ \exists\, T_0 > 0 \qquad \liminf_{\varepsilon\to0} \varepsilon \log \inf_{x\in B(0,\mu)} \mathbb{P}_x\big\{\tau^\varepsilon \leq T_0\big\} > -(\overline{V}+\eta)$$

- ▷ Construct piecewise a continuous exit path φ connecting x_0, 0, $\partial\mathcal{D}$ and some point y at distance μ from $\overline{\mathcal{D}}$ with $I(\varphi) \leq \overline{V} + \eta$
- ▷ Use LDP to estimate probability of x^ε remaining in $\mu/2$-neighbourhood of exit path

Third step

More lemmas in the same spirit . . . (involving exit locations)

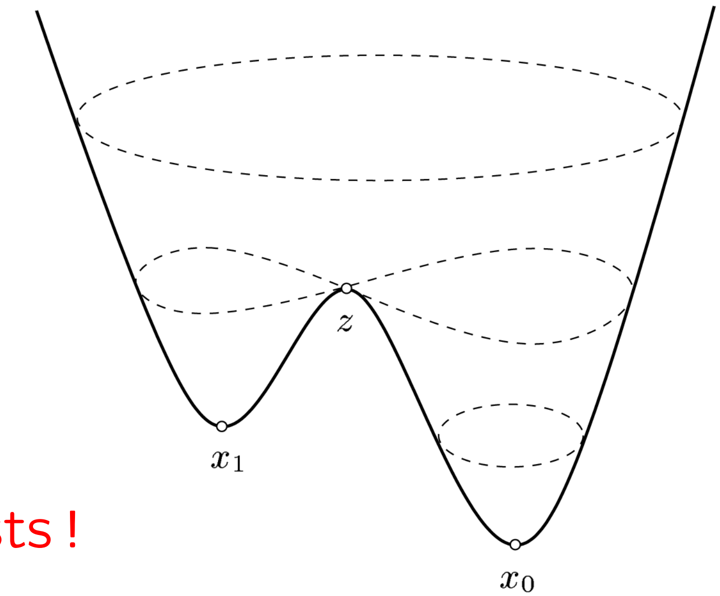
Forth step

Prove Theorem by considering successive trials to leave $\mathcal{D}$ using strong Markov property

Refined results in the gradient case

Simplest case: V double-well potential

First-hitting time τ^{hit} of deeper well



- $\mathbb{E}_{x_1} \tau^{\text{hit}} = c(\sigma)\, \mathrm{e}^{2\,[V(z)-V(x_1)]\,/\,\sigma^2}$

- $\displaystyle\lim_{\sigma\to 0} c(\sigma) = \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x_1)}}$ exists !

  $\lambda_1(z)$ unique negative e.v. of $\nabla^2 V(z)$

  (Physics' literature: [Eyring '35], [Kramers '40];
  rigorous: [Bovier, Gayrard, Eckhoff, Klein '02–'05], [Helffer, Klein, Nier '04])

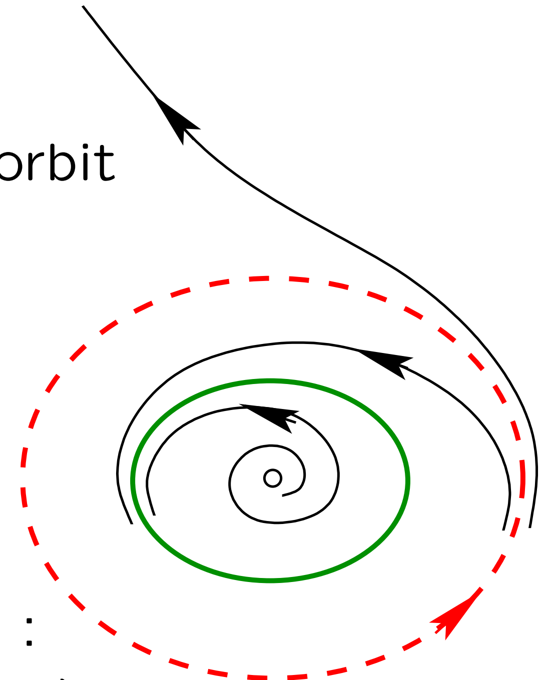
- Subexponential asymptotics known

  Related to geometry at well and saddle / small eigenvalues of the generator

- $\tau^{\text{hit}} \approx$ exp. distributed: $\displaystyle\lim_{\sigma\to 0} \mathbb{P}\Big\{\tau^{\text{hit}} > t\,\mathbb{E}\,\tau^{\text{hit}}\Big\} = \mathrm{e}^{-t}$

  ([Day '82], [Bovier *et al* '02])

New phenomena for drift term not deriving from a potential?



Simplest situation of interest

Nontrivial invariant set which is a single periodic orbit

Assume from now on

$d = 2$, $\partial\mathcal{D} =$ *unstable* periodic orbit

- $\mathbb{E}\tau \sim \mathrm{e}^{\overline{V}/\sigma^2}$ still holds
- Quasipotential $V(\Pi, z) \equiv \overline{V}$ is constant on $\partial\mathcal{D}$:
  Exit equally likely anywhere on $\partial\mathcal{D}$ (on exp. scale)
- Phenomenon of cycling [Day '92]:
  Distribution of x_τ on $\partial\mathcal{D}$ does *not* converge as $\sigma \to 0$
  Density is *translated* along $\partial\mathcal{D}$ proportionally to $|\log\sigma|$.
- In *stationary regime*: (obtained by reinjecting particle)
  Rate of escape $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\Big\{x_t \in \mathcal{D}\Big\}$ has $|\log\sigma|$-periodic prefactor
  [Maier & Stein '96]

Density of the first-passage time at an unstable periodic orbit

Study first-exit time by taking number of revolutions into account

Idea

Density of first-passage time at unstable orbit

$$p(t) = c(t,\sigma)\ \mathrm{e}^{-\overline{V}/\sigma^2} \times \text{transient term} \times \text{ geometric decay per period}$$

Identify $c(t,\sigma)$ as periodic component in first-passage density

Notations

- Value of quasipotential on unstable orbit: $\overline{V}$
- Period of unstable orbit: $T = 2\pi/\varepsilon$
- Curvature at unstable orbit: $a(t) = -\dfrac{\partial^2}{\partial x^2} V(x^{\text{unst}}(t), t)$
- Lyapunov exponent of unstable orbit: $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(t)\ \mathrm{d}t$

Universality in first-passage-time distributions

Theorem ([Berglund & G '04], [Berglund & G '05], work in progress)

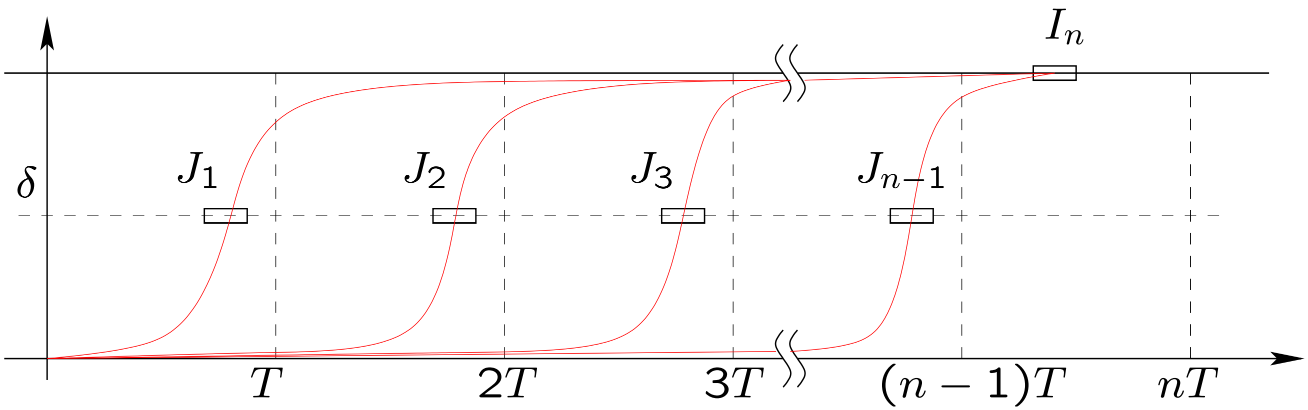
For any $\Delta \geqslant \sqrt{\sigma}$ and all $t \geqslant t_0$

$$\mathbb{P}\{\tau \in [t, t+\Delta]\} = \int_t^{t+\Delta} p(s, t_0)\, \mathrm{d}s \left[1 + \mathcal{O}(\sqrt{\sigma})\right]$$

where

- $p(t, t_0) = \dfrac{f_{\text{trans}}(t, t_0)}{\mathcal{N}}\, Q_{\lambda T}\Big(\theta(t) - |\log \sigma|\Big) \dfrac{\theta'(t)}{\lambda T_{\mathsf{K}}(\sigma)}\, \mathrm{e}^{-(\theta(t) - \theta(t_0))\,/\,\lambda T_{\mathsf{K}}(\sigma)}$
- $Q_{\lambda T}(y)$ is a *universal* λT-periodic function
- $\theta(t)$ is a "natural" parametrisation of the boundary: $\theta'(t) > 0$ is explicitely known *model-dependent*, T-periodic fct.; $\theta(t+T) = \theta(t) + \lambda T$
- $T_{\mathsf{K}}(\sigma)$ is the analogue of Kramers' time: $T_{\mathsf{K}}(\sigma) = \dfrac{C}{\sigma}\, e^{\overline{V}/\sigma^2}$
- f_{trans} grows from 0 to 1 in time $t - t_0$ of order $|\log \sigma|$

Idea of the proof



Exit occurs in $I_n = [t, t+\Delta] \subset [(n-1)T, nT]$

$\Longrightarrow$ rate function has n minimizers (of comparable value)

$$\mathbb{P}^{0,0}\Big\{\tau \in I_n\Big\} \simeq \sum_{\ell=1}^{n} \underbrace{\mathbb{P}^{J_\ell,\delta}\Big\{\tau \in I_n\Big\}}_{Q_{n-\ell}(t)} \underbrace{\mathbb{P}^{0,0}\Big\{\tau' \in J_\ell\Big\}}_{P_\ell}$$

$$P_\ell \simeq \text{const}\,\mathrm{e}^{-\ell q} \exp\Big\{-\frac{\overline{V}_1}{\sigma^2}\Big(1-\mathrm{e}^{-2\ell\lambda T}\Big)\Big\}, \qquad q = T\,\mathrm{e}^{-\overline{V}_1/\sigma^2}$$

$$Q_k(t) \simeq C(t)\,\mathrm{e}^{-2k\lambda T} \exp\Big\{-\frac{\overline{V}_2}{\sigma^2}\Big(1-c(t)\,\mathrm{e}^{-2k\lambda T}\Big)\Big\}$$

The different regimes (after time change $\theta(t) \mapsto t$)

$$p(t,t_0) = \frac{f_{\text{trans}}(t,t_0)}{\mathcal{N}}\, Q_{\lambda T}\Bigl(t - |\log\sigma|\Bigr) \frac{1}{\lambda T_{\text{K}}(\sigma)}\, \mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\text{K}}(\sigma)}$$

Transient regime

f_{trans} is increasing; exponentially close to 1 for $t - t_0 > 2|\log\sigma|$

Metastable regime

$$Q_{\lambda T}(y) = 2\lambda T \sum_{k=-\infty}^{\infty} P(y - k\lambda T) \quad \text{where} \quad P(z) = \frac{1}{2}\,\mathrm{e}^{-2z} \exp\Bigl\{-\frac{1}{2}\,\mathrm{e}^{-2z}\Bigr\}$$

kth summand: Path spends

- ▷ k periods near stable periodic orbit
- ▷ $[(t-t_0)/T] - k$ periods near unstable periodic orbit

