WIAS

Weierstraß-Institut für Angewandte Analysis und Stochastik

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Metastability in irreversible diffusion processes and stochastic resonance

Joint work with Nils Berglund (CPT–CNRS Marseille)

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What is stochastic resonance (SR)?

SR $=$ mechanism to amplify weak signals in presence of noise

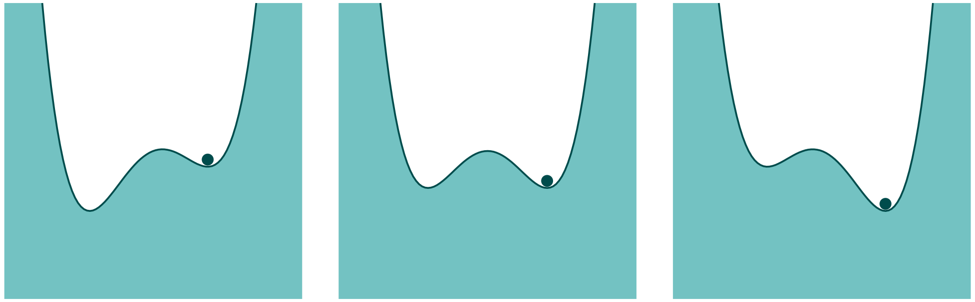
Requirements

- (background) noise
- weak input
- characteristic barrier or threshold (nonlinear system)

Examples

- periodic occurrence of ice ages (?)
- Dansgaard–Oeschger events
- bidirectional ring lasers
- visual and auditory perception
- receptor cells in crayfish
- ...

The paradigm



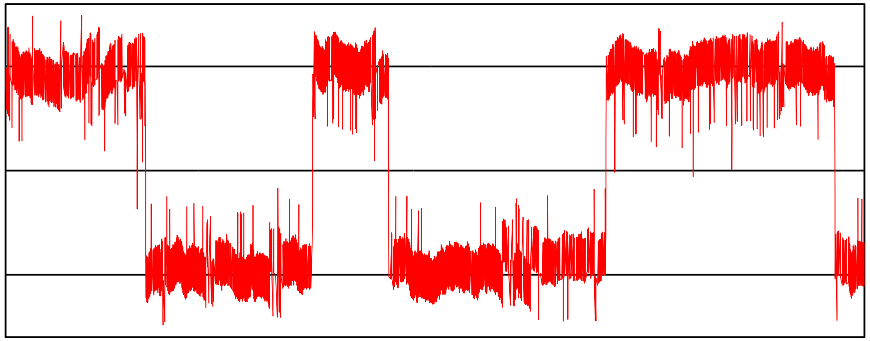
Overdamped motion of a Brownian particle . . .

$$\mathrm{d}x_t = \underbrace{\left[-x_t^3 + x_t + A\cos(\varepsilon t)\right]}_{= -\frac{\partial}{\partial x} V(x_t, \varepsilon t)} \mathrm{d}t + \sigma \, \mathrm{d}W_t$$

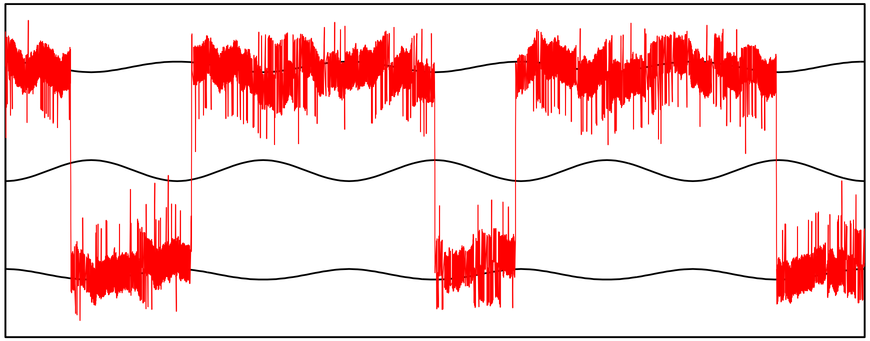
. . . in a periodically modulated double-well potential

$$V(x,s) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A\cos(s)x \,, \qquad A < A_\text{c}$$

