The Effect of Gaussian White Noise on Dynamical Systems

Part I: Diffusion Exit from a Domain

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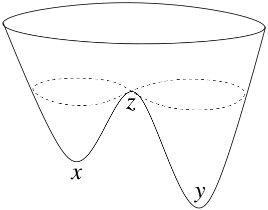
Introduction: A Brownian particle in a potential

Small random perturbations

Gradient dynamics (ODE)

$$\dot{x}_t^{\text{det}} = -\nabla V(x_t^{\text{det}})$$



Random perturbation by Gaussian white noise (SDE)

$$\mathrm{d}x_t^\varepsilon(\omega) = -\nabla V(x_t^\varepsilon(\omega))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}B_t(\omega)$$

Equivalent notation

$$\dot{x}_t^\varepsilon(\omega) = -\nabla V(x_t^\varepsilon(\omega)) + \sqrt{2\varepsilon}\,\xi_t(\omega)$$

with

- $V : \mathbb{R}^d \to \mathbb{R}$: confining potential, growth condition at infinity
- $\{B_t(\omega)\}_{t\geq 0}$: d-dimensional Brownian motion
- $\{\xi_t(\omega)\}_{t\geq 0}$: Gaussian white noise, $\langle \xi_t \rangle = 0$, $\langle \xi_t \xi_s \rangle = \delta(t-s)$

Fokker–Planck equation

Stochastic differential equation (SDE) of gradient type

$$dx_t^\varepsilon(\omega) = -\nabla V(x_t^\varepsilon(\omega))\,dt + \sqrt{2\varepsilon}\,dB_t(\omega)$$

Kolmogorov's forward or Fokker–Planck equation

- Solution $\{x_t^\varepsilon(\omega)\}_t$ is a (time-homogenous) Markov process
- Transition probability densities $(x,t) \mapsto p(x,t|y,s)$ satisfy

$$\frac{\partial}{\partial t}p = \mathcal{L}_\varepsilon p = \nabla \cdot \left[\nabla V(x)p\right] + \varepsilon \Delta p$$

- If $\{x_t^\varepsilon(\omega)\}_t$ admits an invariant density p_0, then $\mathcal{L}_\varepsilon p_0 = 0$
- Easy to verify (for gradient systems)

$$p_0(x) = \frac{1}{Z_\varepsilon}e^{-V(x)/\varepsilon} \qquad \text{with} \qquad Z_\varepsilon = \int_{\mathbb{R}^d} e^{-V(x)/\varepsilon}\,dx$$

Equilibrium distribution

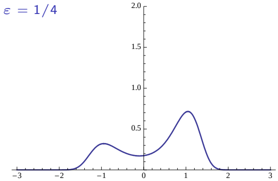
- Invariant measure or equilibrium distribution

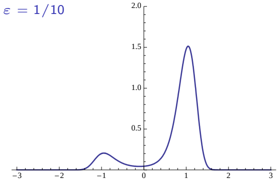
$$\mu_\varepsilon(dx) = \frac{1}{Z_\varepsilon} e^{-V(x)/\varepsilon} \, dx$$

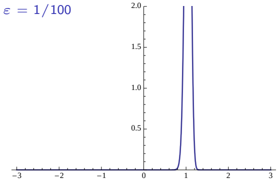
- System is reversible w.r.t. μ_ε (detailed balance)

$$p(y,t|x,0) \, e^{-V(x)/\varepsilon} = p(x,t|y,0) \, e^{-V(y)/\varepsilon}$$

- For small ε, the invariant measure μ_ε concentrates in the minima of V







Timescales

Let V be a double-well potential as before, start in $x_0^\varepsilon = x_-^\star =$ left-hand well

How long does it take until x_t^ε is well described by its invariant distribution?

