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# Large deviations and Wentzell–Freidlin theory

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## Outline

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  - Sample-path large deviations for stochastic differential equations
- ▷ Diffusion exit from a domain
  - Introduction
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Slides available at http://www.wias-berlin.de/people/gentz/misc.html

## Introduction: Small random perturbations

Consider small random perturbation

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,g(x_t^\varepsilon)\,\mathrm{d}W_t, \qquad\qquad x_0^\varepsilon = x_0$$

of ODE

$$\dot{x}_t = b(x_t) \qquad\qquad \text{(with same initial cond.)}$$

We expect $x_t^\varepsilon \approx x_t$ for small $\varepsilon$.

Depends on

- ▷ deterministic dynamics
- ▷ noise intensity $\varepsilon$
- ▷ time scale

Indeed, for $b$ Lipschitz continuous and $g = \mathrm{Id}$

$$\|x_t^\varepsilon - x_t\| \leqslant L \int_0^t \|x_s^\varepsilon - x_s\| \, \mathrm{d}s + \sqrt{\varepsilon}\, \|W_t\|$$

Gronwall's lemma shows

$$\sup_{0 \leqslant s \leqslant t} \|x_s^\varepsilon - x_s\| \leqslant \sqrt{\varepsilon} \sup_{0 \leqslant s \leqslant t} \|W_s\| \, \mathrm{e}^{Lt}$$

Remains to estimate $\sup\limits_{0 \leqslant s \leqslant t} \|W_s\|$

- $d = 1$: Use reflection principle

$$\mathbb{P}\Big\{ \sup_{0 \leqslant s \leqslant t} |W_s| \geqslant r \Big\} \leqslant 2\, \mathbb{P}\Big\{ \sup_{0 \leqslant s \leqslant t} W_s \geqslant r \Big\} \leqslant 4\, \mathbb{P}\{W_t \geqslant r\} \leqslant 2\, \mathrm{e}^{-r^2/2t}$$

- $d > 1$: Reduce to $d = 1$ using independence

$$\mathbb{P}\Big\{ \sup_{0 \leqslant s \leqslant t} \|W_s\| \geqslant r \Big\} \leqslant 2d\, \mathrm{e}^{-r^2/2dt}$$

## Introduction: Small random perturbations

For $\Gamma \subset \mathcal{C} = \mathcal{C}([0,T],\mathbb{R}^d)$ with $\Gamma \subset B((x_s)_s,\delta)^{\mathrm{c}}$ ( $\mathcal{C}$ equipped with sup norm $\|\cdot\|_\infty$ )

$$\mathbb{P}\{x^\varepsilon \in \Gamma\} \leqslant \mathbb{P}\Big\{\sup_{0\leqslant s\leqslant t} \|x^\varepsilon_s - x_s\| \geqslant \delta\Big\} \leqslant \mathbb{P}\Big\{\sup_{0\leqslant s\leqslant t} \|W_s\| \geqslant \frac{\delta}{\sqrt{\varepsilon}}\,\mathrm{e}^{-Lt}\Big\} \leqslant 2d\,\exp\Big\{-\frac{\delta^2\,\mathrm{e}^{-2Lt}}{2\varepsilon d t}\Big\}$$

and

$$\mathbb{P}\{x^\varepsilon \in \Gamma\} \to 0 \qquad \text{as } \varepsilon \to 0$$

- ▷ Event $\{x^\varepsilon \in \Gamma\}$ is atypical: Occurrence a large deviation
- ▷ Question: Rate of convergence as a function of $\Gamma$?
- ▷ Generally not possible, but exponential rate can be found

**Aim:** Find functional $I : \mathcal{C} \to [0,\infty]$ s.t.

$$\mathbb{P}\{\|x^\varepsilon - \varphi\|_\infty < \delta\} \approx \mathrm{e}^{-I(\varphi)/\varepsilon} \qquad \text{for } \varepsilon \to 0$$

- ▷ Provides local description

**Special case:** Scaled Brownian motion, $d = 1$

$$\mathrm{d}W_t^\varepsilon = \sqrt{\varepsilon}\,\mathrm{d}W_t, \qquad \Longrightarrow \qquad W_t^\varepsilon = \sqrt{\varepsilon}\,W_t$$

