Metastability for the Ginzburg–Landau equation with space–time white noise

Barbara Gentz

University of Bielefeld, Germany

Joint work with

Florent Barret (Ecole Polytechnique, Palaiseau, France)
Nils Berglund (Université d'Orléans, France)

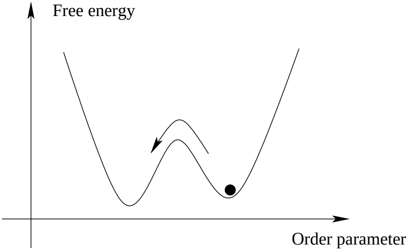
SAMSI, 10 February 2010

Seminar BINGO!
To play, simply print out this bingo sheet and attend a departmental seminar.
Mark over each square that occurs throughout the course of the lecture.
The first one to form a straight line (or all four corners) must yell out BINGO!! to win!
SEMINAR
B I N G O
Speaker bashes previous work
Repeated use of "um..."
Speaker sucks up to host professor
Host Professor falls asleep
Speaker wastes 5 minutes explaining outline
Laptop malfunction
Work ties in to Cancer/HIV or War on Terror
"...et al."
You're the only one in your lab that bothered to show up
Blatant typo
Entire slide filled with equations
"The data clearly shows..."
FREE Speaker runs out of time
Use of Powerpoint template with blue background
References Advisor (past or present)
There's a Grad Student wearing same clothes as yesterday
Bitter Post-doc asks question
"That's an interesting question"
"Beyond the scope of this work"
Master's student bobs head fighting sleep
Speaker forgets to thank collaborators
Cell phone goes off
You've no idea what's going on
"Future work will..."
Results conveniently show improvement
JORGE CHAM © 2007
WWW.PHDCOMICS.COM

Metastability in the real world

Examples

- ▷ Supercooled liquid
- ▷ Supersaturated gas
- ▷ Wrongly magnetized ferromagnet



- ▷ Near first-order phase transitions
- ▷ Nucleation implies crossing of energy barrier

Metastability in stochastic lattice models

Ingredients

- Lattice: $\Lambda \subset \mathbb{Z}^d$
- Configuration space: $\mathcal{X} = S^\Lambda$, S finite set (e.g. $\{-1, 1\}$)
- Hamiltonian: $H : \mathcal{X} \to \mathbb{R}$ (e.g. Ising model or lattice gas)
- Gibbs measure: $\mu_\beta(x) = \mathrm{e}^{-\beta H(x)} / Z_\beta$
- Dynamics: Markov chain with invariant measure μ_β
  (e.g. Metropolis such as Glauber or Kawasaki dynamics)

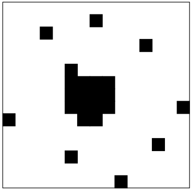
Metastability in stochastic lattice models

Ingredients

- Lattice: $\Lambda \subset \mathbb{Z}^d$
- Configuration space: $\mathcal{X} = S^\Lambda$, S finite set (e.g. $\{-1, 1\}$)
- Hamiltonian: $H : \mathcal{X} \to \mathbb{R}$ (e.g. Ising model or lattice gas)
- Gibbs measure: $\mu_\beta(x) = \mathrm{e}^{-\beta H(x)} / Z_\beta$
- Dynamics: Markov chain with invariant measure μ_β
  (e.g. Metropolis such as Glauber or Kawasaki dynamics)

Results (for $\beta \gg 1$)

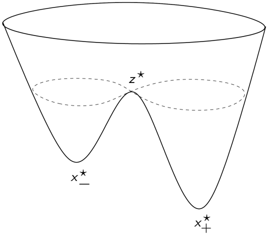
- Transition time from *empty* to *full* configuration
- Typical transition paths
- Shape of critical droplet



References

- Frank den Hollander, *Metastability under stochastic dynamics*, Stochastic Process. Appl. **114** (2004), 1–26
- Enzo Olivieri & Maria Eulália Vares, *Large deviations and metastability*, Cambridge University Press, Cambridge, 2005

Reversible diffusions



Gradient dynamics (ODE)

$$\dot{x}_t^{\text{det}} = -\nabla V(x_t^{\text{det}})$$

Random perturbation by Gaussian white noise (SDE)

$$\mathrm{d}x_t^\varepsilon(\omega) = -\nabla V(x_t^\varepsilon(\omega))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}B_t(\omega)$$

with

- $V : \mathbb{R}^d \to \mathbb{R}$: confining potential, growth condition at infinity
- $\{B_t(\omega)\}_{t\geq 0}$: d-dimensional Brownian motion