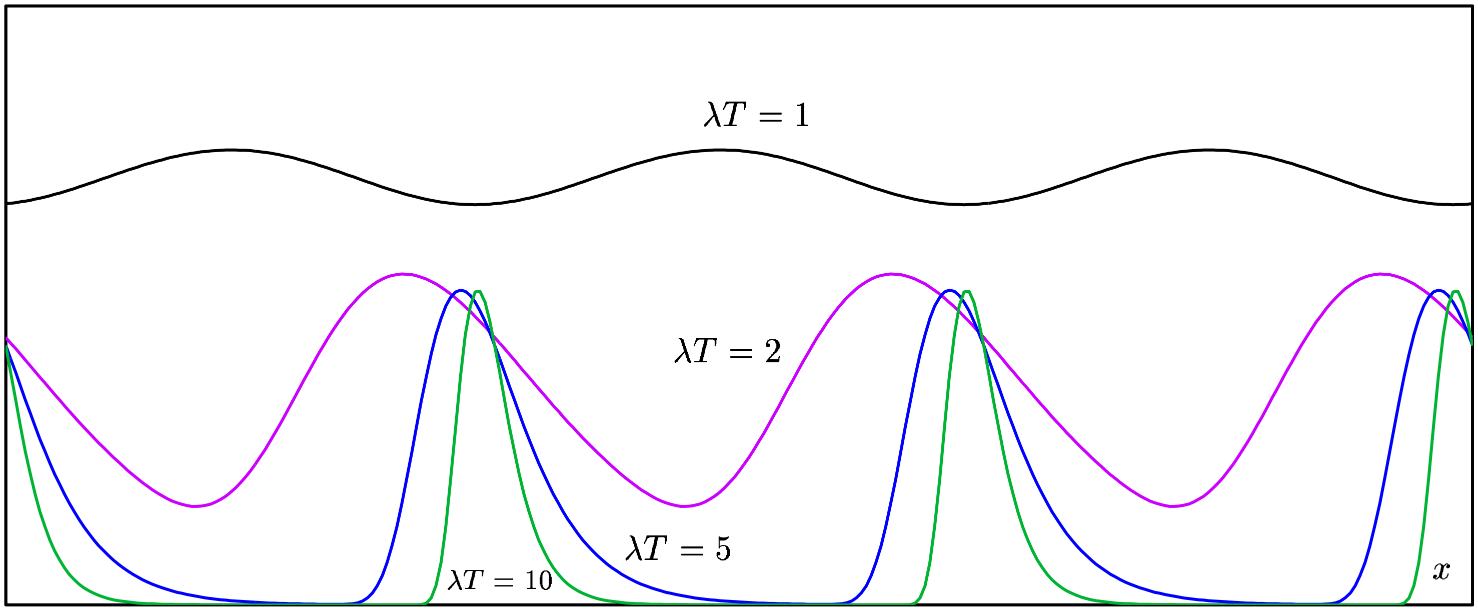
Periodic dependence on $|\log\sigma|$: Peaks $P(z)$ rotate as σ decreases

Asymptotic regime

Significant decay only for $t - t_0 \gg T_{\text{K}}(\sigma)$

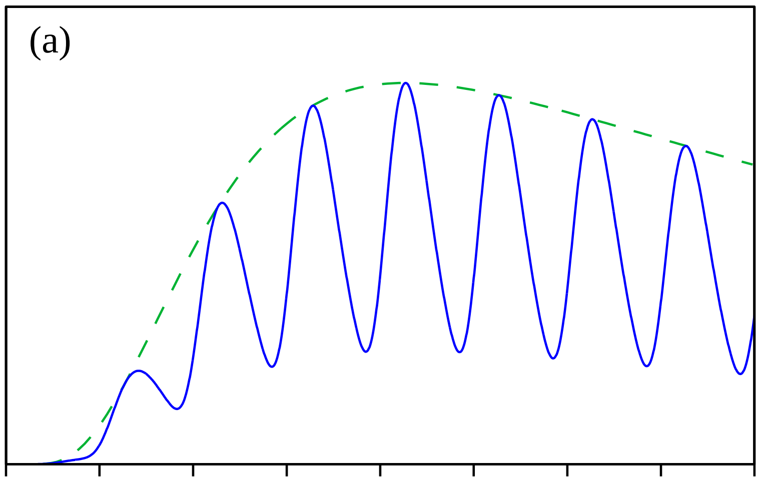
The universal profile

$y \mapsto Q_{\lambda T}(\lambda T y)/2\lambda T$

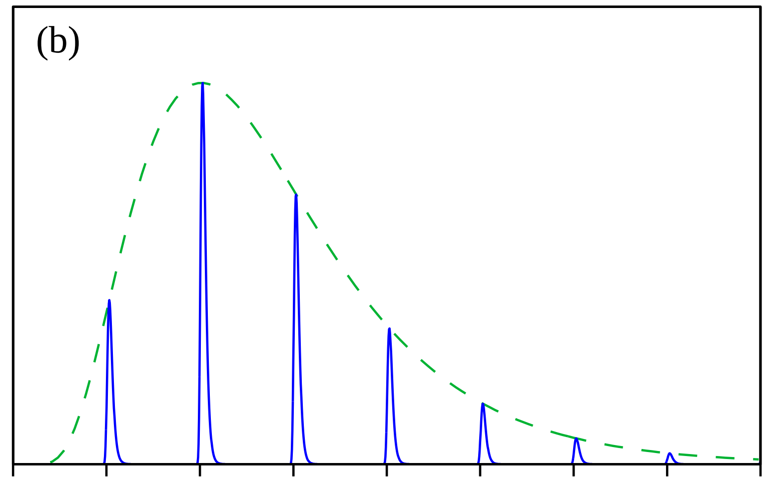


- Profile determines concentration of first-passage times within a period
- Shape of peaks: Gumbel distribution
- The larger λT, the more pronounced the peaks
- For smaller values of λT, the peaks overlap more

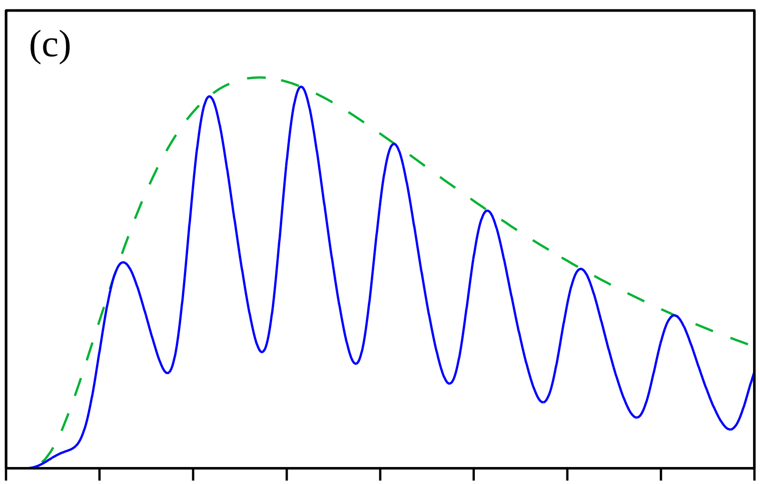
Density of the first-passage time $\overline{V} = 0.5,\ \lambda = 1$



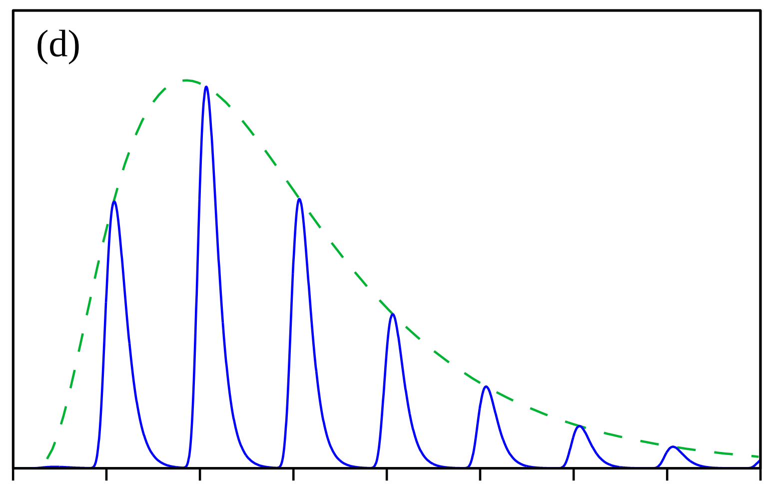
$\sigma = 0.4,\ T = 2$



$\sigma = 0.4,\ T = 20$

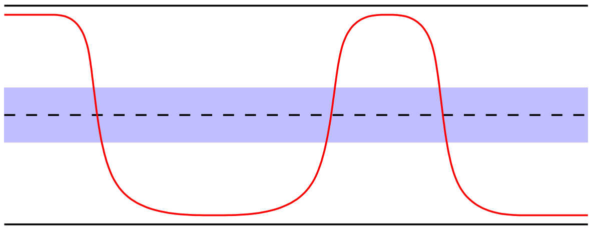


$\sigma = 0.5,\ T = 2$



$\sigma = 0.5,\ T = 5$

Residence-times



x_t crosses unstable periodic orbit $x^{\text{per}}(t)$ at time s

τ: time of first crossing back after time s

- ▷ First-passage-time density:

$$p(t,s) = \frac{\partial}{\partial t}\mathbb{P}^{s,x^{\text{per}}(s)}\big\{\tau < t\big\}$$

- ▷ Asymptotic transition-phase density: (stationary regime)

$$\psi(t) = \int_{-\infty}^{t} p(t,s)\psi(s - T/2)\,\mathrm{d}s = \psi(t+T)$$

- ▷ Residence-time distribution:

$$q(t) = \int_{0}^{T} p(s+t,s)\psi(s - T/2)\,\mathrm{d}s$$

Computation of residence-time distributions

Without forcing $(A = 0)$

$p(t,s) \sim$ exponential, $\psi(t)$ uniform $\Longrightarrow q(t) \sim$ exponential

With forcing $(A \gg \sigma^2)$

- ▷ First-passage-time density:

$$p(t,s) \simeq \frac{f_{\text{trans}}(t,s)}{\mathcal{N}} Q_{\lambda T}(t - |\log \sigma|) \frac{1}{\lambda T_{\text{K}}} \, \mathrm{e}^{-(t-s)/\lambda T_{\text{K}}}$$

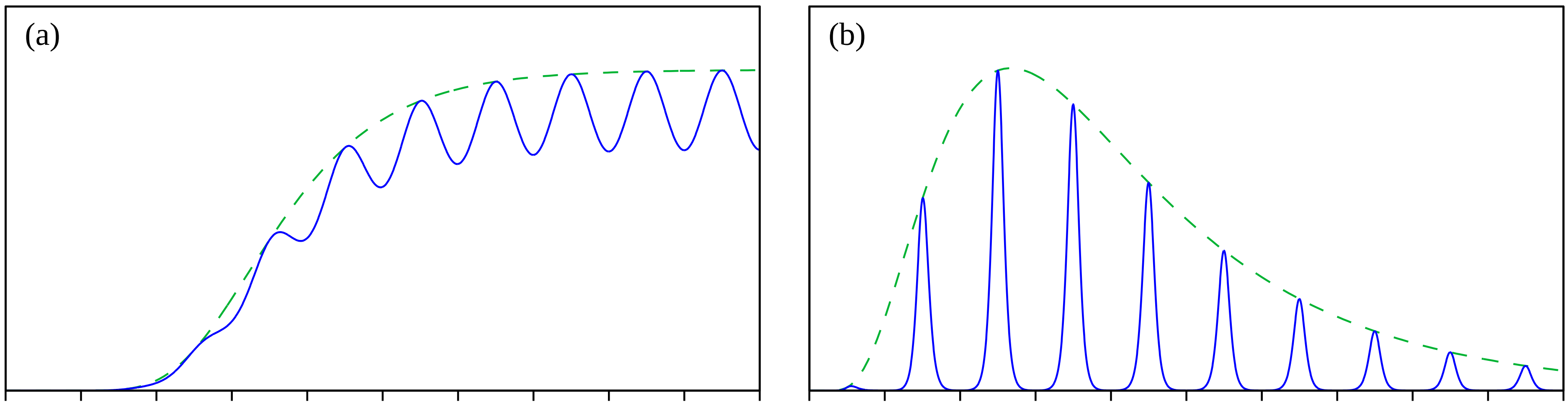
- ▷ Asymptotic transition-phase density:

$$\psi(s) \simeq \frac{1}{\lambda T} Q_{\lambda T}(s - |\log \sigma|) \Big[1 + \mathcal{O}(T/T_{\text{K}}) \Big]$$

- ▷ Residence-time distribution: (no cycling)

$$q(t) \simeq \tilde{f}_{\text{trans}}(t) \frac{\mathrm{e}^{-t/\lambda T_{\text{K}}}}{\lambda T_{\text{K}}} \, \frac{\lambda T}{2} \sum_{k=-\infty}^{\infty} \frac{1}{\cosh^2(t + \lambda T/2 - k\lambda T))}$$

Density of the residence-time distribution $\overline{V} = 0.5, \ \lambda = 1$



$\sigma = 0.2, \ T = 2$ $\qquad$ $\sigma = 0.4, \ T = 10$

- Peaks symmetric
- Shape of peaks: Solitons
- No cycling
- σ fixed, λT increasing: Transition into synchronisation regime

- Picture as for Dansgaard–Oeschger events:
  Periodically perturbed *asymmetric* double-well potential

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PART V

Small-ball probabilities for Brownian motion

- ▷ Small-ball probabilities for Brownian motion
- ▷ Generalizations

Small-ball probabilities for Brownian motion

BM is growing with $\sqrt{t}$ – What does that mean?