▷ Consider endpoint instead of whole path

$$\mathbb{P}\{W_t^\varepsilon \in A\} = \int_A \frac{1}{\sqrt{2\pi\varepsilon t}} \exp\{-x^2/2\varepsilon t\}\,\mathrm{d}x$$

▷ Use Laplace method to evaluate integral

$$\varepsilon \log \mathbb{P}\{W_t^\varepsilon \in A\} \sim -\frac{1}{2} \inf_{x\in A} \frac{x^2}{t} \qquad \text{as } \varepsilon \to 0$$

**Caution**

▷ $|A| = 1$: l.h.s. $= -\infty <$ r.h.s. $\in (-\infty, 0]$

▷ Limit does not necessarily exist

**Remedy:** Use interior and closure $\Longrightarrow$ Large deviation principle

$$-\frac{1}{2} \inf_{x\in A^\circ} \frac{x^2}{t} \leqslant \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{W_t^\varepsilon \in A\} \leqslant \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{W_t^\varepsilon \in A\} \leqslant -\frac{1}{2} \inf_{x\in \bar{A}} \frac{x^2}{t}$$

**Schilder's Theorem** (1966)

Scaled BM satisfies a (full) large deviation principle with good rate function

$$I(\varphi) = I_{[0,T],0}(\varphi) = \begin{cases} \frac{1}{2}\|\varphi\|_{H_1}^2 = \frac{1}{2}\int_{[0,T]} \|\dot\varphi_s\|^2 \, \mathrm{d}s & \text{if } \varphi \in H_1 \text{ with } \varphi_0 = 0 \\ +\infty & \text{otherwise} \end{cases}$$

That is

- ▷ Rate function: $I : \mathcal{C}_0 = \{\varphi \in \mathcal{C} : \varphi_0 = 0\} \to [0, \infty]$ is lower semi-continuous
- ▷ Good rate function: $I$ has compact level sets
- ▷ Upper and lower large-deviation bound:

$$-\inf_{\Gamma^\circ} I \leqslant \liminf_{\varepsilon\to0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in \Gamma\} \leqslant \limsup_{\varepsilon\to0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in \Gamma\} \leqslant -\inf_{\overline{\Gamma}} I \qquad \text{for all } \Gamma \in \mathcal{B}(\mathcal{C}_0)$$

**Remarks**

- ▷ Infinite-dimensional version of Laplace method
- ▷ $W^\varepsilon \notin H^1 \implies I(W^\varepsilon) = +\infty$ (almost surely)
- ▷ $I(0) = 0$ reflects $W^\varepsilon \to 0$ $(\varepsilon \to 0)$

## Large deviations for Brownian motion: Examples

**Example I:** Endpoint again . . . $(d = 1)$ $\quad \Gamma = \{\varphi \in \mathcal{C}_0 \colon \varphi_t \in A\}$

$$\inf_\Gamma I = \inf_{x \in A} \frac{1}{2} \int_0^t \left| \frac{\mathrm{d}}{\mathrm{d}s} \left( \frac{xs}{t} \right) \right|^2 \mathrm{d}s = \inf_{x \in A} \frac{x^2}{2t} = \text{cost to force BM to be in } A \text{ at time } t$$

$$\Longrightarrow \quad \mathbb{P}\{W_t^\varepsilon \in A\} \sim \exp\{-\inf_{x \in A} x^2/2t\varepsilon\}$$

Note: Typical spreading of $W_t^\varepsilon$ is $\sqrt{\varepsilon t}$

**Example II:** BM leaving a small ball $\quad \Gamma = \{\varphi \in \mathcal{C}_0 \colon \|\varphi\|_\infty \geqslant \delta\}$

$$\inf_\Gamma I = \inf_{0 \leqslant t \leqslant T} \inf_{\varphi \in \mathcal{C}_0 \colon \|\varphi_t\| = \delta} I(\varphi) = \inf_{0 \leqslant t \leqslant T} \frac{\delta^2}{2t} = \frac{\delta^2}{2T} = \text{cost to force BM to leave } B(0,\delta) \text{ before } T$$

$$\Longrightarrow \quad \mathbb{P}\{\exists t \leqslant T, \ \|W_t^\varepsilon\| \geqslant \delta\} \sim \exp\{-\delta^2/2T\varepsilon\}$$