Invariant measure or equilibrium distribution (for gradient systems)

$$\mu_\varepsilon(dx) = \frac{1}{Z_\varepsilon}\mathrm{e}^{-V(x)/\varepsilon}\,dx \qquad \text{with} \qquad Z_\varepsilon = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x$$

Dynamics reversible w.r.t. invariant measure μ_ε (detailed balance)

Transition times between potential wells

First-hitting time of a small ball $B_\delta(x_+^\star)$ around minimum $x_+^\star$

$$\tau_+ = \tau^\varepsilon_{x_+^\star}(\omega) = \inf\{t \geq 0 \colon x_t^\varepsilon(\omega) \in B_\delta(x_+^\star)\}$$

Eyring–Kramers Law [Eyring 35, Kramers 40]

- $d = 1$: $\quad \mathbb{E}_{x_-^\star} \tau_+ \simeq \dfrac{2\pi}{\sqrt{V''(x_-^\star)|V''(z^\star)|}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$
- $d \geq 2$: $\quad \mathbb{E}_{x_-^\star} \tau_+ \simeq \dfrac{2\pi}{|\lambda_1(z^\star)|} \sqrt{\dfrac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$

where $\lambda_1(z^\star)$ is the unique negative eigenvalue of $\nabla^2 V$ at saddle $z^\star$

Proving Kramers Law

- Exponential asymptotics and optimal transition paths via large deviations approach [Wentzell & Freidlin 69–72]

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{x_-^\star} \tau_+ = V(z^\star) - V(x_-^\star)$$

Only 1-saddles are relevant for transitions between wells

- Low-lying spectrum of generator of the diffusion (analytic approach) [Helffer & Sjöstrand 85, Miclo 95, Mathieu 95, Kolokoltsov 96, ...]
- Potential theoretic approach [Bovier, Eckhoff, Gayrard & Klein 04]

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{2\pi}{|\lambda_1(z^\star)|} \sqrt{\frac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon} \big[1 + \mathcal{O}\big((\varepsilon|\log\varepsilon|)^{1/2}\big)\big]$$

- Full asymptotic expansion of prefactor [Helffer, Klein & Nier 04]
- Asymptotic distribution of τ_+ [Day 83, Bovier, Gayrard & Klein 05]

$$\lim_{\varepsilon\to 0} \mathbb{P}_{x_-^\star} \{\tau_+ > t \cdot \mathbb{E}_{x_-^\star} \tau_+\} = \mathrm{e}^{-t}$$

Ginzburg–Landau equation

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 + \text{noise}$$

- On finite interval $x \in [0, L]$
- $u(x,t) \in \mathbb{R}$ (one-dimensional, representing e.g. magnetization)
- Boundary conditions
  - Periodic b.c. $u(0,t) = u(L,t)$ and $\partial_x u(0,t) = \partial_x u(L,t)$
  - Neumann b.c. with zero flux $\partial_x u(0,t) = \partial_x u(L,t) = 0$
- Weak space–time white noise

Deterministic dynamics minimizes energy functional

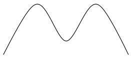
$$V(u) = \int_0^L [\tfrac{1}{2} u'(x)^2 - \tfrac{1}{2} u(x)^2 + \tfrac{1}{4} u(x)^4] \, dx$$

as

$$\partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 = -\frac{\delta V}{\delta u}$$

Stationary states for the deterministic system

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = -u(x) + u(x)^3 = -\frac{\mathrm{d}}{\mathrm{d}u}\Big[\quad\Big]$$



- Uniform stationary states
  - $u_\pm(x) \equiv \pm 1$ (stable; global minima of V)
  - $u_0(x) \equiv 0$ (unstable – when is u_0 a transition state?)
- Periodic b.c.: For $k = 1, 2, \dots$ and $L > 2\pi k$
  - Continuous one-parameter family of stationary states