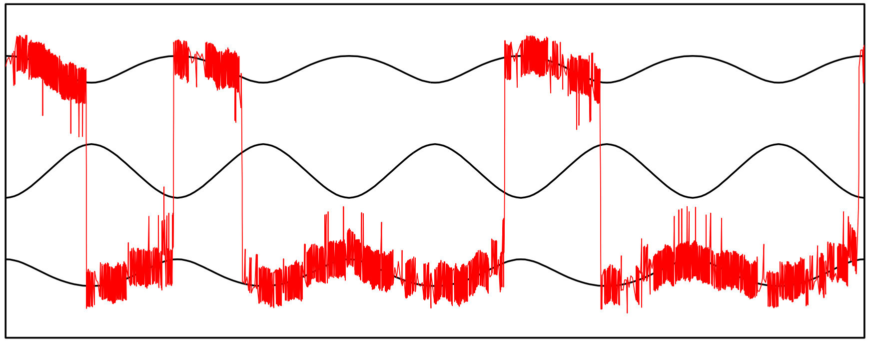
Sample paths



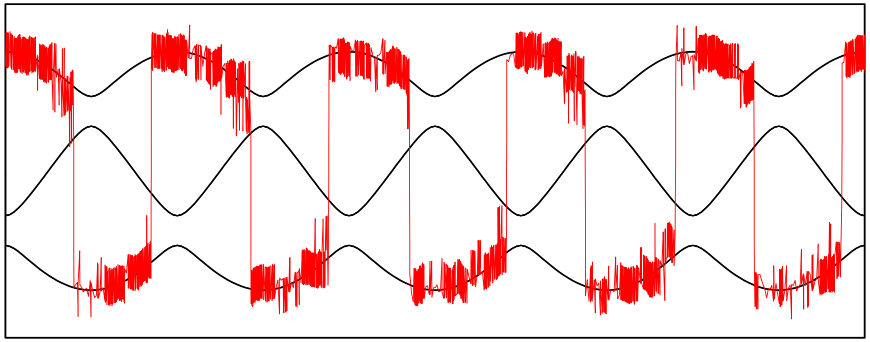
$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$



$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$



$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$



$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

Different parameter regimes

Synchronisation I

- For matching time scales: $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp e^{2H/\sigma^2}$
- Quasistatic approach: Transitions twice per period with high probability

  (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- Requires exponentially long forcing periods

Synchronisation II

- For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$

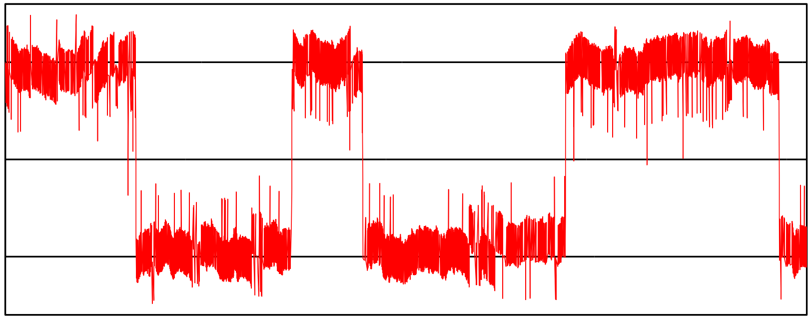
  and close-to-critical forcing amplitude: $A \approx A_{\text{c}}$
- Transitions twice per period with high probability
- Subtle dynamical effects: Effective barrier heights [Berglund & G '02]