**Example III:** BM staying in a cone (similarly . . . )

## Large deviations for Brownian motion: Lower bound

**To show:** Lower bound for open sets

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in G\} \geqslant -\inf_G I \qquad \text{for all open } G \subset \mathcal{C}_0$$

**Lemma** (local variant of lower bound)

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in B(\varphi, \delta)\} \geqslant -I(\varphi) \qquad \text{for all } \varphi \in \mathcal{C}_0 \text{ with } I(\varphi) < \infty, \quad \text{all } \delta > 0$$

- ▷ Lemma $\implies$ lower bound
- ▷ Standard proof of Lemma: uses Cameron–Martin–Girsanov formula

**Cameron–Martin–Girsanov formula** (special case, $d = 1$ )

$\{W_t\}_t$ $\mathbb{P}$–BM $\implies$ $\{\widehat{W}_t\}_t$ $\mathbb{Q}$–BM

where

$$\widehat{W}_t = W_t - \int_0^t h(s)\,\mathrm{d}s, \qquad h \in \mathcal{L}_2$$

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\biggl\{\int_0^t h(s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t h(s)^2\,\mathrm{d}s\biggr\}$$

## First step

$$X_t = \exp\Big\{\int_0^t h(s)\ \mathrm{d}W_s - \frac{1}{2}\int_0^t h(s)^2\ \mathrm{d}s\Big\} \qquad h \in \mathcal{L}_2$$

$$Y_t = \exp\Big\{\int_0^t (\gamma + h(s))\ \mathrm{d}W_s - \frac{1}{2}\int_0^t (\gamma + h(s))^2\ \mathrm{d}s\Big\} = X_t\ \exp\Big\{\gamma\widehat{W}_t - \frac{1}{2}\gamma^2 t\Big\} \qquad \gamma > 0$$

are exponential martingales wrt. $\mathbb{P}$

## Second step

$$\mathbb{E}_{\mathbb{Q}}\left\{Z\ \exp\{\gamma(\widehat{W}_t - \widehat{W}_s)\}\right\} = \mathbb{E}_{\mathbb{P}}\{Z\,X_t\ \exp\{\gamma(\widehat{W}_t - \widehat{W}_s)\}\} = \mathbb{E}_{\mathbb{P}}\Big\{Z\ \exp\Big\{-\gamma\widehat{W}_s + \frac{1}{2}\gamma^2 t\Big\}\,\mathbb{E}_{\mathbb{P}}\{Y_t \mid \mathcal{F}_s\}\Big\}$$

$$= \mathbb{E}_{\mathbb{P}}\Big\{Z\,X_s\ \exp\Big\{\frac{1}{2}\gamma^2(t-s)\Big\}\Big\} = \mathbb{E}_{\mathbb{Q}}\{Z\}\ \exp\Big\{\frac{1}{2}\gamma^2(t-s)\Big\} \qquad \forall\ Z \in \mathcal{F}_s$$

- ▷ $\widehat{W}_t - \widehat{W}_s$ is $\mathbb{Q}$–independent of $\mathcal{F}_s$ $\Longrightarrow$ increments are independent
- ▷ Increments are Gaussian

$\Longrightarrow$ $\widehat{W}_t$ is BM with respect to $\mathbb{Q}$

$d=1, \quad \delta>0, \quad \varphi \in \mathcal{C}_0$ with $I(\varphi)<\infty, \quad \widehat{W}_t = W_t - \varphi_t/\sqrt{\varepsilon}$

$$\mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} = \mathbb{P}\{\|\widehat{W}\|_\infty < \delta/\sqrt{\varepsilon}\} = \int_{\widehat{W}\in B(0,\delta/\sqrt{\varepsilon})} \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot\varphi_s \, \mathrm{d}W_s + \frac{1}{2\varepsilon}\int_0^T \dot\varphi_s^2 \, \mathrm{d}s\biggr\} \, \mathrm{d}\mathbb{Q}$$