- $\triangleright$ $\operatorname{Var}\{W_t\}$ grows like $t \Longrightarrow$ typical spreading at time t is $\sqrt{t}$
- $\triangleright$ $\mathbb{P}\{|W_t| \geq c\sqrt{t}\} \leq \mathrm{e}^{-c^2/2} \ll 1$ for $c \gg 1$
- $\triangleright$ Also lower bound:
  $\mathbb{P}\{|W_t| \leq c\sqrt{t}\} = \sqrt{2/\pi}\, c\, [1 - \mathcal{O}(c^2)] \ll 1$ for $c \ll 1$
- $\triangleright$ These are statements on the endpoint W_t
- $\triangleright$ For the whole sample path, recall LDP: (for small ε)

$$
\begin{aligned}
\mathbb{P}\{\sup_{0\leq t\leq T} |W_t| \geq c\sqrt{t}/\sqrt{\varepsilon}\} &\leq \mathbb{P}\{\sup_{0\leq t\leq T} |W_t| \geq c\sqrt{T}/\sqrt{\varepsilon}\} \\
&= \mathbb{P}\{\sup_{0\leq t\leq T} |\sqrt{\varepsilon} W_t| \geq c\sqrt{T}\} \sim \mathrm{e}^{-c^2/2\varepsilon}
\end{aligned}
$$

Note: The large deviation is realized for sample paths leaving the set as late as possible. Thus: The first two probabilities behave in the same way.

Small-ball probabilities for Brownian motion

What can be said about the probability

$$\mathbb{P}\{\sup_{0\leq t\leq T} |W_t| \leq \varepsilon\}$$

that BM stays for a long time in a small neighbourhood of the origin ("in a small ball")?

Unlikely event!

For the endpoint, we've seen

$$\mathbb{P}\{|W_t| \leq c\sqrt{t}\} = \sqrt{\frac{2}{\pi}}\, c\, [1 - \mathcal{O}(c^2)]$$

Equivalent

$$\mathbb{P}\{|W_t| \leq \varepsilon\} = \sqrt{\frac{2}{\pi}}\, \frac{\varepsilon}{\sqrt{t}} \left[1 - \mathcal{O}\left(\frac{\varepsilon^2}{t}\right)\right]$$

Here, the behaviour of the paths is not dominated by the behaviour of the endpoint as it is easier for the whole path to exit some time than to be outside the ball at time t

Small-ball probabilities for Brownian motion

$\tau_r =$ first-exit time of BM from a centred ball $B(0,r)$ of radius r

Theorem
For $d = 1$ and any $r > 0$,

$$\mathbb{P}\{\sup_{0\leqslant s\leqslant 1} |W_s| < r\} \leqslant \frac{4}{\pi} \mathrm{e}^{-\pi^2/8r^2}$$

For arbitrary dimension d, the distribution function of the first-exit time τ_r can be expressed with the help of an infinite series

Theorem [Ciesielski & Taylor, 1962]

$$\mathbb{P}\{\tau_r > t\} = \mathbb{P}\{\sup_{0\leqslant s\leqslant t} \|W_s\| < r\} = \sum_{l=1}^{\infty} \xi_{d,l}\, \mathrm{e}^{-q_{d,l}^2 t/2r^2}$$

where $q_{d,l}$, $l \geqslant 1$, are the positive roots of the Bessel function J_ν, for $\nu = d/2-1$, and

$$\xi_{d,l} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{d,l}^{\nu-1}}{J_{\nu+1}(q_{d,l})}$$

Generalizations: Weighted norms

Theorem [Berthet & Zhan Shi, 1998 (preprint)] ($d = 1$)

$$\mathbb{P}\Big\{\sup_{0<t\leq 1} \frac{|W_t|}{f(t)} < \varepsilon\Big\} \sim \exp\Big(-\frac{\pi^2}{8\varepsilon^2}\int_0^1 \frac{\mathrm{d}t}{f^2(t)}\Big)$$

There is a condition on the admissible weights f:

- ▷ Admissible are for example $f(t) = t^\alpha$, $-\infty < \alpha < 1/2$, strictly positive f, $f(t) = t^{1/2}(\log(1/t))^\beta$ for $\beta > 1/2$
- ▷ An example of a not admissible function is $f(t) = \sqrt{t \log\log(1/t)}$

- ▷ Generalizations to other norms, to shifted balls
- ▷ Generalizations to Gaussian processes

- ▷ We will use the simplest variant to study escape from a saddle

References for PART V

Here is a brief selection of references:

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PART VI

First-passage of Brownian motion to a (curved) boundary

- ▷ Brownian motion crossing constant levels (reflection principle)
- ▷ Brownian motion crossing a linear boundary
- ▷ A master equation for the distribution of the first-passage time to a general boundary
- ▷ An integral equation for the first-passage density

First passage to a constant level

Recall the reflection principle for BM

$$\mathbb{P}^{0,-b}\{\tau_0 < t\} = 2\mathbb{P}^{0,-b}\{W_t \geq 0\}$$

$\tau_a =$ first-passage time of BM at level $a \geq 0$

Equivalent

$$\mathbb{P}^{0,0}\{\tau_b < t\} = 2\mathbb{P}^{0,0}\{W_t \geq b\} = \frac{1}{\sqrt{2\pi t}} \int_b^\infty \mathrm{e}^{-x^2/2t}\,\mathrm{d}x$$

Differentiate to obtain density of τ_b

$$\begin{aligned}
f(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}^{0,0}\{\tau_b < t\} \\
&= -\frac{1}{\sqrt{2\pi t}}\frac{1}{t}\int_b^\infty e^{-x^2/2t}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi t}}\int_b^\infty \frac{x^2}{t^2}e^{-x^2/2t}\,\mathrm{d}x \\
&= -\frac{1}{\sqrt{2\pi t}}\frac{1}{t}\int_b^\infty e^{-x^2/2t}\,\mathrm{d}x - \frac{1}{\sqrt{2\pi t}}\left[\frac{x}{t}e^{-x^2/2t}\Big|_{x=b}^\infty - \frac{1}{t}\int_b^\infty e^{-x^2/2t}\,\mathrm{d}x\right] \\
&= \frac{1}{\sqrt{2\pi t}}\frac{b}{t}\mathrm{e}^{-b^2/2t} = \frac{b}{t^{3/2}}\varphi\left(\frac{b}{\sqrt{t}}\right) \qquad (\varphi = \text{ standard Normal density})
\end{aligned}$$

Linear boundaries

The formula for the density generalizes to linear boundaries

$$\tau_g := \inf\{t \colon W_t \geq g(t)\} \qquad \text{with} \qquad g(t) := b + ct \qquad (b > 0)$$

τ_g has density

$$f(t) = \frac{b}{t^{3/2}} \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)$$

Note that for $c \geq 0$

$$\mathbb{P}^{0,0}\{\tau_g < \infty\} = \mathrm{e}^{-2cb}$$

For $c > 0$: $\mathbb{P}\{\tau_g = \infty\} > 0 \Longrightarrow f$ no proper density

General boundaries

In general: No closed-form expression for the density of the first-passage time of BM to a curved boundary

$g : (0, \infty) \to \mathbb{R}$ continuous, $g(0+) \geq 0$

Markov property for BM allows to restart upon first passage, yielding

Master equation

$$1 - \Phi\left(\frac{z}{\sqrt{t}}\right) = \int_0^t \left[1 - \Phi\left(\frac{z - g(s)}{\sqrt{t - s}}\right)\right] F(\mathrm{d}s) \qquad \forall\, z \geq g(t)$$

- ▷ F is the distribution function of τ_g
- ▷ Φ is the distribution function of a standard Normal r.v.

From this integral equation, a variety of integral equations for the first-passage distribution or density are derived
Solved either numerically or using fixed-point arguments

General boundaries

Under additional assumptions on g

(g cont. differentiable with $\mathbb{P}\{\tau_g = 0\} = 0$)

Density f of τ_g exists and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[1 - \Phi\left(\frac{g(t)}{\sqrt{t}}\right)\right] = \frac{1}{2} f(t) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[1 - \Phi\left(\frac{g(t) - g(s)}{\sqrt{t-s}}\right)\right] f(s)\,\mathrm{d}s \quad \forall\, t$$

(Proof nontrivial – taking derivatives has to be justified)

References for PART VI

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PART VII

The simplest class of slow–fast systems: Slowly driven systems

- ▷ Concentration of sample paths near the bottom of a well
- ▷ Stochastic resonance
- ▷ Hysteresis cycles
- ▷ Bifurcation delay

Concentration of sample paths near the bottom of a well: Deterministic case

$d = 1$

Overdamped motion in a potential landscape

$$\varepsilon \dot{x}_t = f(x_t, t) \;, \qquad f(x,t) = -\nabla U(x,t) = -\frac{\partial}{\partial x} U(x,t)$$

Assume for the moment that U is a single-well potential for all t
(Otherwise: restrict to a suitable space–time region)

Let $x^\star(t)$ denote the bottom of the well, i.e.,

$$f(x^\star(t), t) = 0 \qquad \forall\, t$$

$t \mapsto x^\star(t)$ is called equilibrium branch

$x^\star(t)$ is called uniformly asymptotically stable if

$$a^\star(t) := \partial_x f(x^\star(t), t) = -\partial_{xx} U(x^\star(t), t) \leq -a_0 < 0 \qquad \forall\, t$$

(Curvature of the well remains bounded away from zero)

Excursion: Static potentials

Assume $U(x,t) = U(x,t_0)$ for all times t ("frozen system")

Dynamics

$$y_t := x_t^{\text{frozen}} - x^\star(t_0)$$

$$\varepsilon \dot{y}_t = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} x_t^{\text{frozen}} = f(x_t^{\text{frozen}}, t_0) = a^\star(t_0) y_t + \mathcal{O}(y_t^2) \;, \qquad a^\star(t_0) < 0$$

This implies

$$|y_t| \leq |y_0| \, \mathrm{e}^{-|a^\star(t_0)|\, t/2\varepsilon} \qquad \text{for } |y_t| \text{ small enough}$$

- x_t^{frozen} approaches $x^\star(t_0)$ exponentially fast
- The speed depends on the curvature of the well:
  The steeper the well, the faster the approach

What happens when the shape of the well changes slowly in time?

Back to slowly driven systems

Theorem [Tihonov 1952, Gradšteĭn 1953]

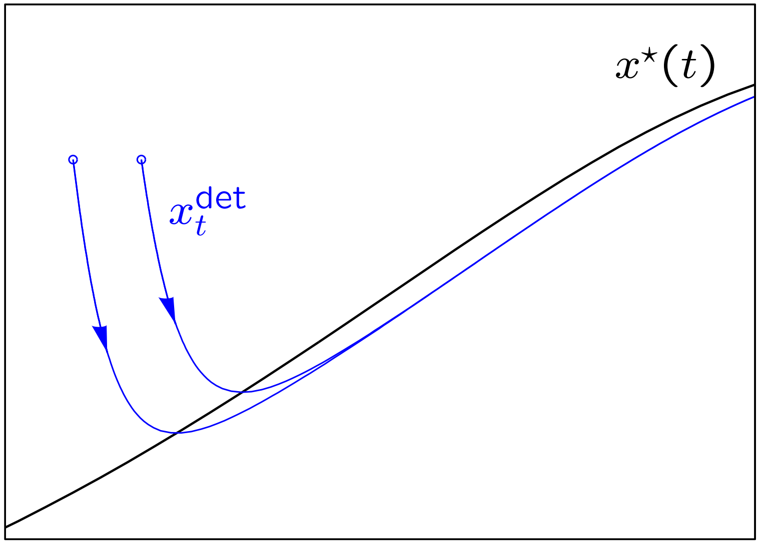
$\exists\, \varepsilon_0, c_0, c_1 \ \forall \varepsilon \leq \varepsilon_0$ (depending only on f) s.t.