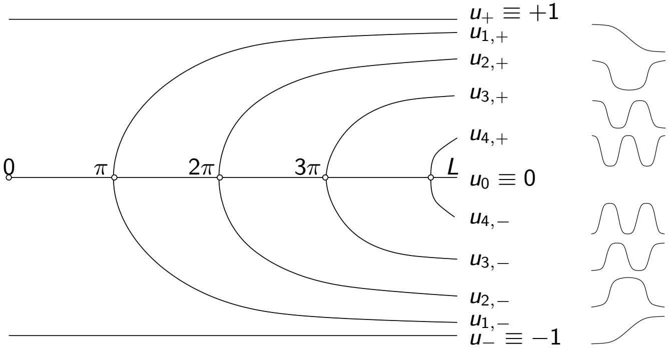
    $$u_{k,\varphi}(x) = \sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \varphi, m\Big) \quad \text{where} \quad 4k\sqrt{m+1}\,\mathrm{K}(m) = L$$
- Neumann b.c.: For $k = 1, 2, \dots$ and $L > \pi k$
  - Two stationary states

    $$u_{k,\pm}(x) = \pm\sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \mathrm{K}(m), m\Big) \quad \text{where} \quad 2k\sqrt{m+1}\,\mathrm{K}(m) = L$$

Stationary states: Neumann b.c.

For $k = 1, 2, \ldots$ and $L > \pi k$:

$$u_{k,\pm}(x) = \pm\sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \mathrm{K}(m), m\Big) \quad \text{where} \quad 2k\sqrt{m+1}\mathrm{K}(m) = L$$



Stability of the stationary states: Neumann b.c.

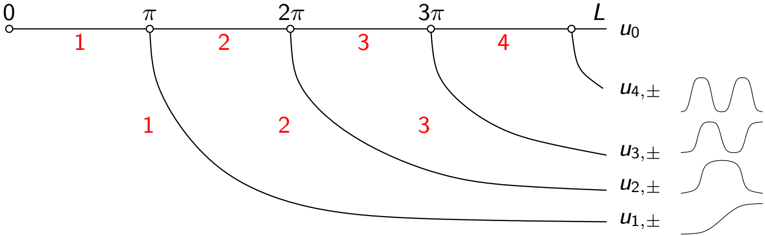
Consider linearization of GL equation at stationary solution $u : [0, L] \to \mathbb{R}$

$$\partial_t v = A[u]v \quad \text{where} \quad A[u] = \frac{d^2}{dx^2} + 1 - 3u^2$$

Stability is determined by the eigenvalues of $A[u]$

- $u_\pm(x) \equiv \pm 1$: $A[u_\pm]$ has eigenvalues $-(2 + (\pi k/L)^2)$, $k = 0, 1, 2, \dots$
- $u_0(x) \equiv 0$: $A[u_0]$ has eigenvalues $1 - (\pi k/L)^2$, $k = 0, 1, 2, \dots$

Counting the number of positive eigenvalues: None for $u_\pm$ and ...



Stability of the stationary states: Neumann b.c.

- For $L < \pi$:
  - $u_\pm(x) \equiv \pm 1$ are stable; global minima
  - $u_0(x) \equiv 0$ is unstable; transition state
  - Activation energy $V(u_0) - V(u_\pm) = L/4$

- For $L > \pi$:
  - $u_\pm(x) \equiv \pm 1$ remain stable; global minima
  - $u_0(x) \equiv 0$ remains unstable; but no longer forms the transition state
  - $u_{1,\pm}(x)$ are the new transition states (of instanton shape)

- Pitchfork bifurcation as L increases through π:
  Uniform transition state u_0 bifurcates into pair of instanton states $u_{1,\pm}$

- Subsequent bifurcations at $L = k\pi$ for $k = 2, 3, \dots$ do not affect transition states

Ginzburg–Landau equation with noise

$$\begin{cases} \partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 + \sqrt{2\varepsilon}\xi(t,x) \\ u(\cdot,0) = \varphi(\cdot) \\ \partial_x u(0,t) = \partial_x u(L,t) = 0 \qquad \text{(Neumann b.c.)} \end{cases}$$

- Space–time white noise $\xi(t,x)$ as formal derivative of Brownian sheet
- Mild / evolution formulation, following [Walsh '86]:

$$u(x,t) = \int_0^L G_t(x,z)\varphi(z)\,\mathrm{d}z + \int_0^t \int_0^L G_{t-s}(x,z)\big[u(s,z) - u(s,z)^3\big]\,\mathrm{d}z\,\mathrm{d}s$$
$$+ \sqrt{2\varepsilon} \int_0^t \int_0^L G_{t-s}(x,z) W(\mathrm{d}s,\mathrm{d}z)$$

where

- G is the fundamental solution of the deterministic equation
- W is the Brownian sheet

Existence and a.s. uniqueness [Faris & Jona-Lasinio 82]