- For ε small, paths will stay in the left-hand well for a long time
- x_t^ε first approaches a Gaussian distribution, centered in $x_-^\star$,

$$T_{\text{relax}} = \frac{1}{V''(x_-^\star)} = \frac{1}{\text{curvature at the bottom of the well}} \qquad (d=1)$$

- With overwhelming probability, paths will remain inside left-hand well, for all times significantly shorter than Kramers' time

$$T_{\text{Kramers}} = \mathrm{e}^{H/\varepsilon}\,, \quad \text{where} \quad H = V(z^\star) - V(x_-^\star) = \text{barrier height}$$

- Only for $t \gg T_{\text{Kramers}}$, the distribution of x_t^ε approaches p_0

The dynamics is thus very different on the different timescales

Diffusion exit from a domain

The more general picture: Diffusion exit from a domain

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t^\varepsilon)\,\mathrm{d}W_t\,, \qquad x_0 \in \mathbb{R}^d$$

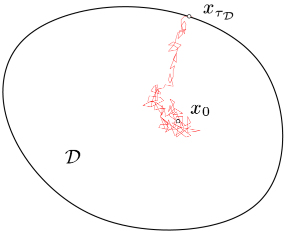
General case: b not necessarily derived from a potential

Consider bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- First-exit time: $\tau = \tau_{\mathcal{D}}^\varepsilon = \inf\{t > 0 \colon x_t^\varepsilon \notin \mathcal{D}\}$
- First-exit location: $x_\tau^\varepsilon \in \partial\mathcal{D}$

Questions

- Does x_t^ε leave $\mathcal{D}$?
- If so: When and where?
- Expected time of first exit?
- Concentration of first-exit time and location?
- Distribution of τ and x_τ^ε ?



First case: Deterministic dynamics leaves $\mathcal{D}$

If x_t leaves $\mathcal{D}$ in finite time, so will x^ε_t. Show that deviation $x^\varepsilon_t - x_t$ is small:

Assume b Lipschitz continuous and $g = \mathrm{Id}$ (isotropic noise)

$$\|x^\varepsilon_t - x_t\| \leq L \int_0^t \|x^\varepsilon_s - x_s\| \,\mathrm{d}s + \sqrt{2\varepsilon}\,\|W_t\|$$

By Gronwall's lemma, for fixed realization of noise ω

$$\sup_{0\leq s\leq t} \|x^\varepsilon_s - x_s\| \leq \sqrt{2\varepsilon} \sup_{0\leq s\leq t} \|W_s\| \,\mathrm{e}^{Lt}$$

- $d = 1$: Use André's reflection principle

$$\mathbb{P}\Big\{\sup_{0\leq s\leq t} |W_s| \geq r\Big\} \leq 2\,\mathbb{P}\Big\{\sup_{0\leq s\leq t} W_s \geq r\Big\} \leq 4\,\mathbb{P}\big\{W_t \geq r\big\} \leq 2\,\mathrm{e}^{-r^2/2t}$$

- $d > 1$: Reduce to $d = 1$ using independence
- General case: Use large-deviation principle

Second case: Deterministic dynamics does not leave $\mathcal{D}$

Assume $\mathcal{D}$ positively invariant under deterministic flow: Study noise-induced exit

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{2\varepsilon}g(x_t^\varepsilon)\,\mathrm{d}W_t\,, \qquad x_0 \in \mathbb{R}^d$$

- b, g locally Lipschitz continuous, bounded-growth condition
- $a(x) = g(x)g(x)^{\mathrm{T}} \geq \frac{1}{M}\,\mathrm{Id}$ (uniform ellipticity)

Infinitesimal generator $\mathcal{A}^\varepsilon$ of diffusion x_t^ε: $\mathcal{A}^\varepsilon v(x) = \lim_{t \searrow 0} \frac{1}{t}\left[\mathbb{E}_x v(x_t) - v(x)\right]$

$$\mathcal{A}^\varepsilon v(x) = \varepsilon \sum_{i,j=1}^{d} a_{ij}(x)\,\frac{\partial^2}{\partial x_i\,\partial x_j}\,v(x) + \langle\, b(x), \nabla v(x)\,\rangle$$

Compare to Fokker–Planck operator: $\mathcal{L}^\varepsilon$ is the adjoint operator of $\mathcal{A}^\varepsilon$

Approaches to the exit problem

- Mean first-exit times and locations via PDEs
- Exponential asymptotics via Wentzell–Freidlin theory

Diffusion exit from a domain: Relation to PDEs

Theorem

- Poisson problem:
  $\mathbb{E}_x\{\tau^\varepsilon_{\mathcal{D}}\}$ is the unique solution of $\begin{cases} \mathcal{A}^\varepsilon u = -1 & \text{in } \mathcal{D} \\ u = 0 & \text{on } \partial\mathcal{D} \end{cases}$

- Dirichlet problem:
  $\mathbb{E}_x\{f(x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}})\}$ is the unique solution of $\begin{cases} \mathcal{A}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases}$
  (for $f : \partial\mathcal{D} \to \mathbb{R}$ continuous)

Remarks

- Expected first-exit times and distribution of first-exit locations obtained exactly from PDEs

Diffusion exit from a domain: Relation to PDEs

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- In principle ...