- $\exists$ particular solution $\hat{x}_t^{\text{det}}$ s.t. $|\hat{x}_t^{\text{det}} - x^\star(t)| \leq c_1 \varepsilon \quad \forall\, t$
- If $|x_0 - x^\star(0)| \leq c_0$ then the solution x_t^{det} starting in x_0 at time $t = 0$ satisfies

$$|x_t^{\text{det}} - \hat{x}_t^{\text{det}}| \leq |x_0 - x^\star(0)|\, \mathrm{e}^{-a_0 t/2\varepsilon} \qquad \forall\, t$$

$\hat{x}_t^{\text{det}}$ is called adiabatic or slow solution

- $\hat{x}_t^{\text{det}}$ attracts nearby solutions
- $\hat{x}_t^{\text{det}}$ tracks $x^\star(t)$ at distance $\leq \varepsilon$
- $\hat{x}_t^{\text{det}}$ is not uniquely determined, we can always start closer to $x^\star(t)$



Sketch of the proof

Part 1: Existence of an adiabatic solution

(compare to the idea of proof in the case of a frozen potential)

For an arbitrary solution x_t, define the deviation $z_t := x_t - x^\star(t)$

A Taylor expansion in the *moving* point $x^\star(t)$ shows

$$\varepsilon \dot{z}_t = a^\star(t) z_t + b^*(z_t, t) - \varepsilon \dot{x}^\star(t) \leq -a_0 z_t + \mathcal{O}(z_t^2) - \varepsilon \dot{x}^\star(t)$$

We need a bound on the speed at which $x^\star(t)$ can change:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(x^\star(t), t) = \partial_x f(x^\star(t), t) \dot{x}^\star(t) + \partial_t f(x^\star(t), t)$$

implies

$$\dot{x}^\star(t) = \frac{\partial_t f(x^\star(t), t)}{|a^\star(t)|} \qquad \text{bounded, as } a^\star(t) \text{ is bounded away from 0}$$

$\Longrightarrow \exists\, K$ s.t. $|\dot{x}^\star(t)| \leq K < \infty$

Sketch of the proof

For small enough z_t, Gronwall's lemma shows

$$\varepsilon \dot{z}_t \leq -\frac{a_0}{2} z_t + \varepsilon K \quad \Longrightarrow \quad \dot{z}_t \leq -\frac{a_0}{2\varepsilon} z_t + K$$

$$\Longrightarrow \quad z_t \leq \left(z_0 - \frac{2\varepsilon}{a_0} K \right) \mathrm{e}^{-a_0 t/2\varepsilon} + \frac{2\varepsilon}{a_0} K$$

Choosing z_0 of order ε yields $|z_t| \leq \textit{const}\, \varepsilon$ for all t. This implies the existence of an adiabatic solution.

Part 2: An adiabatic solution is attracting

Repeating the same kind of arguments, this time using a Taylor expansion around the adiabatic solution $\widehat{x}_t^{\text{det}}$, proves the claim.

The effect of noise

The approach we will present first is not optimal for $d = 1$, but generalisable.

$$\mathrm{d}x_s = -\nabla_x U(x_s, \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

In slow time ($t = \varepsilon s$, $x_t = x_{\varepsilon s}$, $W_t = \sqrt{\varepsilon} W_s$ (in distribution))

$$\begin{aligned}\mathrm{d}x_t &= -\frac{1}{\varepsilon}\nabla_x U(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t \\ &=: \quad \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\end{aligned}$$

Assume for the moment that the potential $U(x,t)$ is quadratic, i.e.,

$$f(x,t) = a^\star(t)[x - x^\star(t)]$$

(Curvature and location of the bottom of the well change in time with $a^\star(t)$ and $x^\star(t)$)

Effect of noise – quadratic potentials

$$z_t := x_t - x_t^{\text{det}}$$

$$\mathrm{d}z_t = \frac{1}{\varepsilon}[f(x_t,t) - f(x_t^{\text{det}},t)]\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} = \frac{1}{\varepsilon}a^\star(t)z_t\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

We can solve the non-autonomous SDE for z_t

$$z_t = z_0 e^{\alpha^\star(t)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}}\int_0^t e^{\alpha^\star(t,s)/\varepsilon}\,\mathrm{d}W_s$$

where $\alpha^\star(t) = \int_0^t a^\star(s)\,\mathrm{d}s$ and $\alpha^\star(t,s) = \alpha^\star(t) - \alpha^\star(s)$

Therefore, z_t is a Gaussian r.v. with variance

$$v^\star(t) = \mathrm{Var}(z_t) = \frac{\sigma^2}{\varepsilon}\int_0^t e^{2\alpha^\star(t,s)/\varepsilon}\,\mathrm{d}s$$

For any fixed time t, z_t has a typical spreading of $\sqrt{v^\star(t)}$, and a standard estimate shows

$$\mathbb{P}\{|z_t| \geq h\} \leq \mathrm{e}^{-h^2/2v^\star(t)}$$

Effect of noise – quadratic potentials

Goal: Similar estimate for the whole sample path

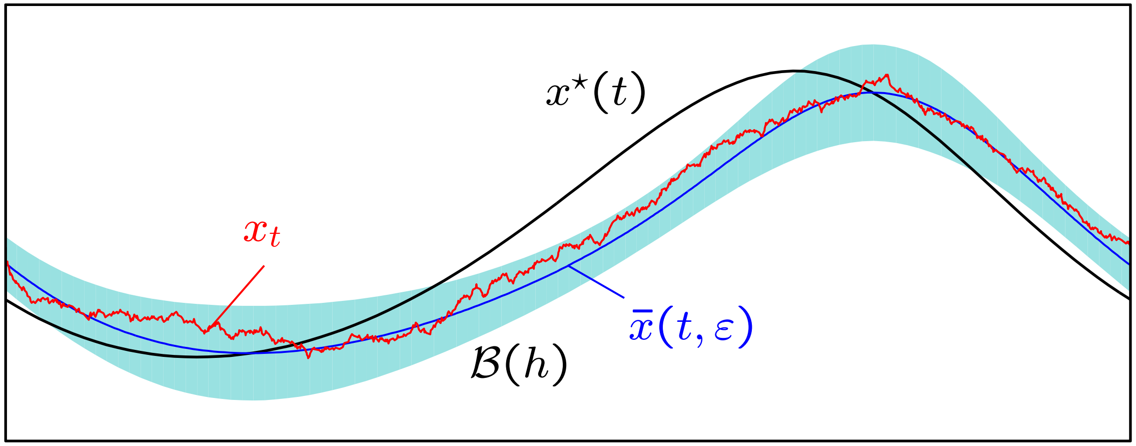
As $v^\star(0) = 0$, we need to find a better idea near the origin. We will replace $v^\star(t)$ by its "asymptotic value", pretending that we started the process at time $t_0 \to -\infty$.

Crucial observation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{v^\star(t)}{\sigma^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varepsilon}\int_0^t \mathrm{e}^{2\alpha^\star(t,s)/\varepsilon}\,\mathrm{d}s = \frac{1}{\varepsilon} + \frac{2a^\star(t)}{\varepsilon}\frac{v^\star(t)}{\sigma^2}$$

- ▷ $v^\star(t)/\sigma^2$ satisfies a singularly perturbed ODE
- ▷ Actual variance $v^\star(t)/\sigma^2$ is the particular solution starting in 0
- ▷ $\exists$ adiabatic solution $\zeta(t)$, tracking $\zeta^\star(t) = 1/2|a^\star(t)|$
- ▷ $v^\star(t)/\sigma^2$ is attracted exponentially fast by $\zeta(t)$s
- ▷ $\operatorname{Var} z_t = v^\star(t) = \sigma^2[\zeta(t) - \zeta(0)\,\mathrm{e}^{2\alpha^\star(t)/\varepsilon}]$

Introducing space–time sets



$$\mathcal{B}(h) := \Big\{(z,t)\colon |z| \leq h\sqrt{\zeta}\Big\}$$

For $h = \sigma$, at each t the "breathing" strip $\mathcal{B}(h)$ has a width equal to the typical spreading of z_t

For $h > \sigma$, we expect z_t to remain in $\mathcal{B}(h)$ for quite a while

How long will it take until z_t exits?

A first result for the first-exit time $\tau_{\mathcal{B}(h)}$

$\forall \gamma \in (0, 1/2)\ \forall t$

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} = C_{h/\sigma}(t,\varepsilon)\,\mathrm{e}^{-h^2/2\sigma^2}$$

with $C_{h/\sigma}(t,\varepsilon) \leq 2\left\lceil\dfrac{|\alpha^\star(t)|}{\varepsilon\gamma}\right\rceil \mathrm{e}^{\gamma[1+\mathcal{O}(\varepsilon)]h^2/\sigma^2}$

- ▷ $\mathrm{e}^{-h^2/2\sigma^2}$ becomes small as soon as $h \gg \sigma$
- ▷ $a^\star(t)$ bounded $\Longrightarrow \alpha^\star(t) \sim t \Longrightarrow C_{h/\sigma}(t,\varepsilon) = \mathit{const}\,\dfrac{t}{\varepsilon\gamma}\,\mathrm{e}^{\gamma h^2[1+\mathcal{O}(\varepsilon)]/\sigma^2}$

  The probability of exit remains small for all times t which are comparable to Kramers' time

Idea for the proof

- ▷ Consider a partition of the time interval s.t. $|\alpha^\star(t_{j+1}, t_j)| = \varepsilon\gamma$
- ▷ $\lceil \dots \rceil$ is the number of intervals in the partition
- ▷ On these short time intervals, approximate z_t by a Gaussian martingale
- ▷ Use Bernstein-type inequality to estimate probability of exit during a short time interval

The behaviour of the first-exit time $\tau_{\mathcal{B}(h)}$ ($d = 1$)

In the special case $d = 1$ the preceding result on the first-exit time from a neighbourhood of a quadratic potential well can be improved:

Theorem [Berglund & G '05]
$\exists c_0, r_0 > 0$ s.t. whenever

$$r = r(h/\sigma, t, \varepsilon) := \frac{\sigma}{h} + \frac{t}{\varepsilon}\,\mathrm{e}^{-c_0 h^2/\sigma^2} \leq r_0$$

then

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} = C_{h/\sigma}(t,\varepsilon) e^{-h^2/2\sigma^2}$$

with

$$C_{h/\sigma}(t,\varepsilon) = \sqrt{\frac{2}{\pi}}\,\frac{|\alpha(t)|}{\varepsilon}\,\frac{h}{\sigma}\Big[1 + \mathcal{O}(r) + \varepsilon + \frac{\varepsilon}{|\alpha(t)|}\log(1 + h/\sigma)\Big]$$

Idea of the proof
Proceed as before, considering the approximating Gaussian martingale as a time-changed BM. Use results on first passage of BM to a curved boundary.

The behaviour of the first-exit time $\tau_{\mathcal{B}(h)}$ ($d = 1$)

For general single-well potentials with non-vanishing curvature, as long as $t < \tau_{cB(h)}$, the solution of the SDE is well approximated by the solution of the linearized SDE.

The error made scales with the width h of $\mathcal{B}(h)$.

Theorem [Berglund & G '05]
$\exists c_0, r_0 > 0$ s.t. whenever

$$r = r(h/\sigma, t, \varepsilon) := \frac{\sigma}{h} + \frac{t}{\varepsilon} \mathrm{e}^{-c_0 h^2/\sigma^2} \leq r_0$$

then

$$C_{h/\sigma}(t,\varepsilon) e^{-[1+\mathcal{O}(h)]h^2/2\sigma^2} \leq \mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C_{h/\sigma}(t,\varepsilon) e^{-[1-\mathcal{O}(h)]h^2/2\sigma^2}$$

with the prefactor $C_{h/\sigma}(t,\varepsilon)$ as above

Repetition: One-dimensional slowly driven systems

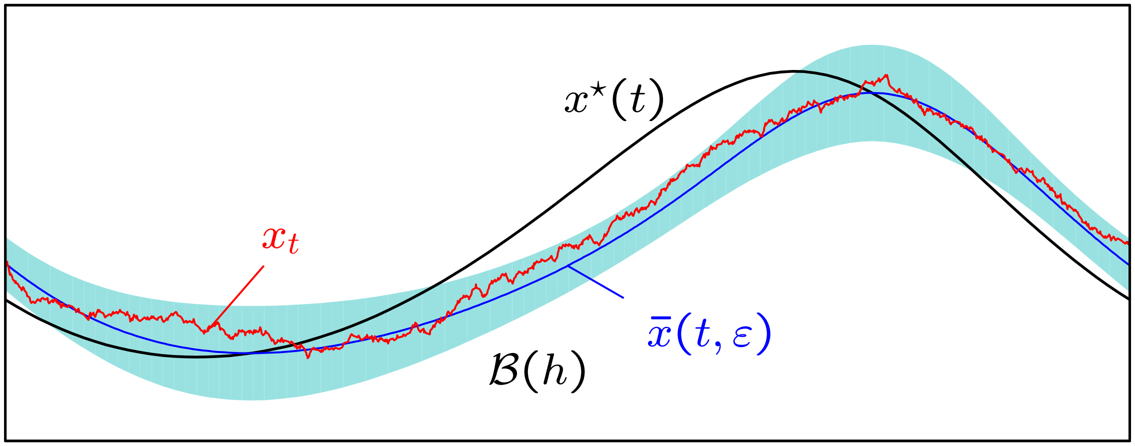
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Uniformly asymptotically stable equilibrium branch $x^\star(t)$:

$$f(x^\star(t), t) = 0\ , \qquad a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0$$

Adiabatic solution:
$\bar{x}(t,\varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$

$\mathcal{B}(h)$: strip around $\bar{x}(t,\varepsilon)$
of width $\simeq h/2|a^\star(t)|$



Theorem [Berglund & G '02], [Berglund & G '05]

$$\mathbb{P}\Big\{x_t \text{ leaves } \mathcal{B}(h) \text{ before time } t\Big\} \simeq \sqrt{\frac{2}{\pi}}\,\frac{1}{\varepsilon}\Big|\int_0^t a^\star(s)\,\mathrm{d}s\Big|\,\frac{h}{\sigma}\ \mathrm{e}^{-h^2/2\sigma^2}$$

Idea

Behaviour of $y_t = x_t - \bar{x}(t,\varepsilon)$?