Question

How long does a noise-induced transition from the global minimum $u_-(x) \equiv -1$ to (a neighbourhood of) $u_+(x) \equiv 1$ take?

$$\tau_{u_+} = \text{first hitting time of such a neighbourhood}$$

Metastability: We expect $\mathbb{E}_{u_-} \tau_{u_+} \sim e^{\text{const}/\varepsilon}$

We seek

- Activation energy ΔW
- Transition rate prefactor Γ_0^{-1}
- Exponent α of error term

such that

$$\mathbb{E}_{u_-} \tau_{u_+} = \Gamma_0^{-1} \, e^{\Delta W/\varepsilon} [1 + \mathcal{O}(\varepsilon^{\alpha})]$$

Large deviations for the Ginzburg–Landau equation

Large deviation principle [Faris & Jona–Lasinio '82]:

- For $L \leq \pi$:

$$\Delta W = V(u_0) - V(u_-) = L/4$$

- For $L > \pi$:

$$\Delta W = V(u_{1,\pm}) - V(u_-) = \frac{1}{3\sqrt{1+m}}\left[8\mathrm{E}(m) - \frac{(1-m)(3m+5)}{1+m}\mathrm{K}(m)\right]$$

Formal computation of the prefactor for the GL equation

Consider $L < \pi$

- Transition state: $u_0(x) \equiv 0$, $V[u_0] = 0$
- Activation energy: $\Delta W = V[u_0] - V[u_-] = L/4$
- Eigenvalues at stable state $u_-(x) \equiv -1$: $\mu_k = 2 + (\pi k/L)^2$
- Eigenvalues at transition state $u_0 \equiv 0$: $\lambda_k = -1 + (\pi k/L)^2$

Thus formally [Maier & Stein '01, '03]

$$\Gamma_0 \simeq \frac{|\lambda_0|}{2\pi} \sqrt{\prod_{k=0}^{\infty} \frac{\mu_k}{|\lambda_k|}} = \frac{1}{2^{3/4}\pi} \sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}$$

For $L > \pi$: Spectral determinant computed by Gelfand's method

Problems

- What happens when $L \nearrow \pi$? (Approaching bifurcation)
- Is the formal computation correct in infinite dimension?

Ginzburg–Landau equation: Introducing Fourier variables

- Fourier series

$$u(x,t) = \frac{1}{\sqrt{L}} y_0(t) + \frac{2}{\sqrt{L}} \sum_{k=1}^{\infty} y_k(t) \cos(\pi k x/L) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} \tilde{y}_k(t)\, e^{i\, k\pi x/L}$$

- Rewrite energy functional V in Fourier variables

$$V(y) = \frac{1}{2} \sum_{k=0}^{\infty} \lambda_k y_k^2 + V_4(y) \,, \qquad \lambda_k = -1 + (\pi k/L)^2$$

where

$$V_4(y) = \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} \tilde{y}_{k_4}$$

- Resulting system of SDEs

$$\dot{y}_k = -\lambda_k y_k - \frac{1}{L} \sum_{k_1+k_2+k_3=k} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} + \sqrt{2\varepsilon}\, \dot{W}_t^{(k)}$$

with i.i.d. Brownian motions $W_t^{(k)}$

Truncating the Fourier series

- Truncate Fourier series (projected equation)

$$u_d(x,t) = \frac{1}{\sqrt{L}} y_0(t) + \frac{2}{\sqrt{L}} \sum_{k=1}^{d} y_k(t) \cos(\pi k x / L)$$

- Retain only modes $k \leq d$ in the energy functional V

$$V^{(d)}(y) = \frac{1}{2} \sum_{k=0}^{d} \lambda_k y_k^2 + V_4^{(d)}(y)$$

where

$$V_4^{(d)}(y) = \frac{1}{4L} \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_i \in \{-d,\dots,0,\dots,+d\}}} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} \tilde{y}_{k_4}$$

- Resulting d-dimensional system of SDEs

$$\dot{y}_k = -\lambda_k y_k - \frac{1}{L} \sum_{\substack{k_1+k_2+k_3=k \\ k_i \in \{-d,\dots,0,\dots,+d\}}} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} + \sqrt{2\varepsilon} \dot{W}_t^{(k)}$$

Reduction to finite-dimensional system

- Show the following result for the projected finite-dimensional systems

$$\varepsilon^\gamma C(d)\, e^{\Delta W^{(d)}/\varepsilon}[1 - R_d^-(\varepsilon)] \leq \mathbb{E}_{u_-^{(d)}} \tau_{u_+^{(d)}} \leq \varepsilon^\gamma C(d)\, e^{\Delta W^{(d)}/\varepsilon}[1 + R_d^+(\varepsilon)]$$