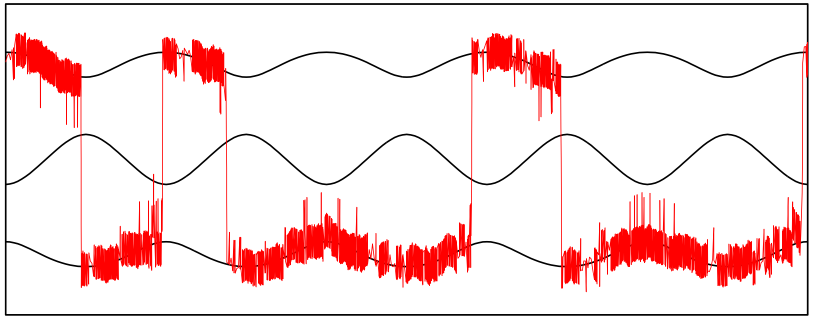
SR outside synchronisation regimes

- Only occasional transitions
- But transition times localised within forcing periods

Unified description / understanding of transition between regimes ?

Qualitative measures for SR





How to measure combined effect of periodic and random perturbations?

Spectral-theoretic approach

- ▷ Power spectrum
- ▷ Spectral power amplification
- ▷ Signal-to-noise ratio

Probabilistic approach

- ▷ Distribution of interspike times
- ▷ Distribution of first-passage times
- ▷ Distribution of residence times

Look for periodic component in density of these distributions

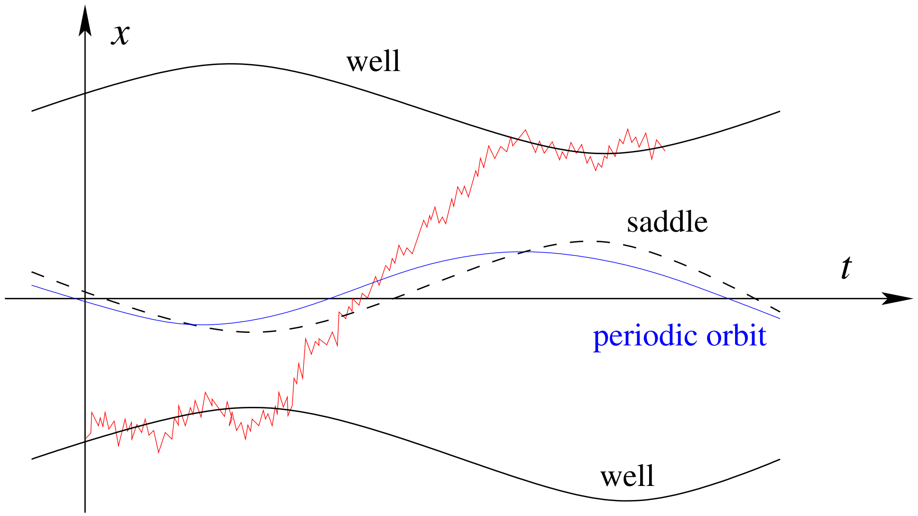
Interwell transitions

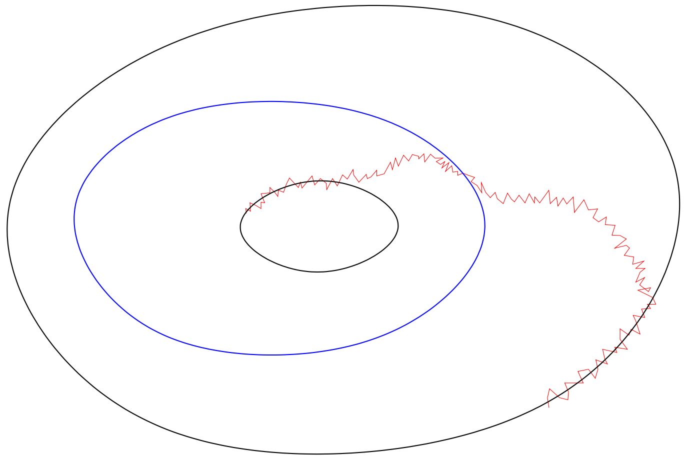
Deterministic motion in a periodically modulated double-well potential