Linearizing the drift coefficent $\longrightarrow$ nonautonomous linear SDE

$$\mathrm{d}y_t^0 = \frac{1}{\varepsilon} a(t) y_t^0 \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t\,, \qquad y_0 = 0$$

$$a(t) = \partial_x f(\bar{x}(t,\varepsilon),t) = \text{curvature}\,; \qquad \alpha(t,s) := \int_s^t a(u)\,du$$

Solution $y_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}W_s$ is a Gaussian process

Variance $v(t) = \dfrac{\sigma^2}{\varepsilon} \displaystyle\int_0^t \mathrm{e}^{2\alpha(t,s)/\varepsilon}\,\mathrm{d}s \sim \dfrac{\sigma^2}{\text{curvature}}$

Concentration result for y_t^0: $\mathbb{P}\{|y_t^0| > \delta\} \le \mathrm{e}^{-\delta^2/2v(t)}$

Theorem: Analogous resultat for the whole path $\{y_t\}_{t\ge 0}$

Example I: Stochastic resonance

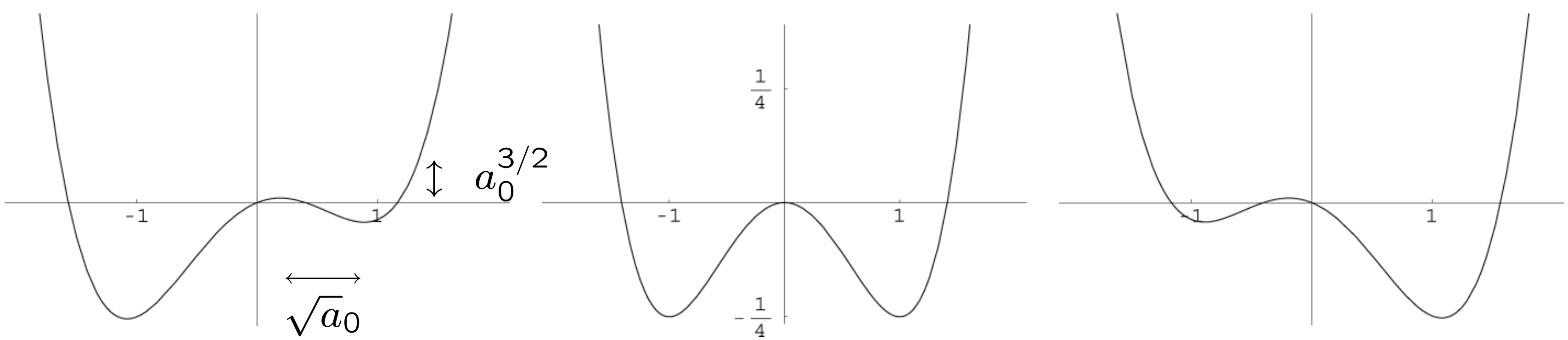
Recall the energy-balance model from the first lecture

Overdamped motion of a Brownian particle

$$\mathrm{d}x_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

in a periodically modulated potential

$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_{\mathrm{c}} - a_0)\cos(2\pi\varepsilon s)x$$



$V(x, 0)$ $\quad V(x, 1/4) = V(x, 3/4)$ $\quad V(x, 1/2)$

Example I: Stochastic resonance

3 small parameters :

$0 < \sigma \ll 1$, $0 < \varepsilon \ll 1$, $0 < a_0 \ll 1$

Equation of motion of a Brownian particle

$$\mathrm{d}x_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s)\ \mathrm{d}s + \sigma\,\mathrm{d}W_s$$

$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_c - a_0)\cos(2\pi\varepsilon s)x\ , \qquad \lambda_c = \tfrac{2}{3\sqrt{3}}$$

Rewrite in slow time $t = \varepsilon s$:

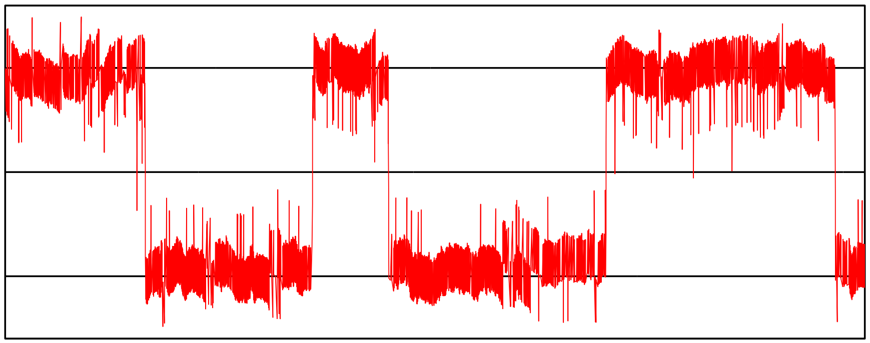
$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\ \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

with drift term

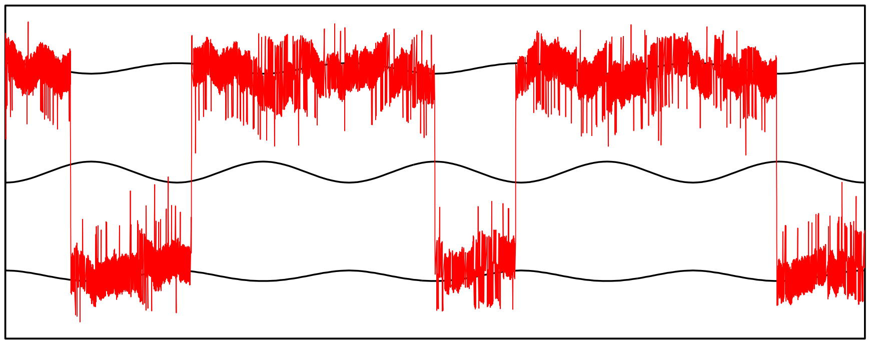
$$f(x, t) = -\frac{\partial}{\partial x} V(x, t) = x - x^3 - (\lambda_c - a_0)\cos(2\pi t)$$

Sample paths

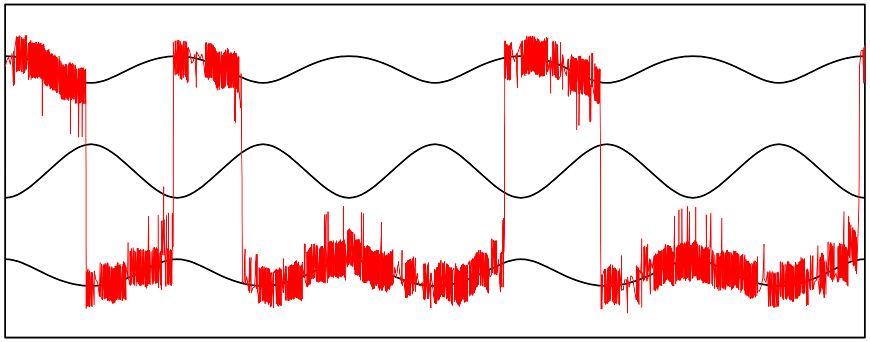
Amplitude of modulation $A = \lambda_c - a_0$
Speed of modulation ε
Noise intensity σ



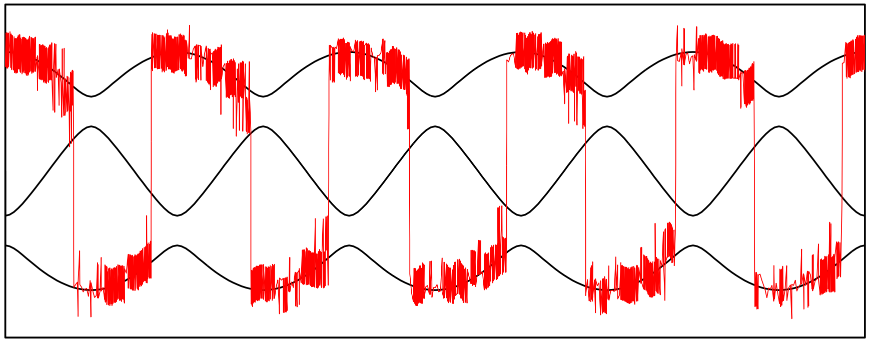
$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$



$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$

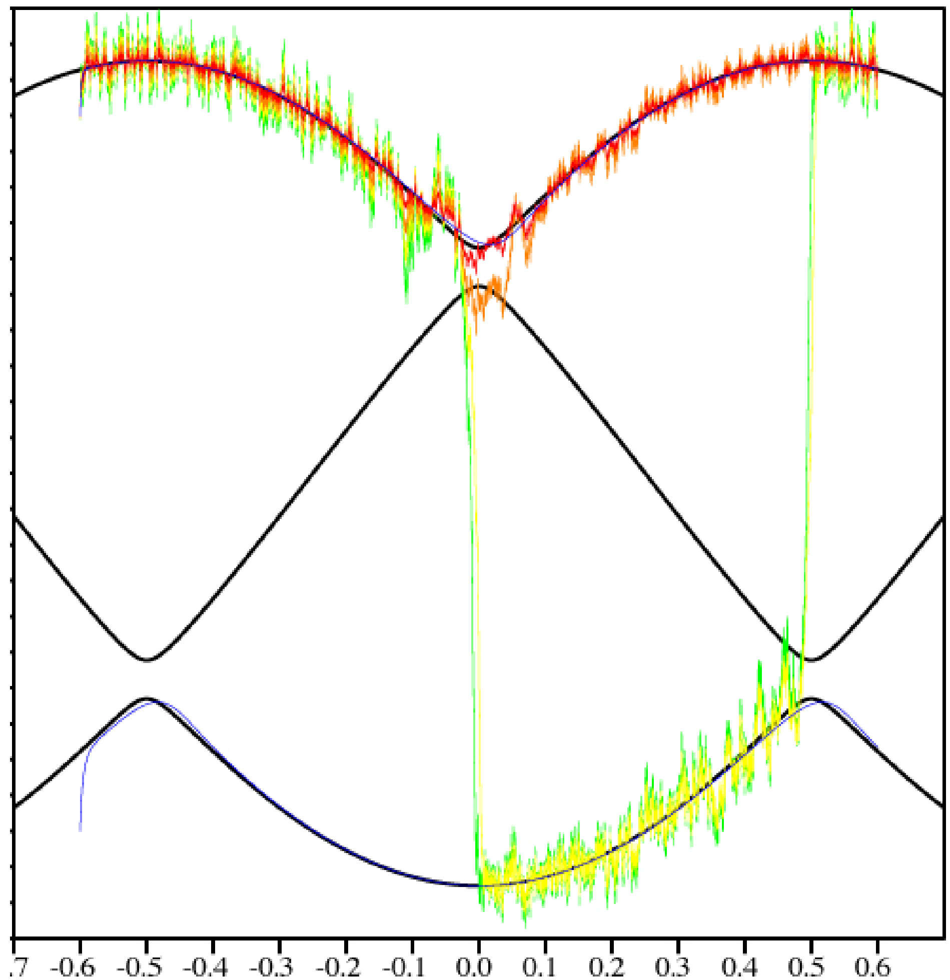


$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

Small-barrier-height regime



System Stochastic resonance

Epsilon	0.005	0.005	0.005	0.005	0.005
Sigma	0	0.03	0.06	0.09	0.12
Gap	0.005	0.005	0.005	0.005	0.005