(The contribution ε^γ is only present at bifurcation points / non-quadratic saddles)

- The following limits exist and are finite

$$\lim_{d\to\infty} C(d) =: C(\infty) \qquad \text{and} \qquad \lim_{d\to\infty} \Delta W^{(d)} =: \Delta W^{(\infty)}$$

- Important: Uniform control of error terms (uniform in d):

$$R^\pm(\varepsilon) := \sup_d R_d^\pm(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Away from bifurcation points, c.f. [Barret, Bovier & Méleard 09]

Taking the limit $d \to \infty$

- For any ε, distance between $u(x,t)$ and solution $u^{(d)}(x,t)$ of the projected equation becomes small [Liu '03] on any finite time interval $[0,T]$
- Uniform error bounds and large deviation results allow to decouple limits of small ε and large d
- Yielding

$$\varepsilon^{\gamma} C(\infty)\, \mathrm{e}^{\Delta W^{(\infty)}/\varepsilon}[1 - R^{-}(\varepsilon)] \leq \mathbb{E}_{u_-} \tau_{u_+} \leq \varepsilon^{\gamma} C(\infty)\, \mathrm{e}^{\Delta W^{(\infty)}/\varepsilon}[1 + R^{+}(\varepsilon)]$$

Result for the Ginzburg–Landau equation

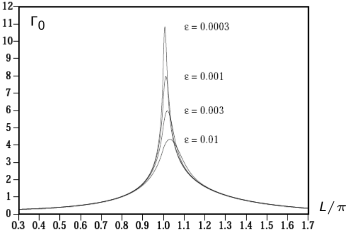
Theorem [Barret, Berglund & G., in preparation]

For the Ginzburg–Landau equation with Neumann b.c., $L < \pi$

(Similar expression for $L > \pi$)

$$\mathbb{E}_{u_-} \tau_{u_+} = \frac{1}{\Gamma_0(L)} \, \mathrm{e}^{L/4\varepsilon} [1 + \mathcal{O}((\varepsilon |\log \varepsilon|)^{1/4})]$$

where the rate prefactor satisfies
(recall: $\lambda_1 = -1 + (\pi/L)^2$)



$$\Gamma_0(L) = \frac{1}{2^{3/4}\pi} \sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}} \sqrt{\frac{\lambda_1}{\lambda_1 + \sqrt{3\varepsilon/4L}}} \Psi_+ \Bigl(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}} \Bigr)$$

$$\longrightarrow \frac{\Gamma(1/4)}{2(3\pi^7)^{1/4}} \sqrt{\sinh(\sqrt{2}\pi)} \, \varepsilon^{-1/4} \qquad \text{as } L \nearrow \pi$$

Towards a proof in the finite-dimensional case: Potential theory for Brownian motion I

First-hitting time $\tau_A = \inf\{t > 0 \colon B_t \in A\}$ of $A \subset \mathbb{R}^d$

Fact I: The expected first-hitting time $w_A(x) = \mathbb{E}_x \tau_A$ is a solution to the Dirichlet problem

$$\begin{cases} \Delta w_A(x) = 1 & \text{for } x \in A^c \\ w_A(x) = 0 & \text{for } x \in A \end{cases}$$

and can be expressed with the help of the Green function $G_{A^c}(x,y)$ as

$$w_A(x) = \int_{A^c} G_{A^c}(x,y)\,\mathrm{d}y$$

Potential theory for Brownian motion II

The equilibrium potential (or capacitor) $h_{A,B}$ is a solution to the Dirichlet problem

$$\begin{cases} \Delta h_{A,B}(x) = 0 & \text{for } x \in (A \cup B)^c \\ h_{A,B}(x) = 1 & \text{for } x \in A \\ h_{A,B}(x) = 0 & \text{for } x \in B \end{cases}$$

Fact II: $h_{A,B}(x) = \mathbb{P}_x[\tau_A < \tau_B]$

The equilibrium measure (or surface charge density) is the unique measure $\rho_{A,B}$ on ∂A s.t.