Estimate integral by Jensen's inequality

$$\dots = \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\} \mathbb{Q}\{\widehat{W} \in B(0,\delta/\sqrt{\varepsilon})\} \times \frac{1}{\mathbb{Q}\{\dots\}} \int_{\widehat{W}\in B(0,\delta/\sqrt{\varepsilon})} \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot\varphi_s \, \mathrm{d}\widehat{W}_s\biggr\} \, \mathrm{d}\mathbb{Q}$$

$$\geqslant \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\} \mathbb{P}\{W \in B(0,\delta/\sqrt{\varepsilon})\} \times \exp\biggl\{-\frac{1}{\sqrt{\varepsilon}\,\mathbb{P}\{\dots\}} \int_{W\in B(0,\delta/\sqrt{\varepsilon})} \int_0^T \dot\varphi_s \, \mathrm{d}W_s \, \mathrm{d}\mathbb{P}\biggr\}$$

$$= \exp\biggl\{-\frac{I(\varphi)}{\varepsilon}\biggr\} \mathbb{P}\{W \in B(0,\delta/\sqrt{\varepsilon})\} \times 1$$

Finally note

$$\mathbb{P}\{W \in B(0,\delta/\sqrt{\varepsilon})\} \nearrow 1 \quad (\varepsilon \searrow 0) \qquad \Longrightarrow \qquad \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{\|W^\varepsilon - \varphi\|_\infty < \delta\} \geqslant -I(\varphi)$$

Approximate $W^\varepsilon$ by the random polygon $W^{n,\varepsilon}$ joining $(0, W_0^\varepsilon),\ (T/n, W_{T/n}^\varepsilon),\ \dots\ ,\ (T, W_T^\varepsilon)$

**To show:** $W^{n,\varepsilon}$ is a good approximation to $W^\varepsilon$

$$\mathbb{P}\big\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geqslant \delta\big\} \leqslant n\,\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant T/n} \|W_s^\varepsilon - W_s^{n,\varepsilon}\| \geqslant \delta\Big\} \leqslant n\,\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant T/n} \|W_s^\varepsilon\| \geqslant \frac{\delta}{2}\Big\}$$

$$= n\,\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant T/n} \|W_s\| \geqslant \frac{\delta}{2\sqrt{\varepsilon}}\Big\} \leqslant 2nd\,\exp\Big\{-\frac{n\delta^2}{8\varepsilon dT}\Big\} \qquad \text{(standard estimate)}$$

$\Longrightarrow$ Difference is negligible:

$$\limsup_{n\to\infty}\ \limsup_{\varepsilon\to 0}\ \varepsilon\log\mathbb{P}\big\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geqslant \delta\big\} = -\infty \qquad \text{for all } \delta > 0$$

## Large deviations for Brownian motion: Proof of the upper bound

$F \subset \mathcal{C}_0$ closed, $\delta > 0$, $\ell_\delta = \inf_{F^{(\delta)}} I = \inf\{I(\varphi) : \varphi \in F^{(\delta)}\}$, $n \in \mathbb{N}$

$$\mathbb{P}\{W^\varepsilon \in F\} \leqslant \mathbb{P}\{W^{n,\varepsilon} \in F^{(\delta)}\} + \mathbb{P}\{\|W^\varepsilon - W^{n,\varepsilon}\|_\infty \geqslant \delta\} \leqslant \mathbb{P}\{I(W^{n,\varepsilon}) \geqslant \ell_\delta\} + \text{negligible term}$$

$W^{n,\varepsilon}$ being a polygon yields

$$I(W^{n,\varepsilon}) = \frac{1}{2}\int_0^T \|\dot{W}_s^{n,\varepsilon}\|^2 \, \mathrm{d}s = \frac{1}{2}\sum_{k=1}^n \frac{T}{n}\Big\|\frac{n}{T}\big(W^{n,\varepsilon}_{kT/n} - W^{n,\varepsilon}_{(k-1)T/n}\big)\Big\|^2 \overset{(\mathcal{D})}{=} \frac{\varepsilon}{2}\sum_{k=1}^{nd} \xi_i^2 \qquad (\xi_i \sim \mathcal{N}(0,1) \text{ i.i.d.})$$

