The Effect of Gaussian White Noise on Dynamical Systems

Part II: Reduced Dynamics

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17 March 2014


Seminar BINGO!
To play, simply print out this bingo sheet and attend a departmental seminar.
Mark over each square that occurs throughout the course of the lecture.
The first one to form a straight line (or all four corners) must yell out BINGO!! to win!
BINGO!!
SEMINAR
B I N G O
Speaker bashes previous work
Repeated use of "um…"
Speaker sucks up to host professor
Host Professor falls asleep
Speaker wastes 5 minutes explaining outline
Laptop malfunction
Work ties in to Cancer/HIV or War on Terror
"…et al."
You're the only one in your lab that bothered to show up
Blatant typo
Entire slide filled with equations
"The data *clearly* shows…"
FREE Speaker runs out of time
Use of Powerpoint template with blue background
References Advisor (past or present)
There's a Grad Student wearing same clothes as yesterday
Bitter Post-doc asks question
"That's an interesting question"
"Beyond the scope of this work"
Master's student bobs head fighting sleep
Speaker forgets to thank collaborators
Cell phone goes off
You've no idea what's going on
"Future work will…"
Results conveniently show improvement
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General slow–fast systems

General slow–fast systems

Fully coupled SDEs on well-separated time scales

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \ \sigma' \ G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

- $\{W_t\}_{t\geq 0}$ k-dimensional (standard) Brownian motion
- $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- $F : \mathcal{D} \to \mathbb{R}^{n\times k}$, $G : \mathcal{D} \to \mathbb{R}^{m\times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Small parameters

- $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε:
  $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leq 1$

Singular limits for deterministic slow–fast systems

In slow time t	$\overset{t \mapsto s}{\Longleftrightarrow}$	In fast time $s = t/\varepsilon$
$\varepsilon\dot{x} = f(x,y)$ $\dot{y} = g(x,y)$		$x' = f(x,y)$ $y' = \varepsilon g(x,y)$
$\downarrow \varepsilon \to 0$		$\downarrow \varepsilon \to 0$
Slow subsystem	$\not\Longleftrightarrow$	**Fast subsystem**
$0 = f(x,y)$ $\dot{y} = g(x,y)$		$x' = f(x,y)$ $y' = 0$

Study slow variable y on slow manifold $f(x,y) = 0$

Study fast variable x for frozen slow variable y

Near slow manifolds: Assumptions on the fast variables

- Existence of a slow manifold $f(x, y) = 0$:

  $\exists \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists x^\star : \mathcal{D}_0 \to \mathbb{R}^n$
  s.t. $(x^\star(y), y) \in \mathcal{D}$ and $f(x^\star(y), y) = 0$ for $y \in \mathcal{D}_0$

- Slow manifold is attracting:

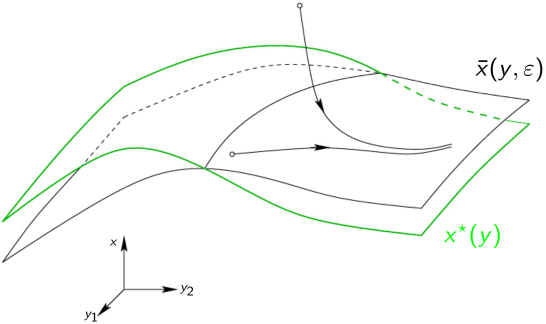
  Eigenvalues of $A^\star(y) := \partial_x f(x^\star(y), y)$ satisfy $\operatorname{Re} \lambda_i(y) \leq -a_0 < 0$
  (uniformly in $\mathcal{D}_0$)

Fenichel's theorem

Theorem [Tihonov 1952, Fenichel 1979]

There exists an *adiabatic manifold*:
$\exists\, \bar{x}(y,\varepsilon)$ s.t.