$$h_{A,B}(x) = \int_{\partial A} G_{B^c}(x,y)\, \rho_{A,B}(\mathrm{d}y)$$

Capacities

Key observation: For a small ball $C = B_\delta(x)$,

$$\int_{A^c} h_{C,A}(y)\,\mathrm{d}y = \int_{A^c}\int_{\partial C} G_{A^c}(y,z)\,\rho_{C,A}(\mathrm{d}z)\,\mathrm{d}y$$
$$= \int_{\partial C} w_A(z)\,\rho_{C,A}(\mathrm{d}z) \simeq w_A(x)\,\mathrm{cap}_C(A)$$

where $\mathrm{cap}_C(A) = \int_{\partial C} \rho_{C,A}(\mathrm{d}y)$ denotes the capacity

$$\Rightarrow \quad \mathbb{E}_x \tau_A = w_A(x) \simeq \frac{1}{\mathrm{cap}_{B_\delta(x)}(A)} \int_{A^c} h_{B_\delta(x),A}(y)\,\mathrm{d}y$$

Variational representation via Dirichlet form

$$\mathrm{cap}_C(A) = \int_{(C\cup A)^c} \|\nabla h_{C,A}(x)\|^2\,\mathrm{d}x = \inf_{h\in\mathcal{H}_{C,A}} \int_{(C\cup A)^c} \|\nabla h(x)\|^2\,\mathrm{d}x$$

where $\mathcal{H}_{C,A}$ = set of sufficiently smooth functions h satisfying b.c.

General case

$$dx_t^\varepsilon = -\nabla V(x_t^\varepsilon)\,dt + \sqrt{2\varepsilon}\,dB_t$$

What changes as the generator Δ is replaced by $\varepsilon\Delta - \nabla V \cdot \nabla$?

$$\operatorname{cap}_C(A) = \varepsilon \inf_{h\in\mathcal{H}_{C,A}} \int_{(C\cup A)^c} \|\nabla h(x)\|^2 \,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x$$

$$\mathbb{E}_x\tau_A = w_A(x) \simeq \frac{1}{\operatorname{cap}_{B_\delta(x)}(A)} \int_{A^c} h_{B_\delta(x),A}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y$$

It remains to investigate capacity and integral.

Assume, $x = x_-^\star$ is a quadratic minimum. Use rough *a priori* bounds on h

$$\int_{A^c} h_{B_\delta(x_-^\star),A}(y)\,\mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y \simeq \frac{(2\pi\varepsilon)^{d/2}\,\mathrm{e}^{-V(x_-^\star)/\varepsilon}}{\sqrt{\det \nabla^2 V(x_-^\star)}}$$

Estimating the capacity

For the truncated energy functional

$$V^{(d)}(y) = \frac{1}{2}\sum_{k=0}^{d}\lambda_k y_k^2 + V_4^{(d)}(y) = -\frac{1}{2}y_0^2 + u_1(y_1) + \frac{1}{2}\sum_{k=2}^{d}\lambda_k y_k^2 + \dots$$

where

$$u_1(y_1) = \frac{1}{2}\lambda_1 y_1^2 + \frac{3}{8}y_1^4$$

To show

$$\mathrm{cap}_C(A) = \varepsilon \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-u_1(y_1)/\varepsilon}\,\mathrm{d}y_2}{\sqrt{2\pi\varepsilon}} \prod_{j=2}^{d}\sqrt{\frac{2\pi\varepsilon}{\lambda_j}}\,\left[1 + \mathcal{O}(R(\varepsilon))\right]$$

where $R(\varepsilon) = \mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})$ is uniformly bounded in d

Sketch of the proof

Proof follows along the lines of [Bovier, Eckhoff, Gayrard & Klein 04]

- Upper bound: Use Dirichlet form representation of capacity

$$\text{cap} = \inf_h \Phi(h) \leqslant \Phi(h_+) = \Phi(h_+) = \varepsilon \int \|\nabla h_+(y)\|^2 \, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y$$

  Choose $\delta = \sqrt{c\varepsilon|\log \varepsilon|}$ and

$$h_+(z) = \begin{cases} 1 & \text{for } y_0 < -\delta \\ f(y_0) & \text{for } -\delta < y_0 < \delta \\ 0 & \text{for } y_0 > \delta \end{cases}$$

  where $\varepsilon f''(y_0) + \partial_{y_0} V(y_0, 0) f' y_0) = 0$ with b.c. $f(\pm\delta) = 0$ or 1, resp.

- Lower bound: Bound Dirichlet form for capacity from below by
  - restricting domain
  - taking only 1*st* component of ∇h
  - using b.c. derived from *a priori* bound on $h_{C,A}$