By Chebychev's inequality, for $\gamma < 1/2$

$$\mathbb{P}\{I(W^{n,\varepsilon}) \geqslant \ell_\delta\} = \mathbb{P}\Big\{\sum_{k=1}^{nd} \xi_i^2 \geqslant \frac{2\ell_\delta}{\varepsilon}\Big\} \leqslant \exp\Big\{-\frac{2\gamma\ell_\delta}{\varepsilon}\Big\}\Big(\mathbb{E}\exp\{\gamma\xi_1^2\}\Big)^{nd} = \exp\Big\{-\frac{2\gamma\ell_\delta}{\varepsilon}\Big\}(1-2\gamma)^{-nd/2}$$

$\gamma < 1/2$ being arbitrary and the lower semi-continuity of $I$ show

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{W^\varepsilon \in F\} \leqslant \limsup_{n\to\infty}\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\{I(W^{n,\varepsilon}) \geqslant \ell_\delta\} \leqslant -\ell_\delta = -\inf_{F^{(\delta)}} I \searrow -\inf_F I$$

## Large deviations for solutions of SDEs: Special case

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t, \qquad x_0^\varepsilon = x_0 \qquad (\,b\text{ Lipschitz, bounded growth, } g(x) \equiv \text{ identity matrix}\,)$$

Define $F : \mathcal{C}_0 \to \mathcal{C}$ by $\varphi \mapsto F(\varphi) = f$, $f$ being the unique solution in $\mathcal{C}$ to

$$f(t) = x_0 + \int_0^t b(f(s))\,\mathrm{d}s + \varphi(t).$$

- ▷ $F(W^\varepsilon) = x^\varepsilon$
- ▷ $F$ is continuous (use Gronwall's lemma)

Define $J : \mathcal{C} \to [0,\infty]$ by $J(f) = \inf\{I(\varphi) \colon \varphi \in \mathcal{C}_0,\ F(\varphi) = f\}$

**Contraction principle** (trivial version)

$I$ good rate fct, governing LDP for $W^\varepsilon$ $\Longrightarrow$ $J$ good rate fct, governing LDP for $x^\varepsilon = F(W^\varepsilon)$

Identify $J$:

$$J(f) = J_{[0,T],x_0}(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{[0,T]} \|\dot{f}_s - b(f_s)\|^2\,\mathrm{d}s & \text{if } f \in H_1 \text{ with } f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,g(x_t^\varepsilon)\,\mathrm{d}W_t, \qquad x_0^\varepsilon = x_0$$

**Assumptions**

- $b$, $g$ Lipschitz continuous
- bounded growth: $\|b(x)\| \leqslant M\,(1+\|x\|^2)^{1/2}$, $a(x) = g(x)g(x)^{\mathrm{T}} \leqslant M\,(1+\|x\|^2)\,\mathrm{Id}$
- ellipticity: $a(x) > 0$

**Theorem** (Wentzell–Freidlin)

$x^\varepsilon$ satisfies a LDP with good rate function

$$J(f) = J_{[0,T],x_0}(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{[0,T]} \Big\| a(f_s)^{-1/2}\big[\dot{f}_s - b(f_s)\big]\Big\|^2 \,\mathrm{d}s & \text{if } f \in H_1 \text{ with } f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

**Remark**

If $a(x)$ is only positive semi-definite: LDP remains valid with good rate function but identification of $J$ may fail;

$$J(f) = \inf\Big\{ I(\varphi) \colon \varphi \in H_1,\ \ f_t = x_0 + \int_0^t b(f_s)\,\mathrm{d}s + \int_0^t a(f_s)^{1/2}\dot{\varphi}_s\,\mathrm{d}s,\ t \in [0,T] \Big\}$$