- $\bar{x}(y,\varepsilon)$ is invariant manifold for deterministic dynamics
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,0) = x^\star(y)$
- $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$



Consider now *stochastic* system under these assumptions

Random slow–fast systems: Slowly driven systems

Typical neighbourhoods for the stochastic fast variable

Special case: One-dimensional slowly driven systems

$$dx_t = \frac{1}{\varepsilon} f(x_t, t)\, dt + \frac{\sigma}{\sqrt{\varepsilon}}\, dW_t\ , \qquad x \in \mathbb{R}$$

Stable slow manifold / stable equilibrium branch $x^\star(t)$:

$$f(x^\star(t), t) = 0\ , \qquad a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0 < 0$$

Linearize SDE for deviation $x_t - \bar{x}(t, \varepsilon)$ from adiabatic solution $\bar{x}(t, \varepsilon) \approx x^\star(t)$

$$dz_t = \frac{1}{\varepsilon} a(t) z_t\, dt + \frac{\sigma}{\sqrt{\varepsilon}}\, dW_t$$

We can solve the non-autonomous SDE for z_t

$$z_t = z_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,s)/\varepsilon}\, dW_s$$

where $\alpha(t) = \int_0^t a(s)\, ds$, $\alpha(t, s) = \alpha(t) - \alpha(s)$ and $a(t) = \partial_x f(\bar{x}(t, \varepsilon), t)$

Typical spreading

$$z_t = z_0 e^{\alpha(t)/\varepsilon} + \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,s)/\varepsilon} \, dW_s$$

z_t is a Gaussian r.v. with variance

$$v(t) = \mathrm{Var}(z_t) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha(t,s)/\varepsilon} \, ds \approx \frac{\sigma^2}{|a(t)|}$$

For any fixed time t, z_t has a typical spreading of $\sqrt{v(t)}$, and a standard estimate shows

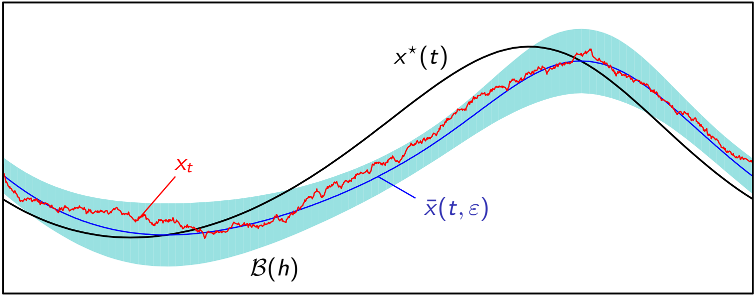
$$\mathbb{P}\{|z_t| \geq h\} \leq e^{-h^2/2v(t)}$$

Goal: Similar concentration result for the whole sample path

Define a strip $\mathcal{B}(h)$ around $\bar{x}(t,\varepsilon)$ of width $\simeq h/\sqrt{|a(t)|}$

$$\mathcal{B}(h) = \left\{(x,t) \colon |x - \bar{x}(t,\varepsilon)| < h/\sqrt{|a(t)|}\right\}$$

Concentration of sample paths



Theorem [Berglund & G 2002, 2006]

$$\mathbb{P}\{(x_s)_s \text{ leaves } \mathcal{B}(h) \text{ before time } t\} \simeq \sqrt{\frac{2}{\pi}}\,\frac{1}{\varepsilon}\left|\int_0^t a(s)\,\mathrm{d}s\right|\,\frac{h}{\sigma}\;\mathrm{e}^{-h^2[1-\mathcal{O}(\varepsilon)-\mathcal{O}(h)]/2\sigma^2}$$

Fully coupled random slow–fast systems

Typical spreading in the general case

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n) \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \;\sigma'\; G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

- Consider deterministic process $(x_t^{\mathrm{det}} = \bar{x}(y_t^{\mathrm{det}}, \varepsilon), y_t^{\mathrm{det}})$ on adiabatic manifold
- Deviation $\xi_t := x_t - x_t^{\mathrm{det}}$ of fast variables from adiabatic manifold
- Linearize SDE for ξ_t ; resulting process ξ_t^0 is Gaussian