Time step 0.001
Seeds 0.534154541 0.355564852

Effective barrier heights and scaling of small parameters

Theorem [Berglund & G, Annals of Appl. Probab. '02]
(informal version; exact formulation uses first-exit times from space–time sets)

$$\exists \text{ threshold value } \sigma_c = (a_0 \vee \varepsilon)^{3/4}$$

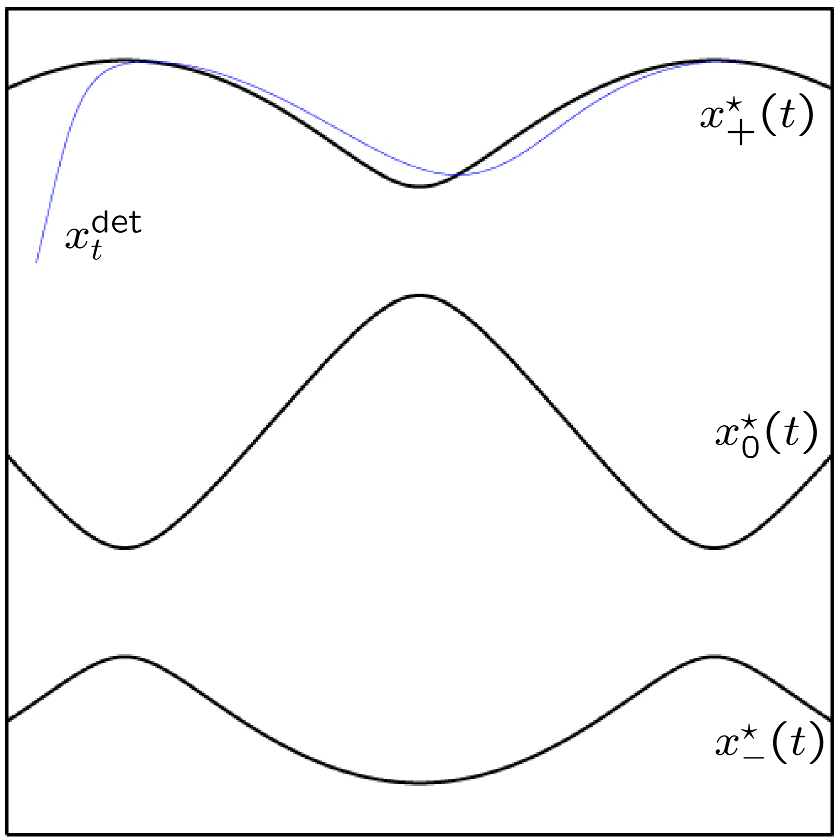
Below: $\sigma \leq \sigma_c$

- ▷ Transitions unlikely
- ▷ Sample paths concentrated in one well
- ▷ Typical spreading $\asymp \dfrac{\sigma}{\left(|t|^2 \vee a_0 \vee \varepsilon\right)^{1/4}} \asymp \dfrac{\sigma}{\left(\text{curvature}\right)^{1/2}}$
- ▷ Probability to observe a transition $\leq e^{-const\, \sigma_c^2/\sigma^2}$

Above: $\sigma \gg \sigma_c$

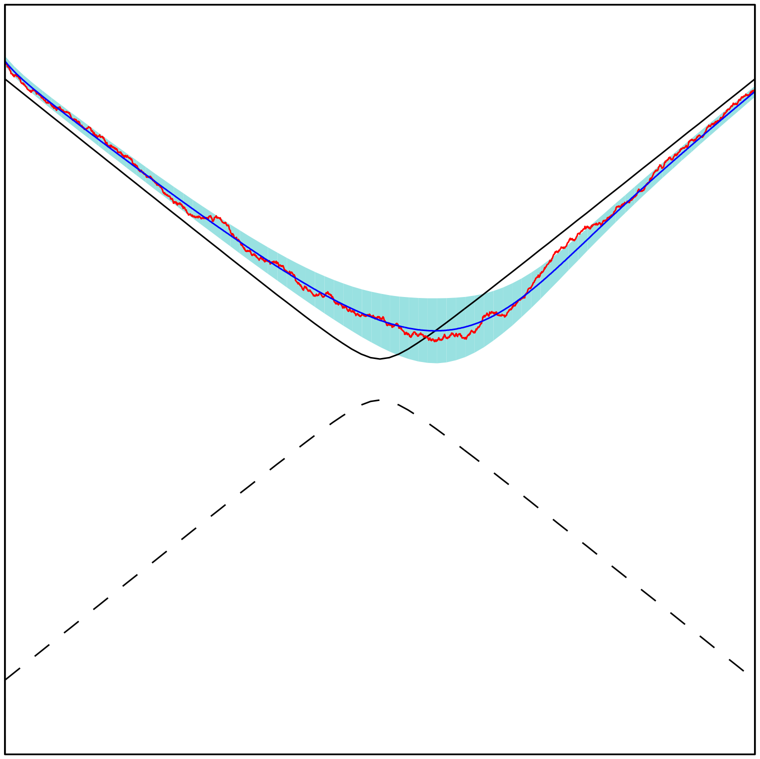
- ▷ 2 transitions per period likely (back and forth)
- ▷ with probability $\geq 1 - e^{-const\, \sigma^{4/3}/\varepsilon|\log \sigma|}$
- ▷ Transtions occur near instants of minimal barrier height
- ▷ Transition window $\asymp \sigma^{2/3}$

Step 1: Deterministic dynamics



- For $t \leqslant -const$:
  x_t^{det} reaches ε-nbhd of $x^\star_+(t)$
  in time $\asymp \varepsilon|\log \varepsilon|$ (Tihonov '52)
- For $-const \leqslant t \leqslant -(a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x^\star_+(t) \asymp \varepsilon/|t|$
- For $|t| \leqslant (a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x^\star_0(t) \asymp (a_0 \vee \varepsilon)^{1/2} \geqslant \sqrt{\varepsilon}$
  (effective barrier height)
- For $(a_0 \vee \varepsilon)^{1/2} \leqslant t \leqslant +const$:
  $x_t^{\text{det}} - x^\star_+(t) \asymp -\varepsilon/|t|$
- For $t \geqslant +const$:
  $|x_t^{\text{det}} - x^\star_+(t)| \asymp \varepsilon$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



$$v(t) \sim \frac{\sigma^2}{\text{curvature}} \sim \frac{\sigma^2}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/2}}$$

$$\zeta(t) := \frac{v(t)}{\sigma^2}$$

$$\mathcal{B}(h) := \Big\{ (x,t) \colon |x - x_t^{\text{det}}| < h\sqrt{\zeta(t)} \Big\}$$

$\tau_{\mathcal{B}(h)} =$ first-exit time of (x_t, t) from $\mathcal{B}(h)$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$

Theorem ([Berglund & G '02], [Berglund & G '05])

$\exists\ h_0, c_1, c_2, c_3 > 0 \quad \forall h \leq h_0$

$$C(h/\sigma, t, \varepsilon)\,\mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leq \mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < t\Big\} \leq C(h/\sigma, t, \varepsilon)\,\mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

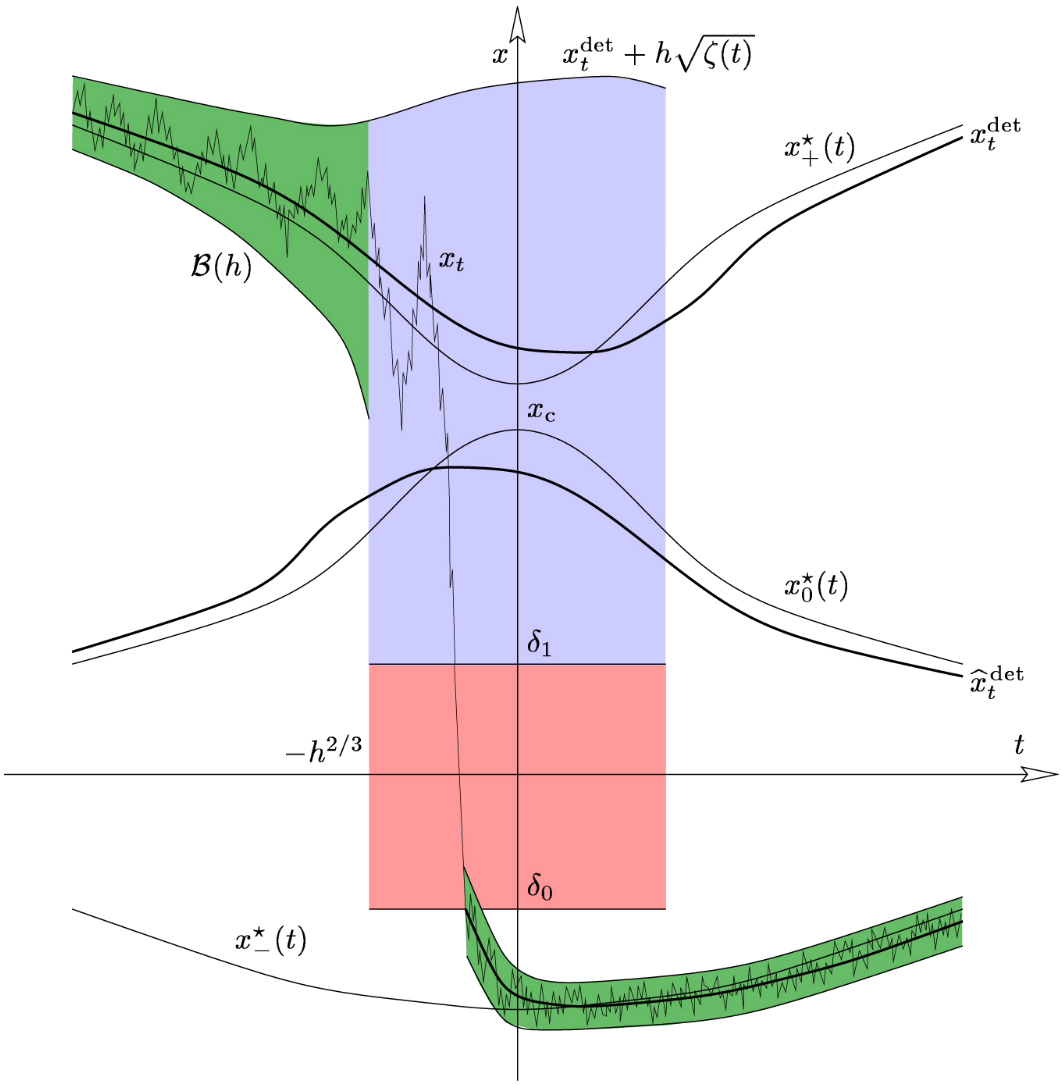
with $\kappa_+ = 1 - c_1 h\ , \quad \kappa_- = 1 + c_1 h + c_1\,\mathrm{e}^{-c_2 t/\varepsilon}\ ;$

$$C(h/\sigma, t, \varepsilon) = \sqrt{\frac{2}{\pi}}\,\frac{|\alpha(t)|}{\varepsilon}\,\frac{h}{\sigma}\Big[1 + \mathcal{O}\Big(\frac{\sigma}{h}\Big) + \frac{t}{\varepsilon}\,\mathrm{e}^{-c_3 h^2/\sigma^2} + \mathrm{e}^{-c_1 t/\varepsilon} + \varepsilon\Big]$$

Basic idea

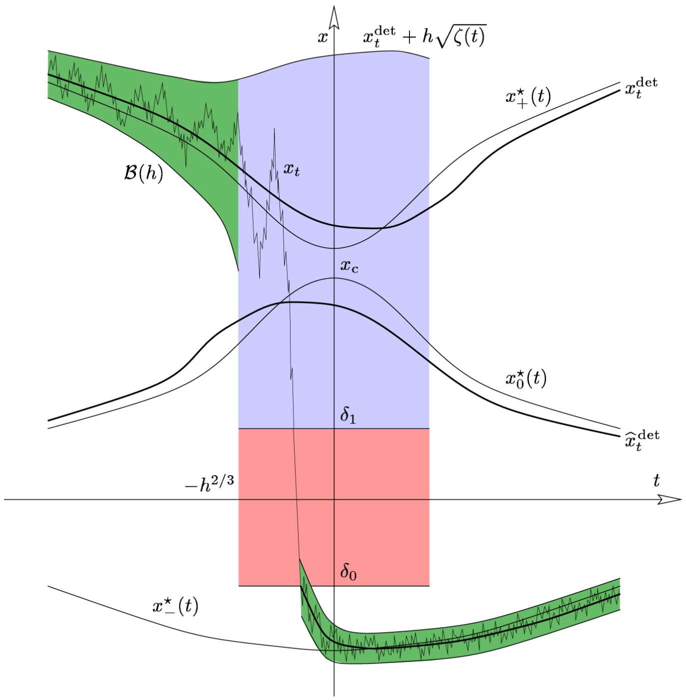
local approximation of y_t by y_t^0; Gaussian process is a rescaled Brownian motion; results on the density of the first-passage time for BM through nonlinear curves