## Large deviations for solutions of SDEs: Sketch of the proof for the general case

- Difficulty: Cannot apply contraction principle directly
- Introduce Euler approximations

$$x_t^{n,\varepsilon} = x_0 + \int_0^t b(x_s^{n,\varepsilon})\, \mathrm{d}s + \sqrt{\varepsilon} \int_0^t g(x_{T_n(s)}^{n,\varepsilon})\, \mathrm{d}W_s, \qquad T_n(s) = \frac{[ns]}{n}$$

- Schilder's Theorem and contraction principle imply LDP for $x^{n,\varepsilon}$ with good rate function $J^n$

$$J^n(f) = \begin{cases} \frac{1}{2} \int_{[0,T]} \left\| a(f_{T_n(s)})^{-1/2} \left[ \dot{f}_s - b(f_s) \right] \right\|^2 \mathrm{d}s & \text{if } f \in H_1 \text{ with } f_0 = x_0 \\ +\infty & \text{otherwise} \end{cases}$$

- To show:
  (1) $x^{n,\varepsilon}$ is a good approximation to $x^\varepsilon$ (not difficult but tedious, uses Itô's formula)
  (2) $J^n$ approximates $J$: $\lim_{n\to\infty} \inf_\Gamma J^n = \inf_\Gamma J$ for all $\Gamma$

## Large deviations for solutions of SDEs: Varadhan's Lemma

### Assumptions

- $\phi : \mathcal{C} \to \mathbb{R}$ continuous
- Tail condition

$$\lim_{L\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \int_{\phi(x^\varepsilon)\geqslant L} \exp\{\phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P} = -\infty$$

**Theorem** (Varadhan's Lemma)

$$\lim_{\varepsilon\to 0} \varepsilon \log \int \exp\{\phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P} = \sup_{\varphi} \left[\phi(\varphi) - J(\varphi)\right]$$

### Remarks

- Moment condition

$$\sup_{0<\varepsilon\leqslant 1} \left( \int \exp\{\alpha\, \phi(x^\varepsilon)/\varepsilon\} \, \mathrm{d}\mathbb{P} \right)^{\varepsilon} < \infty \qquad \text{for some } \alpha \in (1,\infty)$$

  implies tail condition.
- Infinite-dimensional analogue of Laplace method
- Holds in great generality — as long as $x^\varepsilon$ satisfies a LDP with a good rate function $J$

**Noise-induced exit** from a domain $\mathcal{D}$ (bounded, open, smooth boundary)

Consider small random perturbation

$$\mathrm{d}x_t^\varepsilon = b(x_t^\varepsilon)\,\mathrm{d}t + \sqrt{\varepsilon}\,g(x_t^\varepsilon)\,\mathrm{d}W_t, \qquad x_0^\varepsilon = x_0 \in \mathcal{D}$$

of ODE

$$\dot{x}_t = b(x_t) \qquad \text{(with same initial cond.)}$$

First-exit time

$$\tau^\varepsilon = \inf\{t \geqslant 0 \colon x_t^\varepsilon \notin \mathcal{D}\}$$

**Questions**

- Does $x_t^\varepsilon$ leave $\mathcal{D}$?
- If so: When and where?
- Expected time of first exit?
- Concentration of first-exit time and location?

**Towards answers**

- If $x_t$ leaves $\mathcal{D}$, so will $x_t^\varepsilon$. Use LDP to estimate deviation $x_t^\varepsilon - x_t$.
- Later on: Assume $x_t$ does *not* leave $\mathcal{D}$. Study noise-induced exit.

## Diffusion exit from a domain: Relation to PDEs

**Assumptions** (from now on)

- ▷ $b$, $g$ Lipschitz cont., bounded growth
- ▷ $a(x) = g(x)g(x)^T \geqslant (1/M)\,\mathrm{Id}$ (uniform ellipticity)
- ▷ $\mathcal{D}$ bounded domain, smooth boundary

Infinitesimal generator $\mathcal{L}^\varepsilon$ of diffusion $x^\varepsilon$

$$\mathcal{L}^\varepsilon v(t,x) = \frac{\varepsilon}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} v(t,x) + \langle b(x), \nabla v(t,x) \rangle$$