Key observation

$\dfrac{1}{\sigma^2} \operatorname{Cov} \xi_t^0$ is a particular solution of the deterministic slow–fast system

$$(*) \qquad \begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\mathrm{det}}) X(t) + X(t) A(y^{\mathrm{det}})^{\mathrm{T}} + F_0(y^{\mathrm{det}}) F_0(y^{\mathrm{det}})^{\mathrm{T}} \\ \dot{y}_t^{\mathrm{det}} \;\; = g(\bar{x}(y_t^{\mathrm{det}}, \varepsilon), y_t^{\mathrm{det}}) \end{cases}$$

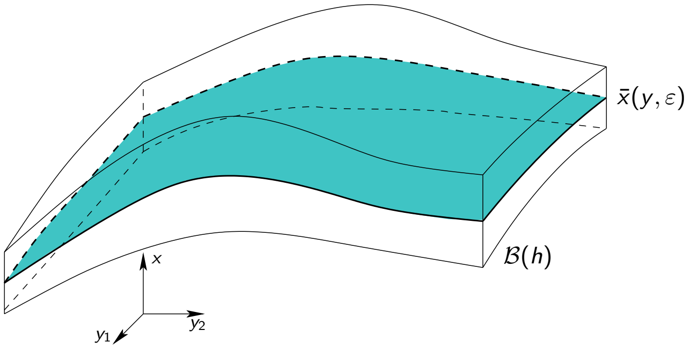
with $A(y) = \partial_x f(\bar{x}(y, \varepsilon), y)$ and F_0 $0th$-order approximation to F

Typical neighbourhoods in the general case

Typical neighbourhoods

$$\mathcal{B}(h) := \left\{ (x,y) \colon \left\langle \left[x - \bar{x}(y,\varepsilon) \right], \bar{X}(y,\varepsilon)^{-1} \left[x - \bar{x}(y,\varepsilon) \right] \right\rangle < h^2 \right\}$$

where $\bar{X}(y,\varepsilon)$ denotes the adiabatic manifold for the system $(*)$



Concentration of sample paths

Define (random) first-exit times

$$\tau_{\mathcal{B}(h)} := \inf\{s > 0 \colon (x_s, y_s) \notin \mathcal{B}(h)\}$$

$$\tau_{\mathcal{D}_0} := \inf\{s > 0 \colon y_s \notin \mathcal{D}_0\}$$

Theorem [Berglund & G 2003]

Assume $\|\bar{X}(y,\varepsilon)\|$, $\|\bar{X}(y,\varepsilon)^{-1}\|$ uniformly bounded in $\mathcal{D}_0$

Then $\exists\, \varepsilon_0 > 0 \quad \exists\, h_0 > 0 \quad \forall\, \varepsilon \leqslant \varepsilon_0 \quad \forall\, h \leqslant h_0$

$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\Bigr\} \leqslant C_{n,m}(t)\, \exp\biggl\{-\frac{h^2}{2\sigma^2}\bigl[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\bigr]\biggr\}$$

where $C_{n,m}(t) = \bigl[C^m + h^{-n}\bigr]\Bigl(1 + \dfrac{t}{\varepsilon^2}\Bigr)$

Reduced dynamics

Reduction to adiabatic manifold $\bar{x}(y,\varepsilon)$:

$$\mathrm{d}y_t^0 = g(\bar{x}(y_t^0,\varepsilon),y_t^0)\,\mathrm{d}t + \sigma' G(\bar{x}(y_t^0,\varepsilon),y_t^0)\,\mathrm{d}W_t$$

Theorem [Berglund & G 2006] (informal version)

y_t^0 approximates y_t to order $\sigma\sqrt{\varepsilon}$ up to Lyapunov time of $\dot{y}^{\text{det}} = g(\bar{x}(y^{\text{det}},\varepsilon)y^{\text{det}})$