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Remarks

- Expected first-exit times and distribution of first-exit locations obtained exactly from PDEs
- In principle . . .
- Smoothness assumption for $\partial\mathcal{D}$ can be relaxed to "exterior-ball condition"

An example in $d = 1$

Motion of a Brownian particle in a quadratic single-well potential

$$dx_t^\varepsilon = b(x_t^\varepsilon)\, dt + \sqrt{2\varepsilon}\, dW_t$$

where $b(x) = -\nabla V(x)$, $V(x) = ax^2/2$ with $a > 0$

- ▹ Drift pushes particle towards bottom at $x = 0$
- ▹ Probability of x_t^ε leaving $\mathcal{D} = (\alpha_1, \alpha_2) \ni 0$ through α_1?

Solve the (one-dimensional) Dirichlet problem

$$\begin{cases} \mathcal{A}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases} \qquad \text{with} \qquad f(x) = \begin{cases} 1 & \text{for } x = \alpha_1 \\ 0 & \text{for } x = \alpha_2 \end{cases}$$

$$\mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}} = \alpha_1\} = \mathbb{E}_x f(x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}}) = w(x) = \int_x^{\alpha_2} e^{V(y)/\varepsilon}\, dy \Bigg/ \int_{\alpha_1}^{\alpha_2} e^{V(y)/\varepsilon}\, dy$$

An example in $d = 1$: Small noise limit?

$$\mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}} = \alpha_1\} = \int_x^{\alpha_2} \mathrm{e}^{V(y)/\varepsilon}\, \mathrm{d}y \Bigg/ \int_{\alpha_1}^{\alpha_2} \mathrm{e}^{V(y)/\varepsilon}\, \mathrm{d}y$$

What happens in the small-noise limit?

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}} = \alpha_1\} = 1 \qquad \text{if } V(\alpha_1) < V(\alpha_2)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}} = \alpha_1\} = 0 \qquad \text{if } V(\alpha_2) < V(\alpha_1)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon_{\mathcal{D}}} = \alpha_1\} = \frac{1}{|V'(\alpha_1)|} \Bigg/ \left(\frac{1}{|V'(\alpha_1)|} + \frac{1}{|V'(\alpha_2)|}\right) \qquad \text{if } V(\alpha_1) = V(\alpha_2)$$

Information is more precise than results relying on a LDP provide

Large deviations: Wentzell–Freidlin theory

Alexander Wentzell (*1937), Eugene Dynkin (*1924), Joseph Doob (1910–2004), Mark Freidlin (*1934)
(May 1994)

Exponential asymptotics via large deviations

- Large-deviation rate function

$$I(\varphi) = I_{[0,T]}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_s - b(\varphi_s)\|^2 \, \mathrm{d}s & \text{for } \varphi \in \mathcal{H}_1 \\ +\infty & \text{otherwise} \end{cases}$$

- Large deviation principle reduces est. of probabilities to variational principle: For any set Γ of paths on $[0, T]$

$$-\inf_{\Gamma^\circ} I \leq \liminf_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}\{(x_t^\varepsilon)_t \in \Gamma\} \leq \limsup_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}\{(x_t^\varepsilon)_t \in \Gamma\} \leq -\inf_{\overline{\Gamma}} I$$

- In short: Probability of observing sample paths being close to a given path $\varphi : [0, T] \to \mathbb{R}^d$ behaves like $\sim \exp\{-2I(\varphi)/\varepsilon\}$
- Assume domain $\mathcal{D}$ has unique asymptotically stable equilibrium point $x_-^\star$
- Quasipotential *with respect to* $x_-^\star$ = cost to reach z against the flow

$$V(x_-^\star, z) := \inf_{t>0} \inf\{I_{[0,t]}(\varphi) \colon \varphi \in \mathcal{C}([0,t], \mathcal{D}), \ \varphi_0 = x_-^\star, \ \varphi_t = z\}$$