- ▷ 2 stable periodic orbits tracking bottoms of wells
- ▷ 1 unstable periodic orbit tracking saddle
- ▷ Unstable periodic orbit separates basins of attraction

Brownian particle in a periodically modulated double-well potential

- ▷ Interwell transitions characterised by crossing of unstable orbit





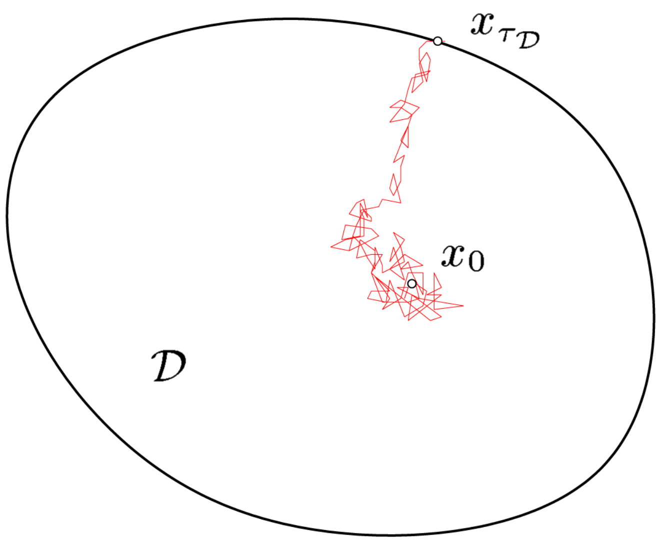
Exit problem

Deterministic ODE	$\dot{x}_t^{\text{det}} = f(x_t^{\text{det}})$	$x_0 \in \mathbb{R}^d$
Small random perturbation	$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$	(same initial cond. x_0)

Bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- *first-exit time* $\tau = \tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
- *first-exit location* $x_\tau \in \partial\mathcal{D}$

Distribution of τ and x_τ ?



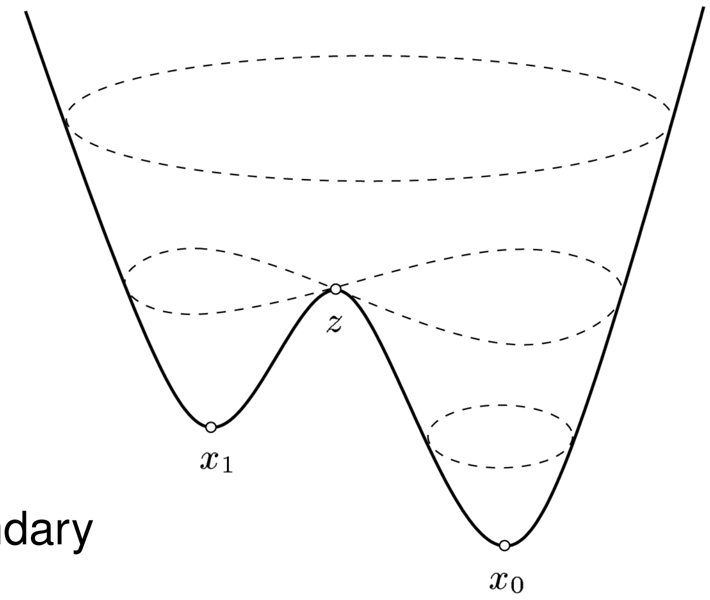
Interesting case

$\mathcal{D}$ positively invariant under deterministic flow

Approaches

- Mean first-exit times and locations via PDEs
- Exponential asymptotics via Wentzell–Freidlin theory

Gradient case (for simplicity: V double-well potential)



Exit from neighbourhood of shallow well

▷ Mean first-hitting time τ^{hit} of deeper well

$$\mathbb{E}_{x_1} \tau^{\text{hit}} = c(\sigma)\, \mathrm{e}^{\overline{V}/\sigma^2}$$