Remark

For $\dfrac{\sigma'}{\sigma} < \sqrt{\varepsilon}$, the deterministic reduced dynamics provides a better approximation

Longer time scales

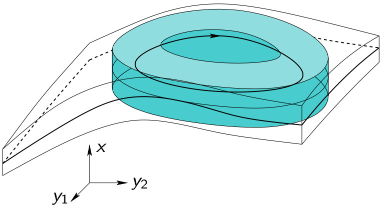
Behaviour of g or behaviour of y_t and y_t^{det} becomes important

Example

y_t^{det} following a stable periodic orbit

- $y_t \sim y_t^{\mathrm{det}}$ for $t \leqslant \dfrac{const}{\sigma \vee \varrho^2 \vee \varepsilon}$

  linear coupling $\rightarrow$ ε
  nonlinear coupling $\rightarrow$ σ
  noise acting on slow variable $\rightarrow$ ϱ



- On longer time scales: Markov property allows to restart

  y_t stays exponentially long in a neighbourhood of the periodic orbit (with probability close to 1)

The main idea of deterministic averaging

Which timescale should be studied?

Simple example

$$\dot{y}^\varepsilon_s = \varepsilon b(y^\varepsilon_s, \xi_s) \,, \quad y^\varepsilon_0 = y \in \mathbb{R}^m$$

- $b : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$
- $\xi : [0, \infty) \to \mathbb{R}^n$
- $0 \leq \varepsilon \ll 1$

If b is not increasing too fast then

$y^\varepsilon_s \to y^0_s \equiv y$ as $\varepsilon \to 0$ uniformly on any finite time interval $[0, T]$

Not the relevant timescale! . . . need to look at time intervals of length $\geq 1/\varepsilon$

- Introduce slow time $t = \varepsilon s$
- Note that $t \in [0, T] \Leftrightarrow s \in [0, T/\varepsilon]$
- Rewrite equation

$$\dot{y}^\varepsilon_t = b(y^\varepsilon_t, \xi_{t/\varepsilon}) \,, \qquad y^\varepsilon_0 = y \in \mathbb{R}^m$$

Deterministic averaging

Assumptions (simplest setting)

- $\|b(y_1,\xi) - b(y_2,\xi)\| \leq K\|y_1 - y_2\|$ for all $\xi \in \mathbb{R}^n$ (Lipschitz condition)
- $\displaystyle\lim_{T\to\infty} \frac{1}{T}\int_0^T b(y,\xi_t)\,\mathrm{d}t = \overline{b}(y)$ uniformly in $y \in \mathbb{R}^m$ (e.g. periodic ξ_t)

Can we obtain an autonomous equation for y_t^ε? Can we replace b by $\overline{b}$?

For small time steps Δ: Freeze y_t^ε

$$y_\Delta^\varepsilon - y = \int_0^\Delta b(y_t^\varepsilon,\xi_{t/\varepsilon})\,\mathrm{d}t = \int_0^\Delta b(y,\xi_{t/\varepsilon})\,\mathrm{d}s + \int_0^\Delta \big[b(y_t^\varepsilon,\xi_{t/\varepsilon}) - b(y,\xi_{t/\varepsilon})\big]\,\mathrm{d}t$$

1. integral $\displaystyle = \Delta\frac{\varepsilon}{\Delta}\int_0^{\Delta/\varepsilon} b(y,\xi_s)\,\mathrm{d}s \approx \Delta\overline{b}(y)$ as $\varepsilon/\Delta \to 0$
2. integral $= \mathcal{O}(\Delta^2)$ (using Lipschitz continuity and leading order)

With a little work: y_t^ε converges uniformly on $[0,T]$ towards solution of $\dot{\overline{y}}_t = \overline{b}(\overline{y}_t)$