Step 3: Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



- ▷ Typical paths stay below $x_t^{\rm det} + h\sqrt{\zeta(t)}$
- ▷ For $t \ll -\sigma^{2/3}$:
  Transitions unlikely;
  as below threshold
- ▷ At time $t = -\sigma^{2/3}$:
  Typical spreading satisfies $\sigma^{2/3} \gg x_t^{\rm det} - x_0^\star(t)$;
  Transitions become likely
- ▷ Near saddle:
  Diffusion dominated dynamics
- ▷ Levels $\delta_1 > \delta_0$ with $f \asymp -1$;
  δ_0 in domain of attr. of $x_-^\star(t)$;
  Drift dominated dynamics
- ▷ Below δ_0: beh. as for small σ

Step 3: Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



Idea of the proof

With probability $\geq \delta > 0$, in time $\asymp \varepsilon |\log \sigma| / \sigma^{2/3}$, the path reaches

- x_t^{det} if above
- then the saddle
- finally the level δ_1

In time $\sigma^{2/3}$ there are $\dfrac{\sigma^{4/3}}{\varepsilon|\log \sigma|}$ attempts possible

During a subsequent time span of length ε, level δ_0 is reached (with probability $\geq \delta$)

Finally, the path reaches the new well

Result

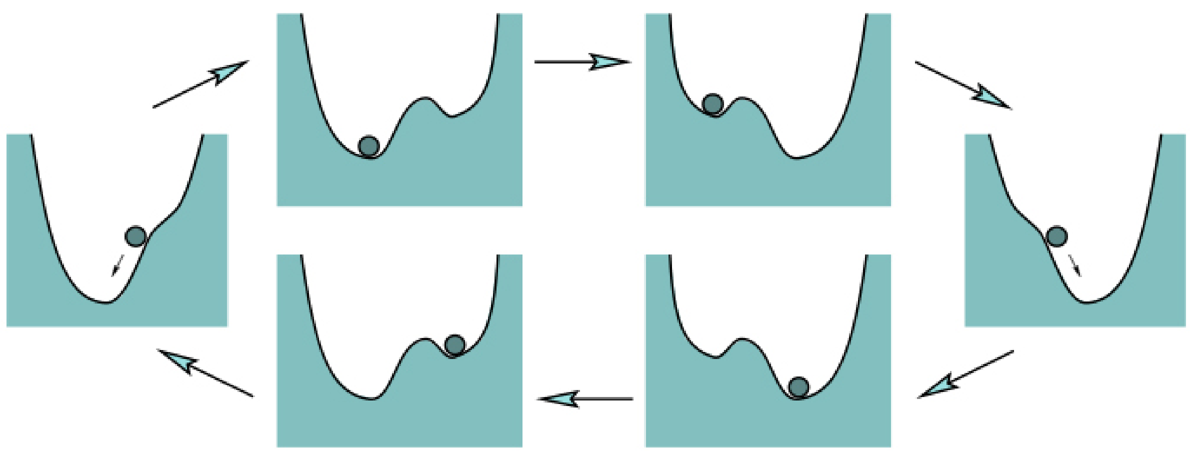
$$\mathbb{P}\Big\{x_s > \delta_0 \quad \forall s \in [-\sigma^{2/3}, t]\Big\} \leq \mathrm{e}^{-const\ \sigma^{4/3}/\varepsilon|\log \sigma|} \qquad (t \geq -\gamma\sigma^{2/3},\ \gamma \text{ small})$$

Example II: Hysteresis cycles

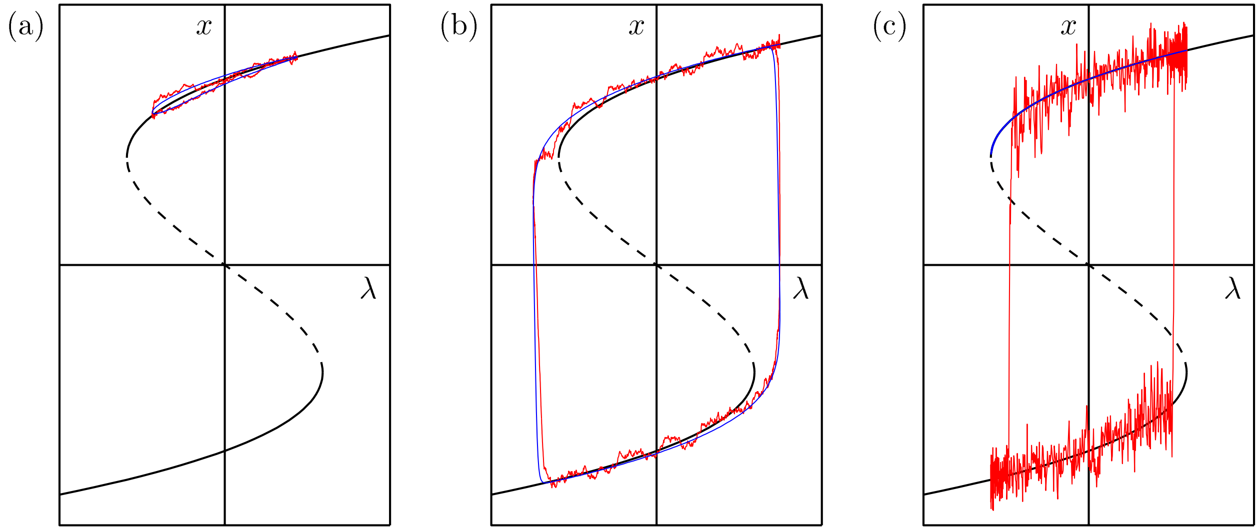
Recall the possibly periodic forcing of the freshwater flux in Stommel's box model

Periodically modulated double-well potential, where we now allow for above-threshold forcing amplitude

In this case, it becomes possible for the deterministic particle to switch wells
(provided the barrier vanishes for a sufficiently long time span ($\geq \gamma\varepsilon$))



Example II: Hysteresis cycles



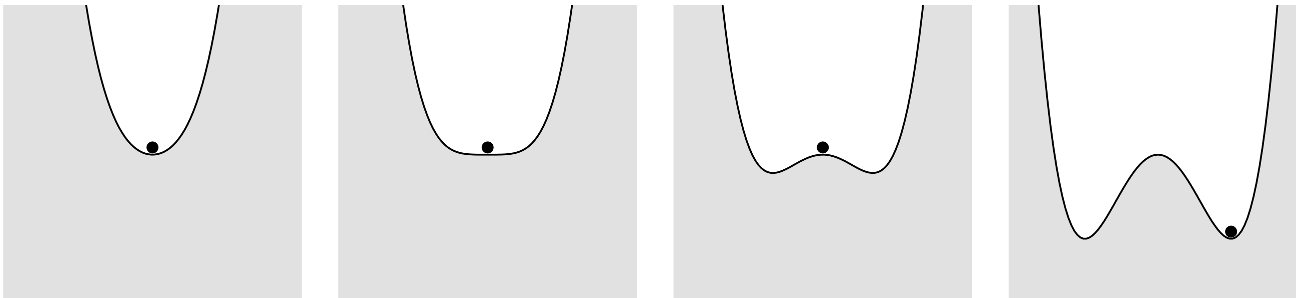
Theorem [Berglund & G '02]

- Small amplitude, small noise: Transitions unlikely during one cycle (However: Concentration of transition times within each period)
- Large amplitude, small noise: Hysteresis cycles
  Area = static area + $\mathcal{O}(\varepsilon^{2/3})$ (as in deterministic case)
- Large noise: Stoch. resonance / noise-induced synchronization
  Area = static area − $\mathcal{O}(\sigma^{4/3})$ (reduced due to noise)

Example III: Bifurcation delay

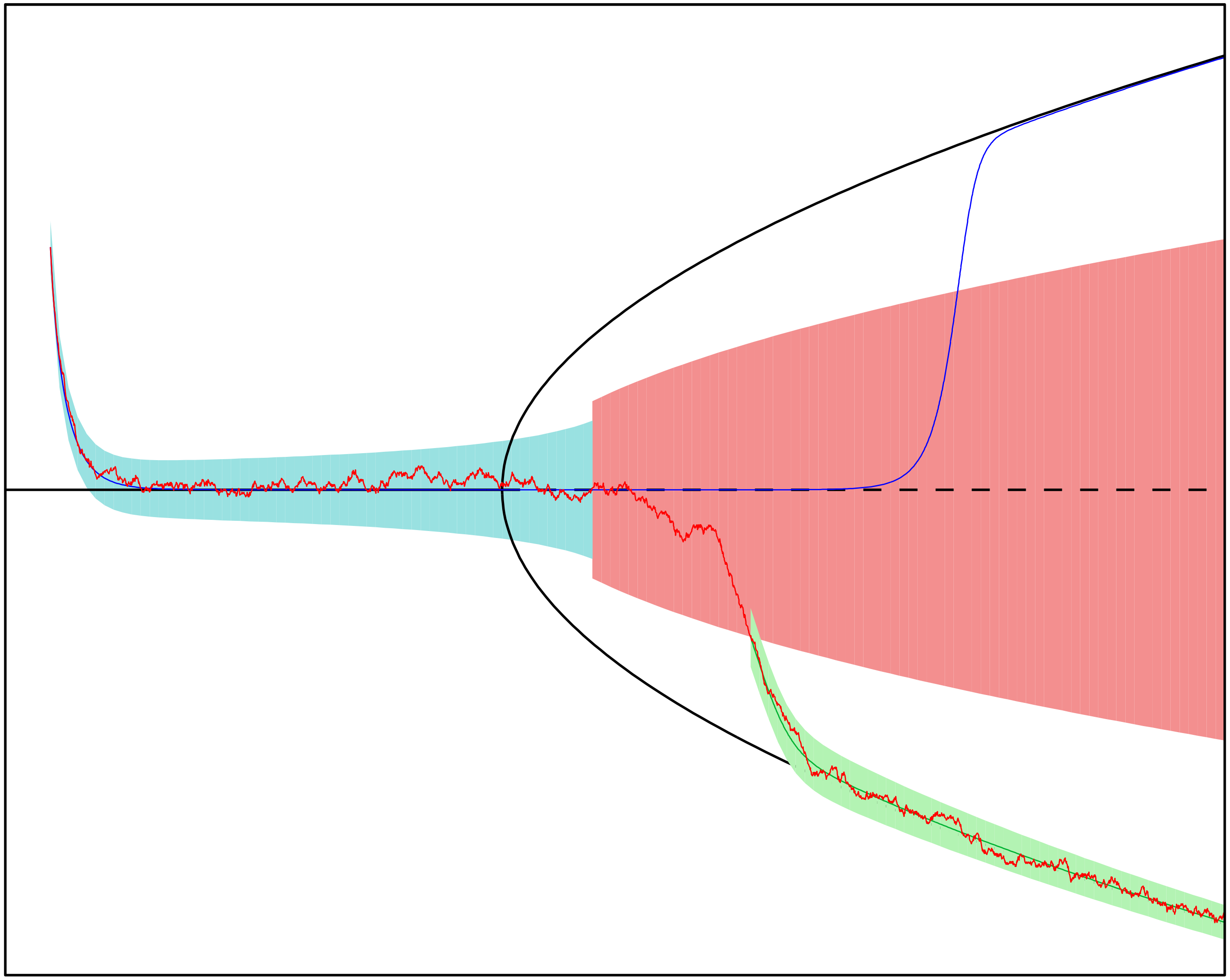
Symmetry breaking; try to measure bifurcation diagram

Slowly modulated potential, changing from single- to double-well



- ▷ What happens, if there is noise in the system?
- ▷ In which well will the particle finally settle?
- ▷ When is the decision taken?