Minimum $\overline{V} = 2[V(z) - V(x_1)]$ of (quasi-)potential on boundary

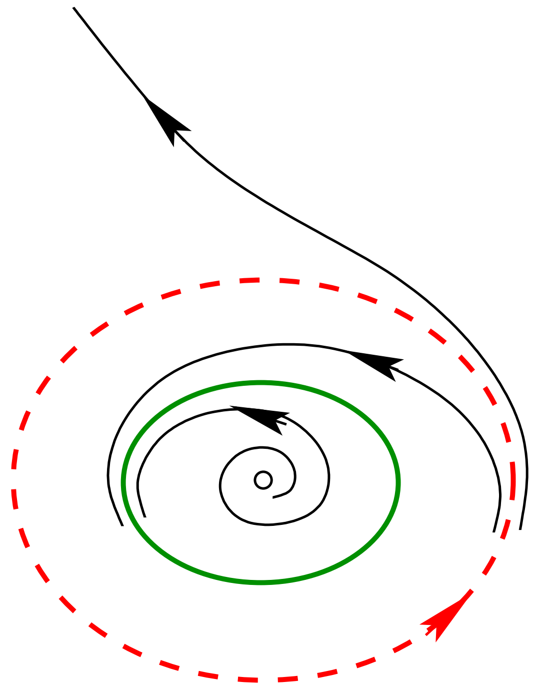
▷ $\lim_{\sigma\to 0} c(\sigma) = \frac{2\pi}{\lambda_1(z)} \sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x_1)}}$ exists !

$\lambda_1(z)$ unique negative e.v. of $\nabla^2 V(z)$

(Physics' literature: [Eyring '35], [Kramers '40];
rigorous results: [Bovier, Gayrard, Eckhoff, Klein '04/'05], [Helffer, Klein, Nier '04])

▷ Subexponential asymptotics known
Related to geometry at well and saddle / small eigenvalues of the generator

New phenomena for drift not deriving from a potential?



Simplest situation of interest

Nontrivial invariant set which is a single periodic orbit

Assume from now on

$d = 2$, $\partial\mathcal{D} =$ *unstable* periodic orbit

- $\mathbb{E}\tau \sim e^{\overline{V}/\sigma^2}$ still holds
- Quasipotential $V(\Pi, z) \equiv \overline{V}$ is constant on $\partial\mathcal{D}$:
  Exit equally likely anywhere on $\partial\mathcal{D}$ (on exp. scale)
- Phenomenon of cycling [Day '92]:
  Distribution of x_τ on $\partial\mathcal{D}$ generally does *not* converge as $\sigma \to 0$.
  Density is *translated* along $\partial\mathcal{D}$ proportionally to $|\log\sigma|$.
- In *stationary regime*: (obtained by reinjecting particle)
  Rate of escape $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{x_t \in \mathcal{D}\}$ has $|\log\sigma|$-periodic prefactor [Maier & Stein '96]

Density of the first-passage time at an unstable periodic orbit

Taking number of revolutions into account

Idea

Density of first-passage time at unstable orbit

$$p(t) = c(t,\sigma)\, \mathrm{e}^{-\overline{V}/\sigma^2} \times \text{ transient term} \times \text{ geometric decay per period}$$

Identify $c(t,\sigma)$ as periodic component in first-passage density

Notations

- ▷ Value of quasipotential on unstable orbit: $\overline{V}$
  (measures cost of going from stable to unstable periodic orbit; based on large-deviations rate function)
- ▷ Period of unstable orbit: $T = 2\pi/\varepsilon$
- ▷ Curvature at unstable orbit: $a(t) = -\dfrac{\partial^2}{\partial x^2} V(x^{\text{unst}}(t), t)$
- ▷ Lyapunov exponent of unstable orbit: $\lambda = \dfrac{1}{T}\displaystyle\int_0^T a(t)\, \mathrm{d}t$

Universality in first-passage-time distributions

Theorem ([Berglund & G '04], [Berglund & G '05], work in progress)