Averaging principle

Slow variable y_t^ε and fast variable ξ_t^ε (now allowed to depend on y_t^ε)

$$\dot{y}_t^\varepsilon = b_1(y_t^\varepsilon, \xi_t^\varepsilon) \,, \qquad y_0^\varepsilon = y \in \mathbb{R}^m$$

$$\dot{\xi}_t^\varepsilon = \frac{1}{\varepsilon} b_2(y_t^\varepsilon, \xi_t^\varepsilon) \,, \qquad \xi_0^\varepsilon = \xi \in \mathbb{R}^n$$

Freeze slow variable y and consider

$$\dot{\xi}_t(y) = b_2(y, \xi_t(y)) \,, \qquad \xi_0(y) = \xi$$

Assume $\lim\limits_{T\to\infty} \dfrac{1}{T} \displaystyle\int_0^T b_1(y, \xi_t(y)) \, \mathrm{d}t = \overline{b}_1(y)$ exists (and is independent of ξ)

Averaging principle

The slow variable y_t^ε is well approximated by $\dot{\overline{y}}_t = \overline{b}_1(\overline{y}_t), \; \overline{y}_0 = y$

Random fast motion:
The main idea of stochastic averaging

Random fast motion

Consider again assumption form last slide

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T b_1(y, \xi_t(y))\,\mathrm{d}t = \overline{b}_1(y) \quad \text{exists}$$

Convergence of time averages: Resembles *Law of Large Numbers*!

Our goal: Consider ξ_t given by a random motion

The general setting

$$\dot{y}_t^\varepsilon = b(\varepsilon, t, y_t^\varepsilon, \omega) \,, \qquad y_0^\varepsilon = y \in \mathbb{R}^m$$

$\omega \in \Omega$ indicates the random influence; underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Assumptions

- $(t, y) \mapsto b(\varepsilon, t, y, \omega)$ is continuous for almost all ω and all ε
- $\sup_{\varepsilon>0} \sup_{t\geq 0} \mathbb{E}\|b(\varepsilon, t, y, \omega)\|^2 < \infty$
- $\|b(\varepsilon, t, x, \omega) - b(\varepsilon, t, y, \omega)\| \leq K\|x - y\|$
  for almost all ω, all $x, y \in \mathbb{R}^m$, all $t \geq 0$ and $\varepsilon > 0$
- There exists $\overline{b}(y, t)$, continuous in (y, t), s.t. $\forall\delta > 0\ \forall T > 0\ \forall y \in \mathbb{R}^m$

$$\lim_{\varepsilon\to 0} \mathbb{P}\biggl\{\biggl\|\int_{t_0}^{t_0+T} b(\varepsilon, t, y, \omega)\,\mathrm{d}t - \int_{t_0}^{t_0+T} \overline{b}(t, y)\,\mathrm{d}t\biggr\| \geq \delta\biggr\} = 0$$

uniformly in $t_0 \geq 0$

Stochastic averaging

Theorem (c.f. [WF 1984])

Under the assumptions on the previous slide,

$$\dot{\overline{y}}_t = \overline{b}(t, \overline{y}_t) , \qquad \overline{y}_0 = y$$

has a unique solution, and

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{ \max_{t \in [0,T]} \| y_t^\varepsilon - \overline{y}_t \| \geq \delta \right\} = 0$$

for all $T > 0$ and all $\delta > 0$.

Remarks

- Convergence in probability is a rather weak notion
- Stronger assumptions yield stronger result

Idea of the proof I

$$\|y_t^\varepsilon - \overline{y}_t\| \leq \int_0^t \|b(\varepsilon, s, y_s^\varepsilon, \omega) - b(\varepsilon, s, \overline{y}_s, \omega)\| \,\mathrm{d}s + \left\| \int_0^t [b(\varepsilon, s, \overline{y}_s, \omega) - \overline{b}(s, \overline{y}_s)] \,\mathrm{d}s \right\|$$