Wentzell–Freidlin theory

Theorem [Wentzell & Freidlin 1969–72, 1984] (general case as on previous slide)

For arbitrary initial condition in $x \in \mathcal{D}$

- Mean first-exit time: $\mathbb{E}_x \tau_{\mathcal{D}}^{\varepsilon} \sim \mathrm{e}^{\overline{V}/2\varepsilon}$ as $\varepsilon \to 0$
- Concentration of first-exit times:

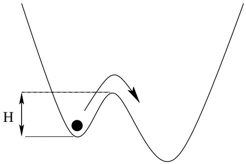
  $\mathbb{P}_x \Big\{ \mathrm{e}^{(\overline{V}-\delta)/2\varepsilon} \leqslant \tau_{\mathcal{D}}^{\varepsilon} \leqslant \mathrm{e}^{(\overline{V}+\delta)/2\varepsilon} \Big\} \to 1$ as $\varepsilon \to 0$ (for arbitrary $\delta > 0$)
- Concentration of exit locations near minima of quasipotential

Gradient case (reversible diffusion)

- $b = -\nabla V$, $g = \mathrm{Id}$
- Quasipotential $V(x_-^\star, z) = 2[V(z) - V(x_-^\star)]$
- Cost for leaving potential well:

  $\overline{V} = \inf_{z \in \partial\mathcal{D}} V(x_-^\star, z) = 2[V(z^\star) - V(x_-^\star)] = 2H$
- Attained for paths going against the deterministic flow:

  $\dot{\varphi}_t = +\nabla V(\varphi_t)$



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MFO Oberwolfach

Remarks for the gradient case

- Arrhenius Law [van't Hoff 1885, Arrhenius 1889] follows as a corollary

$$\mathbb{E}_{x_-^\star} \tau_+ \simeq const\ \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$$

where τ_+ = first-hitting time of small ball $B_\delta(x_+^\star)$ around other minimum $x_+^\star$

$$\tau_+ = \tau^\varepsilon_{x_+^\star}(\omega) = \inf\{t \geq 0 \colon x_t^\varepsilon(\omega) \in B_\delta(x_+^\star)\}$$

- Exponential asymptotics depends only on barrier height
- LDP also provides information on optimal transition paths
- Only 1-saddles are relevant for transitions between wells
- Multiwell case described by hierarchy of "cycles"
- Nongradient case: Work with variational problem
- Prefactor cannot be obtained by this approach

Subexponential asymptotics

Refined results in the gradient case: Kramers' law

First-hitting time of a small ball $B_\delta(x_+^\star)$ around minimum $x_+^\star$

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Arrhenius Law [van't Hoff 1885, Arrhenius 1889] – see previous slide

$$\mathbb{E}_{x_-^\star} \tau_+ \simeq \mathit{const}\; \mathrm{e}^{[V(z^\star) - V(x_-^\star)]/\varepsilon}$$

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Eyring–Kramers Law [Eyring 1935, Kramers 1940]

- $d = 1$: $\quad \mathbb{E}_{x_-^\star}\tau_+ \simeq \dfrac{2\pi}{\sqrt{V''(x_-^\star)|V''(z^\star)|}}\,\mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$

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Eyring–Kramers Law [Eyring 1935, Kramers 1940]

- $d = 1$: $$\mathbb{E}_{x_-^\star}\tau_+ \simeq \frac{2\pi}{\sqrt{V''(x_-^\star)|V''(z^\star)|}}\,\mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$$
- $d \geq 2$: $$\mathbb{E}_{x_-^\star}\tau_+ \simeq \frac{2\pi}{|\lambda_1(z^\star)|}\sqrt{\frac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}}\,\mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$$

  where $\lambda_1(z^\star)$ is the unique negative eigenvalue of $\nabla^2 V$ at saddle $z^\star$

Proving Kramers' law (multiwell potentials)

- Low-lying spectrum of generator of the diffusion (analytic approach) [Helffer & Sjöstrand 1985, Miclo 1995, Mathieu 1995, Kolokoltsov 1996, ...]
- Potential theoretic approach [Bovier, Eckhoff, Gayrard & Klein 2004]

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{2\pi}{|\lambda_1(z^\star)|} \sqrt{\frac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon} \left[1 + \mathcal{O}\big((\varepsilon|\log\varepsilon|)^{1/2}\big)\right]$$