Example III: Bifurcation delay



Deterministic system: Macroscopic bifurcation delay

Example III: Bifurcation delay

In the presence of noise:

- ▷ $\sigma \leq \mathrm{e}^{-K/\varepsilon}$: Bifurcation delay remains of order 1
- ▷ $\sigma = \varepsilon^{p/2}$ for $p > 1$: Bifurcation delay becomes microscopic, delay $= \sqrt{(p-1)\varepsilon|\log\varepsilon|}$
- ▷ $\sigma \geq \sqrt{\varepsilon}$: Spreading of paths is of order $\sqrt{\sigma}$ during a window of size σ around the bifurcation point

References for PART VII

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PART VIII

Random perturbations of general slow–fast systems

- ▷ Controlling the random fluctuations of the fast variables
- ▷ Reduced dynamics

General slow–fast systems

Recall the model for the North-Atlantic thermohaline circulation from the first lecture

Fully coupled SDEs on well-separated time scales

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\, \mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\, \mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\, \mathrm{d}t + \sigma' \, G(x_t, y_t)\, \mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

- ▷ $\{W_t\}_{t \geq 0}$ k-dimensional (standard) Brownian motion
- ▷ $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- ▷ $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- ▷ $F : \mathcal{D} \to \mathbb{R}^{n \times k}$, $G : \mathcal{D} \to \mathbb{R}^{m \times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Small parameters

- ▷ $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- ▷ $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε:
  $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leq 1$

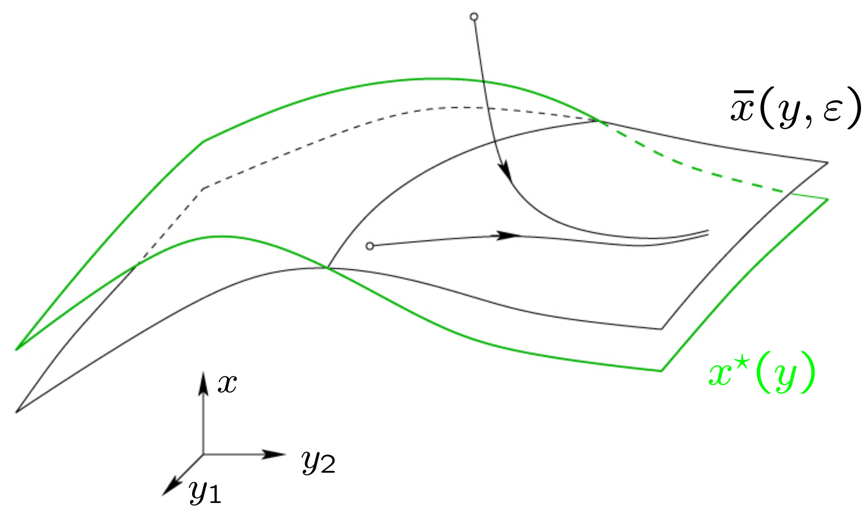
Near slow manifolds: Assumptions on the fast variables

Existence of a slow manifold: $\exists\, \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists\, x^\star : \mathcal{D}_0 \to \mathbb{R}^n$
s.t $\quad (x^\star(y), y) \in \mathcal{D} \quad$ and $\quad f(x^\star(y), y) = 0 \quad$ for $y \in \mathcal{D}_0$

Slow manifold is attracting: Eigenvalues of $A^\star(y) := \partial_x f(x^\star(y), y)$
satisfy $\operatorname{Re} \lambda_i(y) \leq -a_0 < 0$, uniformly in $\mathcal{D}_0$

Theorem ([Tihonov '52], [Fenichel '79])

There exists an *adiabatic manifold*:
$\exists\, \bar{x}(y, \varepsilon)$ s.t.



- ▷ $\bar{x}(y, \varepsilon)$ is invariant manifold for deterministic dynamics
- ▷ $\bar{x}(y, \varepsilon)$ attracts nearby solutions
- ▷ $\bar{x}(y, 0) = x^\star(y)$ and $\bar{x}(y, \varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$

Consider now *stochastic* system under these assumptions

Typical neighbourhoods of adiabatic manifolds

▷ Consider deterministic process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on (invariant) adiabatic manifold
▷ Dev. $\xi_t := x_t - x_t^{\text{det}}$ of fast variables from adiabatic manifold
▷ Linearize SDE for ξ_t; resulting process ξ_t^0 is Gaussian

Key observation

$\dfrac{1}{\sigma^2} \operatorname{Cov} \xi_t^0$ is a particular sol. of the det. slow–fast system

$$\begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\text{det}}) X(t) + X(t) A(y^{\text{det}})^{\mathsf{T}} + F_0(y^{\text{det}}) F_0(y^{\text{det}})^{\mathsf{T}} \\ \dot{y}_t^{\text{det}} = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

with $A(y) = \partial_x f(\bar{x}(y,\varepsilon), y)$ and F_0 0*th*-order approximation to F

▷ System admits an adiabatic manifold $\bar{X}(y, \varepsilon)$

Typical neighbourhoods

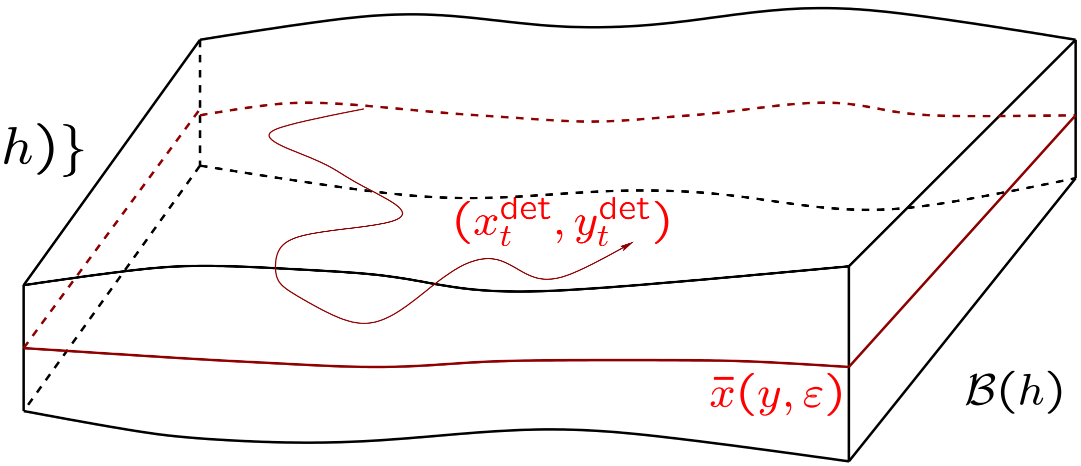
$$\mathcal{B}(h) := \Big\{ (x,y) \colon \Big\langle \Big[x - \bar{x}(y,\varepsilon) \Big], \bar{X}(y,\varepsilon)^{-1} \Big[x - \bar{x}(y,\varepsilon) \Big] \Big\rangle < h^2 \Big\}$$

Concentration of sample paths near adiabatic manifolds

Define (random) first-exit times

$\tau_{\mathcal{D}_0} := \inf\{s > 0 \colon y_s \notin \mathcal{D}_0\}$

$\tau_{\mathcal{B}(h)} := \inf\{s > 0 \colon (x_s, y_s) \notin \mathcal{B}(h)\}$



Theorem [Berglund & G, J. Differential Equations, 2003]

Assume: $\|\overline{X}(y,\varepsilon)\|$, $\|\overline{X}(y,\varepsilon)^{-1}\|$ uniformly bounded in $\mathcal{D}_0$

Then: $\exists\, \varepsilon_0 > 0 \quad \exists\, h_0 > 0 \quad \forall\, \varepsilon \leqslant \varepsilon_0 \quad \forall\, h \leqslant h_0$

$$\mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\Big\} \leqslant C_{n,m}(t) \exp\left\{-\frac{h^2}{2\sigma^2}\Big[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\Big]\right\}$$

where $C_{n,m}(t) = \Big[C^m + h^{-n}\Big]\left(1 + \frac{t}{\varepsilon^2}\right)$

Random perturbations: General slow–fast systems

$$\begin{cases} \mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma'\, G(x_t, y_t)\,\mathrm{d}W_t \end{cases}$$

Theorem

▷ Previous theorem can be summarized as:

$\mathbb{P}\Big\{(x_t, y_t)$ leaves $\mathcal{B}(h)$ before time $t\Big\} \simeq C_{n,m}(t,\varepsilon)\,\mathrm{e}^{-\kappa h^2/2\sigma^2}$

with $\kappa = 1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)$

(provided y_t does not drive the system away from the region where assumptions are satisfied)

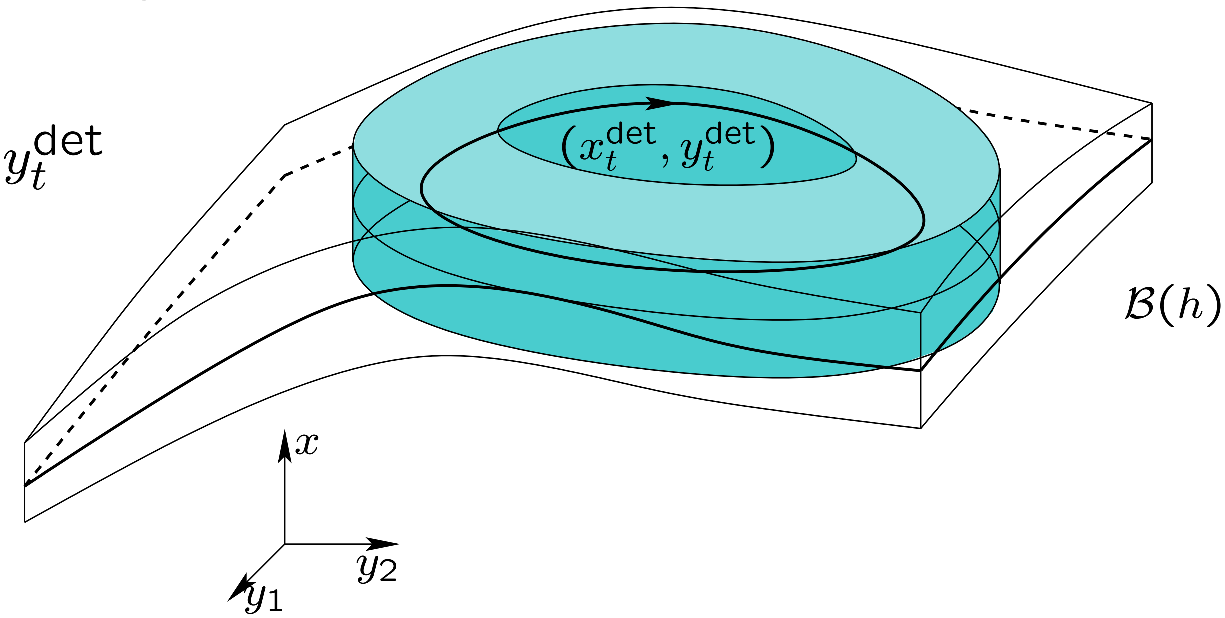
▷ Reduction to adiabatic manifold $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon), y_t^0)\,\mathrm{d}W_t$$

y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y}^{\mathrm{det}} = g(\bar{x}(y^{\mathrm{det}},\varepsilon) y^{\mathrm{det}})$

Near slow manifolds: Longer time scales

▷ Behaviour of g or behaviour of y_t and y_t^{det} becomes important



Example:
y_t^{det} following a stable periodic orbit

▷ $y_t \sim y_t^{\text{det}}$ for $t \leqslant \dfrac{\textit{const}}{\sigma \vee \varrho^2 \vee \varepsilon}$

linear coupling $\rightarrow$ ε

nonlinear coupling $\rightarrow$ σ

noise acting on slow variable $\rightarrow$ ϱ

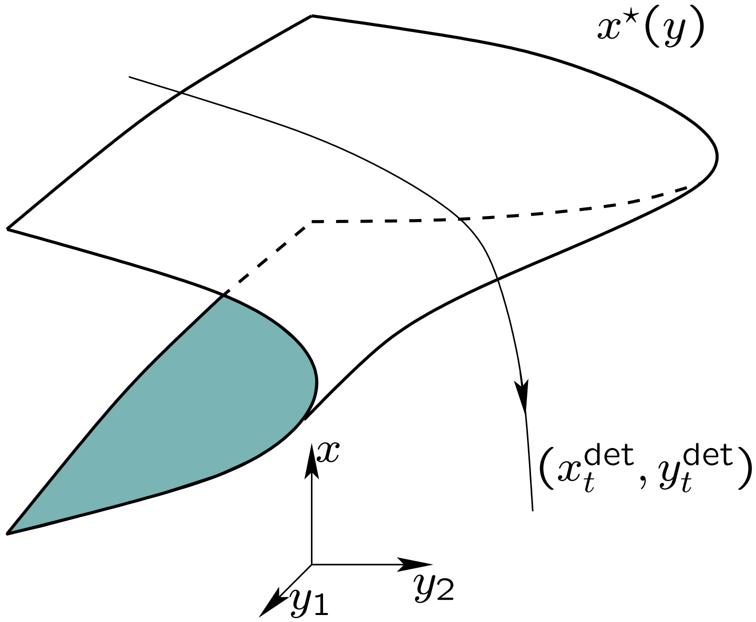
▷ On longer time scales: Markov property allows for restarting y_t stays exp. long in a neighbourhood of the periodic orbit (with probability close to 1)

Bifurcations

Question

What happens if (x_t, y_t) approaches a bifurcation point $(\hat{x}, \hat{y})$ for the deterministic dynamics?

Ex.: Saddle–node bifurcation



General approach

- ▷ Apply centre-manifold theorem
- ▷ Concentration results for deviation from centre manifold ([Berglund & G, 2003])
- ▷ Consider reduced dynamics on centre manifold
- ▷ Concentration results for deviation of reduced system from original variables [Berglund & G, 2003]

References for PART VIII

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