Using Lipschitz condition

$$m(t) := \sup_{s\in[0,t]} \|y_s^\varepsilon - \overline{y}_s\| \leq K \int_0^t m(s) \,\mathrm{d}s + \sup_{s\in[0,t]} \left\| \int_0^s [b(\varepsilon, u, \overline{y}_u, \omega) - \overline{b}(u, \overline{y}_u)] \,\mathrm{d}s \right\|$$

Gronwall's lemma: Sufficient to estimate

$$\mathbb{P}\left\{ \sup_{s\in[0,T]} \left\| \int_0^s [b(\varepsilon, u, \overline{y}_u, \omega) - \overline{b}(u, \overline{y}_u)] \,\mathrm{d}s \right\| \geq \tilde{\delta} \right\}$$

Idea of the proof II

- b Lipschitz continuous $\Rightarrow \overline{b}$ Lipschitz continuous
- On short time intervals $[kT/n, (k+1)T/n]$ replace $\overline{y}_u$ by $\overline{y}_{kT/n}$
- Total error accumulated over all time intervals is still $\mathcal{O}(1/n)$
- Apply assumption on $\overline{b}$ to

$$\int_{kT/n}^{(k+1)T/n} [b(\varepsilon, u, \overline{y}_{kT/n}, \omega) - \overline{b}(u, \overline{y}_{kT/n})] \, ds$$

- It remains to deal with upper integration limits *not* of the form $(k+1)T/n$
- Use: interval short, Tchebyschev's inequality, assumption on second moment

Deviation from the averaged process

Deviations of order $\sqrt{\varepsilon}$

If b is sufficiently smooth & other conditions ...

$$\frac{1}{\sqrt{\varepsilon}}(y_t^\varepsilon - \overline{y}_t) \quad \Rightarrow \quad \text{Gaussian Markov process}$$

Here $\Rightarrow$ denotes convergence in distribution on $[0, T]$

Averaging for stochastic differential equations

$$\begin{cases} \mathrm{d}y_t^\varepsilon = b(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}t + \sigma(y_t^\varepsilon)\,\mathrm{d}W_t & \text{(slow variable} \in \mathbb{R}^m) \\ \mathrm{d}\xi_t^\varepsilon = \dfrac{1}{\varepsilon} f(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}t + \dfrac{1}{\sqrt{\varepsilon}} F(y_t^\varepsilon, \xi_t^\varepsilon)\,\mathrm{d}W_t & \text{(fast variable} \in \mathbb{R}^n) \end{cases}$$

$\sigma = \sigma(y_t^\varepsilon, \xi_t^\varepsilon)$ depending also on ξ_t^ε can be considered
(we refrain from doing so since this would require to introduce additional notations)

Introduce Markov process $\xi_t^{y,\xi}$ for frozen slow variable y

$$\mathrm{d}\xi_t^{y,\xi} = f(y, \xi_t^{y,\xi})\,\mathrm{d}t + F(y, \xi_t^{y,\xi})\,\mathrm{d}W_t\;, \qquad \xi_0^{y,\xi} = \xi$$

Averaging Theorem for SDEs

Assume there exist functions $\bar{b}(y)$ and $\kappa(T)$ s.t. for all $t_0 \geq 0$, $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\mathbb{E}\left\| \frac{1}{T} \int_{t_0}^{t_0+T} b(y, \xi_s^{y,\xi})\,\mathrm{d}s - \bar{b}(y) \right\| \leq \kappa(T) \to 0 \quad \text{as } T \to \infty$$

Let $\bar{y}_t$ denote the solution of

$$\mathrm{d}\bar{y}_t = \bar{b}(\bar{y}_t) + \sigma(\bar{y}_t)\,\mathrm{d}W_t\ , \qquad \bar{y}_0 = y$$

Theorem

For all $T > 0$, $\delta > 0$ and all initial conditions $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$\lim_{\varepsilon \to 0} \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \|y_t^\varepsilon - \bar{y}_t\| > \delta \right\} = 0$$

(convergence in probability)

References

Deterministic slow–fast systems

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Averaging

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