There exists a *model-dependent* time change such that *after performing this time change*, for any $\Delta \geqslant \sqrt{\sigma}$ and all $t \geqslant t_0$,

$$\mathbb{P}\{\tau \in [t, t+\Delta]\} = \int_t^{t+\Delta} p(s, t_0)\,\mathrm{d}s\,\bigl[1 + \mathcal{O}(\sqrt{\sigma})\bigr]$$

where

- $p(t, t_0) = \dfrac{1}{\mathcal{N}}\, Q_{\lambda T}\bigl(t - |\log\sigma|\bigr)\,\dfrac{1}{\lambda T_{\mathsf{K}}(\sigma)}\,\mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\mathsf{K}}(\sigma)}\, f_{\text{trans}}(t, t_0)$
- $Q_{\lambda T}(y)$ is a *universal* λT-periodic function
- $T_{\mathsf{K}}(\sigma)$ is the analogue of Kramers' time: $T_{\mathsf{K}}(\sigma) = \dfrac{C}{\sigma}\, e^{\overline{V}/\sigma^2}$
- f_{trans} grows from 0 to 1 in time $t - t_0$ of order $|\log\sigma|$

The different regimes

$$p(t,t_0) = \frac{1}{\mathcal{N}}\, Q_{\lambda T}\big(t - |\log\sigma|\big)\, \frac{1}{\lambda T_{\mathsf{K}}(\sigma)}\, \mathrm{e}^{-(t-t_0)\,/\,\lambda T_{\mathsf{K}}(\sigma)}\, f_{\text{trans}}(t,t_0)$$

Transient regime

f_{trans} is increasing from 0 to 1; exponentially close to 1 after time $t - t_0 > 2|\log\sigma|$

Metastable regime

$$Q_{\lambda T}(y) = 2\lambda T \sum_{k=-\infty}^{\infty} P(y - k\lambda T) \qquad \text{with peaks} \qquad P(z) = \frac{1}{2}\,\mathrm{e}^{-2z} \exp\Big\{-\frac{1}{2}\,\mathrm{e}^{-2z}\Big\}$$

kth summand: Path spends

- ▷ k periods near stable periodic orbit
- ▷ the remaining $[(t-t_0)/T] - k$ periods near unstable periodic orbit

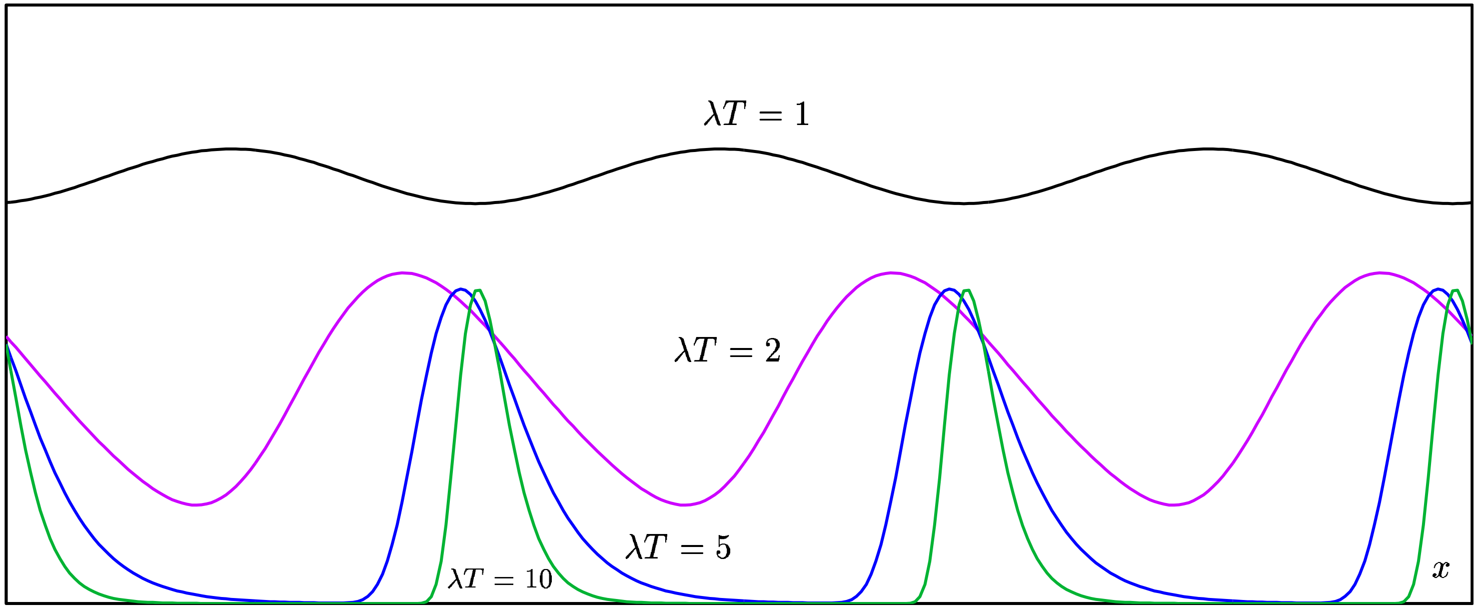
Periodic dependence on $|\log\sigma|$: Peaks rotate as σ decreases