  (obtained from similar asymptotics for eigenvalues of generator)
- Full asymptotic expansion of prefactor [Helffer, Klein & Nier 2004]
- Asymptotic distribution of τ_+ is exponential

$$\lim_{\varepsilon\to 0} \mathbb{P}_{x_-^\star}\left\{\tau_+ > t \cdot \mathbb{E}_{x_-^\star}\tau_+\right\} = \mathrm{e}^{-t}$$

  [Day 1983, Bovier, Gayrard & Klein 2005]

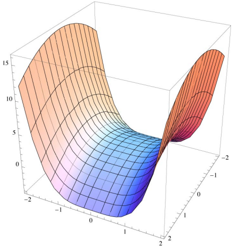
Generalizations: Non-quadratic saddles

What happens if $\det \nabla^2 V(z^\star) = 0$?

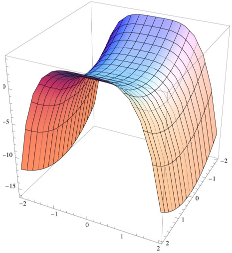
$\det \nabla^2 V(z^\star) = 0 \quad \Rightarrow \quad$ At least one vanishing eigenvalue at saddle $z^\star$

$\Rightarrow \quad$ Saddle has at least one non-quadratic direction

$\Rightarrow \quad$ Kramers Law not applicable



Quartic unstable direction



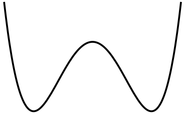
Quartic stable direction

Why do we care about this non-generic situation?

Parameter-dependent systems may undergo bifurcations

Example: Two harmonically coupled particles

$$V_\gamma(x_1, x_2) = U(x_1) + U(x_2) + \frac{\gamma}{2}(x_1 - x_2)^2$$

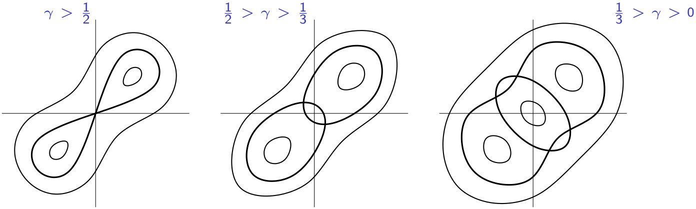


$U(x) = \frac{x^4}{4} - \frac{x^2}{2}$

Change of variable: Rotation by $\pi/4$ yields

$$\widehat{V}_\gamma(y_1, y_2) = -\frac{1}{2}y_1^2 - \frac{1-2\gamma}{2}y_2^2 + \frac{1}{8}\left(y_1^4 + 6y_1^2y_2^2 + y_2^4\right)$$

Note: $\det \nabla^2 \widehat{V}_\gamma(0,0) = 1 - 2\gamma$ $\Rightarrow$ Pitchfork bifurcation at $\gamma = 1/2$



General case of n particles [Berglund, Fernandez & G 2007]

Transition times for non-quadratic saddles

- Assume $x_-^\star$ is a quadratic local minimum of V (non-quadratic minima can be dealt with)
- Assume $x_+^\star$ is another local minimum of V
- Assume $z^\star = 0$ is the relevant saddle for passage from $x_-^\star$ to $x_+^\star$
- Normal form near saddle
$$V(y) = -u_1(y_1) + u_2(y_2) + \frac{1}{2}\sum_{j=3}^{d} \lambda_j y_j^2 + \dots$$
- Assume growth conditions on u_1, u_2

Theorem [Berglund & G 2010]

$$\mathbb{E}_{x_-^\star}\tau_+ = \frac{(2\pi\varepsilon)^{d/2}\,\mathrm{e}^{-V(x_-^\star)/\varepsilon}}{\sqrt{\det\nabla^2 V(x_-^\star)}} \Bigg/ \varepsilon\,\frac{\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-u_2(y_2)/\varepsilon}\,\mathrm{d}y_2}{\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-u_1(y_1)/\varepsilon}\,\mathrm{d}y_1}\prod_{j=3}^{d}\sqrt{\frac{2\pi\varepsilon}{\lambda_j}}$$
$$\times\left[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^\alpha)\right]$$

where $\alpha > 0$ depends on the growth conditions and is explicitly known

Corollary: Pitchfork bifurcation

Pitchfork bifurcation: $V(y) = -\frac{1}{2}|\lambda_1|y_1^2 + \frac{1}{2}\lambda_2 y_2^2 + C_4 y_2^4 + \frac{1}{2}\sum_{j=3}^{d}\lambda_j y_j^2 + \dots$