**Theorem**

For $f : \partial\mathcal{D} \to \mathbb{R}$ continuous

- ▷ $\mathbb{E}_x\{\tau^\varepsilon\}$ is the unique solution of the PDE $\begin{cases} \mathcal{L}^\varepsilon u = -1 & \text{in } \mathcal{D} \\ u = 0 & \text{on } \partial\mathcal{D} \end{cases}$
- ▷ $\mathbb{E}_x\{f(x^\varepsilon_{\tau^\varepsilon})\}$ is the unique solution of the PDE $\begin{cases} \mathcal{L}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases}$

**Remarks**

- ▷ Information on first-exit times and exit locations can be obtained *exactly* from PDEs
- ▷ In principle . . .
- ▷ Smoothness assumption for $\partial\mathcal{D}$ can be relaxed to “exterior-ball condition”

## Overdamped motion of a Brownian particle in a single-well potential

$d = 1$, potential $U$ deriving from $b$, $b(0) = 0$, $x\,b(x) < 0$ for $x \neq 0$, $g(x) \equiv 1$

- ▷ Drift pushes particle towards bottom
- ▷ Probability of $x^\varepsilon$ leaving $\mathcal{D} = (\alpha_1, \alpha_2) \ni 0$ ?

Solve the (one-dimensional) Dirichlet problem

$$\begin{cases} \mathcal{L}^\varepsilon w = 0 & \text{in } \mathcal{D} \\ w = f & \text{on } \partial\mathcal{D} \end{cases} \qquad \text{with} \qquad f(x) = \begin{cases} 1 & \text{for } x = \alpha_1 \\ 0 & \text{for } x = \alpha_2 \end{cases}$$

$$w(x) = \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = \mathbb{E}_x f(x^\varepsilon_{\tau^\varepsilon}) = \int_x^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\,\mathrm{d}y \Big/ \int_{\alpha_1}^{\alpha_2} \mathrm{e}^{2U(y)/\varepsilon}\,\mathrm{d}y$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = 1 \qquad \text{if } U(\alpha_1) < U(\alpha_2)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = 0 \qquad \text{if } U(\alpha_2) < U(\alpha_1)$$

$$\lim_{\varepsilon\to 0} \mathbb{P}_x\{x^\varepsilon_{\tau^\varepsilon} = \alpha_1\} = \frac{1}{|U'(\alpha_1)|} \Big/ \left(\frac{1}{|U'(\alpha_1)|} + \frac{1}{|U'(\alpha_2)|}\right) \qquad \text{if } U(\alpha_1) = U(\alpha_2)$$

**Corollary** (to LDP for $x^\varepsilon$)

$$\lim_{\varepsilon\to0} \varepsilon \log \mathbb{P}_x\bigl\{\tau^\varepsilon \leqslant t\bigr\} = -\inf\bigl\{V(x,y;s) \colon s\in[0,t],\ y\notin\mathcal{D}\bigr\},$$

where

$$\begin{aligned} V(x,y;s) &= \inf\bigl\{J_{[0,s],x}(\varphi)\colon \varphi\in\mathcal{C}([0,s],\mathbb{R}^d),\ \varphi_0=x,\ \varphi_s=y\bigr\} \\ &= \inf\biggl\{\frac12\int_0^s \|h_u\|^2\,\mathrm{d}u : h\in\mathcal{L}_2([0,s],\mathbb{R}^d) \text{ such that} \\ &\qquad \varphi_v = x+\int_0^v b(\varphi_u)\,\mathrm{d}u + \int_0^v g(\varphi_u)h_u\,\mathrm{d}u,\ \ v\in[0,s], \text{ and } \varphi_s=y\biggr\} \\ &= \text{cost of forcing a path to connect } x \text{ and } y \text{ in time } s \end{aligned}$$

**Remarks**

- ▷ Upper and lower LDP bounds coincide $\implies$ limit exists
- ▷ Calculation of asymptotical behaviour reduces to variational problem
- ▷ $V(x,y;s)$ is solution to a Hamilton–Jacobi equation; extremals solution to an Euler equation