Asymptotic regime

Significant decay only for $t - t_0 \gg T_{\mathsf{K}}(\sigma)$

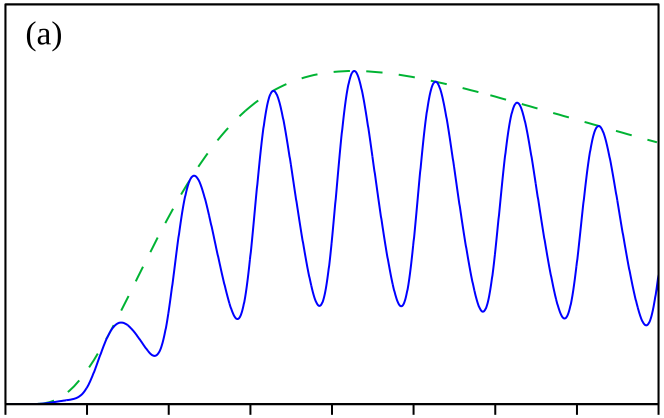
The universal profile

$y \mapsto Q_{\lambda T}(\lambda T y)/2\lambda T$

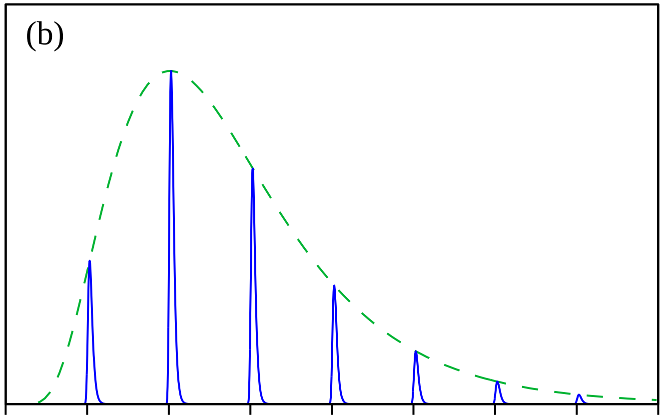


- Profile determines concentration of first-passage times within a period
- Shape of peaks: Gumbel distribution
- The larger λT, the more pronounced the peaks
- For smaller values of λT, the peaks overlap more

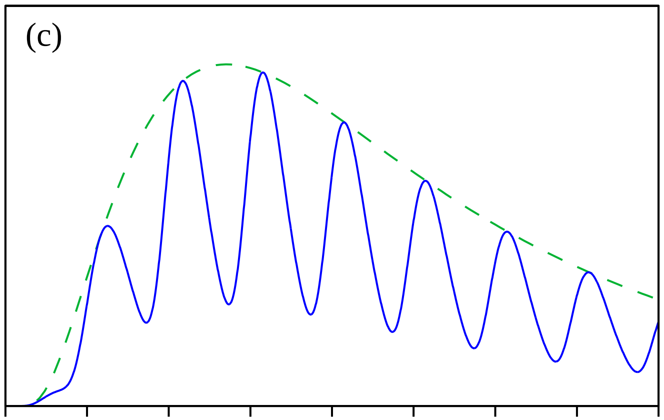
Density of the first-passage time $\overline{V} = 0.5,\ \lambda = 1$