- For $\lambda_2 > 0$ (possibly small wrt. ε):

$$\mathbb{E}_{x_-^\star}\tau_+ = 2\pi\sqrt{\frac{(\lambda_2+\sqrt{2\varepsilon C_4})\lambda_3\dots\lambda_d}{|\lambda_1|\det\nabla^2 V(x_-^\star)}}\,\frac{e^{[V(z^\star)-V(x_-^\star)]/\varepsilon}}{\Psi_+(\lambda_2/\sqrt{2\varepsilon C_4})}\,[1+R(\varepsilon)]$$

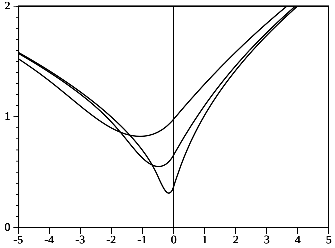
where

$$\Psi_+(\alpha) = \sqrt{\frac{\alpha(1+\alpha)}{8\pi}}\,e^{\alpha^2/16}\,K_{1/4}\Big(\frac{\alpha^2}{16}\Big)$$

$$\lim_{\alpha\to\infty}\Psi_+(\alpha) = 1$$

$K_{1/4} =$ modified Bessel fct. of 2*nd* kind

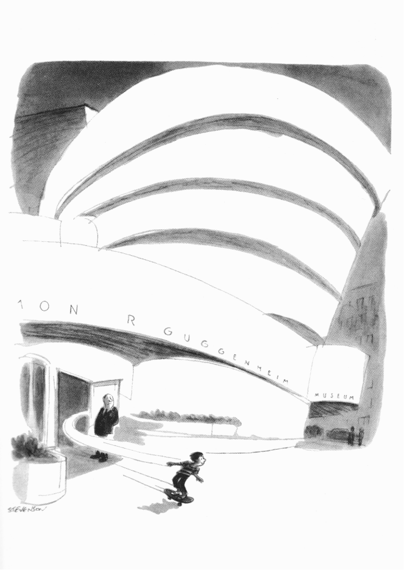
- For $\lambda_2 < 0$: Similar, involving $I_{\pm 1/4}$



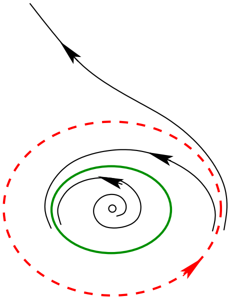
$\lambda_2 \mapsto$ prefactor

$\varepsilon = 0.5$, $\varepsilon = 0.1$, $\varepsilon = 0.01$

Non-gradient case: Cycling



New phenomena in non-gradient case: Cycling



Simplest situation of interest:
Nontrivial invariant set which is a single periodic orbit

Assume from now on:
$d = 2$, $\partial\mathcal{D} =$ *unstable* periodic orbit

- Wentzell–Freidlin theory: $\mathbb{E}\tau_{\mathcal{D}} \sim e^{\overline{V}/2\varepsilon}$ still holds
- Quasipotential $V(\Pi, z) \equiv \overline{V}$ is constant on $\partial\mathcal{D}$:
  Exit equally likely anywhere on $\partial\mathcal{D}$ (on exp. scale)
- Phenomenon of cycling [Day 1992]:
  Distribution of $x_{\tau_{\mathcal{D}}}$ on $\partial\mathcal{D}$ does *not* converge as $\varepsilon \to 0$
  Density is *translated* along $\partial\mathcal{D}$ proportionally to $|\log\varepsilon|$
- In *stationary regime*: (obtained by reinjecting particle)
  Rate of escape $\frac{d}{dt}\mathbb{P}\{x_t \notin \mathcal{D}\}$ has $|\log\varepsilon|$-periodic prefactor [Maier & Stein 1996]

Universality in cycling

Theorem [Berglund & G 2004, 2005, 2014] (informal version)