## Diffusion exit from a domain: Assumptions and the concept of quasipotentials

### Assumptions

- ▷ $\dot{x}_t = b(x_t)$ has a unique stable equilibrium point $x^* = 0$ in $\mathcal{D}$, $x^\star$ is asymptotically stable
- ▷ $\overline{\mathcal{D}}$ is contained in the basin of attraction of $x^* = 0$ (for the deterministic dynamics)
- ▷ $\overline{V} = \inf_{z \in \partial D} V(0, z) < \infty$

with quasipotential

$$V(0, y) = \inf_{t>0} V(0, y; t) = \text{cost of forcing a path starting in } x^\star = 0 \text{ to reach } y \text{ eventually}$$

### Remarks

- ▷ Similar if $\mathcal{D}$ contains for instance a stable periodic orbit
- ▷ Conditions exclude characteristic boundary
- ▷ Uniform-ellipticity condition can be relaxed; requires additional controllability condition
- ▷ Were $\overline{V} = \infty$, all possible exit points would be equally unlikely
- ▷ If $b$ derives from a potential $U$, $g = \mathrm{Id}$:
  Quasipotential satisfies $V(0, y) = 2\left[U(y) - U(0)\right]$ for all $y \in \overline{\mathcal{D}}$ such that $U(y) \leqslant \min_{\partial D} U$

**Arrhenius law:** For $b$ deriving from a potential, $g = \mathrm{Id}$

*The average time to leave potential well is* $\exp\{$*twice the barrier height* / *noise intensity*$\}$

## Diffusion exit from a domain: Main results

**Theorem**

For all initial conditions $x \in \mathcal{D}$ and all $\delta > 0$

- **First-exit time:**

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \Big\{ \exp\{(\overline{V} - \delta)/\varepsilon\} < \tau^\varepsilon < \exp\{(\overline{V} + \delta)/\varepsilon\} \Big\} = 1$$

  and

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x\{\tau^\varepsilon\} = \overline{V}$$

- **First-exit location:** For any closed subset $N \subset \partial\mathcal{D}$ satisfying $\inf_{z \in N} V(0, z) > \overline{V}$

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \{ x^\varepsilon_{\tau^\varepsilon} \in N \} = 0$$

  If $V(0, \cdot)$ has a unique minimum $z^*$ on $\partial\mathcal{D}$, then

$$\lim_{\varepsilon \to 0} \mathbb{P}_x \{ \|x^\varepsilon_{\tau^\varepsilon} - z^*\| < \delta \} = 1$$

**Remarks**

- $x^\varepsilon$ favours exit near boundary points where $V(0, \cdot)$ is minimal
- If $V(0, \cdot)$ has multiple minima on $\partial\mathcal{D}$: corresponding weights cannot be obtained by large-deviation techniques

## Diffusion exit from a domain: Idea of the proof

### First step

$x^\varepsilon$ cannot remain in $\mathcal{D}$ arbitrarily long without hitting a small neighbourhood $B(0,\mu)$ of $0$:

$$\forall\,\mu \qquad \lim_{t\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in\mathcal{D}} \mathbb{P}_x\Big\{ x_s^\varepsilon \in \mathcal{D}\backslash B(0,\mu) \text{ for all } s \leqslant t \Big\} = -\infty$$

$\Longrightarrow$ Restrict to initial conditions in $B(0,\mu)$

### Second step

Lower bound on probability to leave $\mathcal{D}$:

$$\forall\,\eta>0\ \exists\,\mu_0\ \forall\,\mu<\mu_0\ \exists\,T_0>0 \qquad \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{x\in B(0,\mu)} \mathbb{P}_x\big\{\tau^\varepsilon \leqslant T_0\big\} > -(\overline{V}+\eta).$$

- ▷ Construct piecewise a continuous exit path $\varphi$ connecting $x$, $0$, $\partial\mathcal{D}$ and some point $y$ at distance $\mu$ from $\overline{\mathcal{D}}$ with $I(\varphi) \leqslant \overline{V} + \eta$
- ▷ Use LDP to estimate probability of $x^\varepsilon$ remaining in $\mu/2$-neighbourhood of exit path

### Third step

More lemmas in the same spirit ... (involving exit locations)

### Fourth step

Prove Theorem by considering successive attempts to leave $\mathcal{D}$ using strong Markov property

## The end: References


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