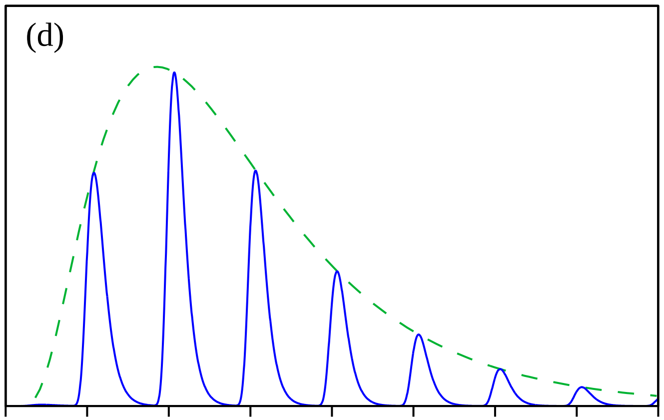
$\sigma = 0.4,\ T = 2$



$\sigma = 0.4,\ T = 20$



$\sigma = 0.5,\ T = 2$



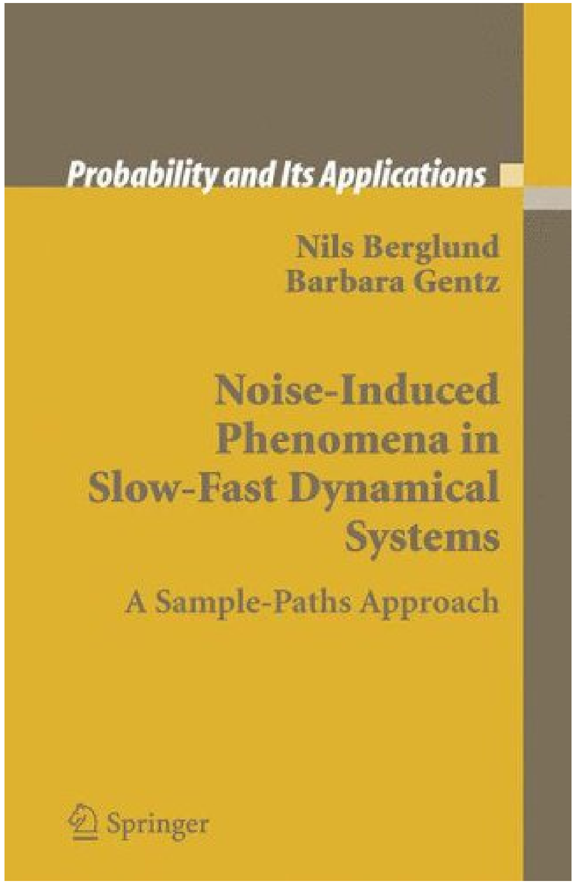
$\sigma = 0.5,\ T = 5$

General results on sample-path behaviour in slow–fast systems

- *Noise-Induced Phenomena in Slow–Fast Dynamical Systems. A Sample-Paths Approach,* “Probability and its Applications”, Springer, London, 2005
- *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)

Case studies: Bifurcations in slowly driven systems

- *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)
- *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab. **12**, 1419–1470 (2002)
- *The effect of additive noise on dynamical hysteresis*, Nonlinearity **15**, 605–632 (2002)



Passage through an unstable periodic orbit

- *On the noise-induced passage through an unstable periodic orbit I: Two-level model*, J. Statist. Phys. **114**, 1577–1618 (2004)
- *Universality of first-passage- and residence-time distributions in non-adiabatic stochastic resonance*, Europhys. Lett. **70**, 1–7 (2005)