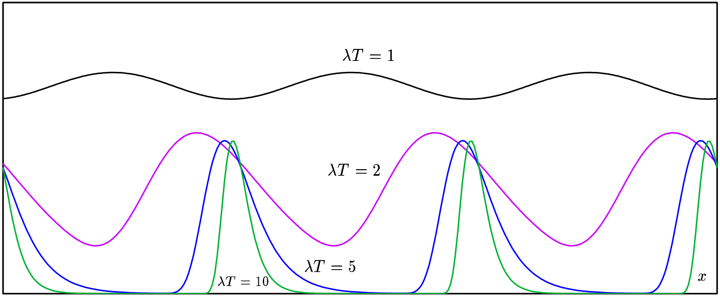
There exists an explicit parametrization of $\partial\mathcal{D}$ s.t. the exit time density is given by

$$p(t,t_0) = \frac{f_{\text{trans}}(t,t_0)}{\mathcal{N}}\; Q_{\lambda T}\big(\theta(t) - \tfrac{1}{2}|\log\varepsilon|\big)\,\frac{\theta'(t)}{\lambda T_{\text{K}}(\varepsilon)}\,\mathrm{e}^{-(\theta(t)-\theta(t_0))\,/\,\lambda T_{\text{K}}(\varepsilon)}$$

- $Q_{\lambda T}(y)$ is a *universal* λT-periodic function
- $\theta(t)$ is a "natural" parametrisation of the boundary:
  $\theta'(t) > 0$ is explicitely known *model-dependent*, T-periodic function;
  $\theta(t+T) = \theta(t) + \lambda T$
- $T_{\text{K}}(\varepsilon)$ is the analogue of Kramers' time: $T_{\text{K}}(\varepsilon) = \dfrac{C}{\sqrt{\varepsilon}}\,\mathrm{e}^{\overline{V}/2\varepsilon}$
- f_{trans} grows from 0 to 1 in time $t - t_0$ of order $|\log\varepsilon|$
- $\mathcal{N}$ is the normalization

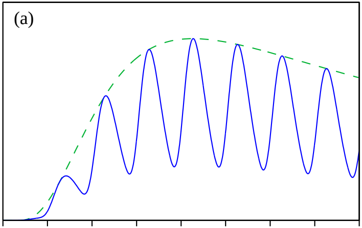
The universal profile

$y \mapsto Q_{\lambda T}(\lambda T y)/2\lambda T$

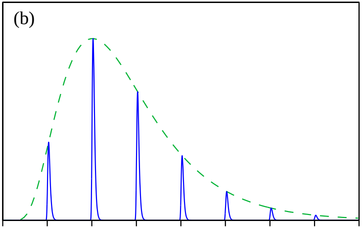


- Profile determines concentration of first-passage times within a period
- Shape of peaks: Gumbel distribution $P(z) = \frac{1}{2}\,\mathrm{e}^{-2z} \exp\bigl\{-\frac{1}{2}\,\mathrm{e}^{-2z}\bigr\}$
- The larger λT, the more pronounced the peaks
- For smaller values of λT, the peaks overlap more

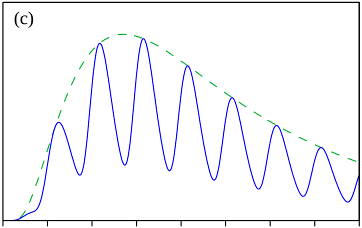
Density of the first-passage time for $\overline{V} = 0.5,\ \lambda = 1$



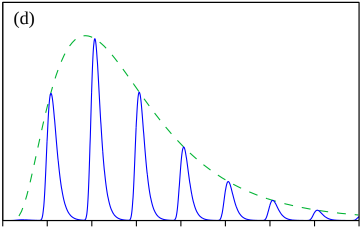
$\varepsilon = 0.4,\ T = 2$



$\varepsilon = 0.4,\ T = 20$



$\varepsilon = 0.5,\ T = 2$



$\varepsilon = 0.5,\ T = 5$

Dependence of exit distribution on the noise intensity

Author: Nils Berglund

- σ decreasing from 1 to 0.0001
- Parameter values: $\lambda_+ = 1$, $T_+ = 4$, $\overline{V} = 1$

Modifications

- System starting in quasistationary distribution (no transitional phase)
- Maximum is chosen to be constant (area under the curve *not* constant)

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