Blatt 1. Abgabe bis 24.10.2025

5. Let M be a C-manifold of dimension n. Let V be a chart on M and E be a subset of V. The compact inclusion $E \Subset V$ can be understood in two ways: in the sense of the topology of M as well as in the sense of the topology of $\mathbb{R}^n$, when identifying V with a subset of $\mathbb{R}^n$. Prove that these two meanings of $E \Subset V$ are equivalent.

6. Prove that, on any C-manifold M, there is a countable *locally finite* family of relatively compact charts covering M.

   Remark. A family $\mathcal{F}$ of subsets of M is called locally finite if any compact subset of M intersects only finitely many sets from $\mathcal{F}$.

7. Fix some positive integers n, m, let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a C^1-function. Consider the null set of F, that is, the set

$$M = \left\{x \in \mathbb{R}^{n+m} : F(x) = 0\right\},$$

   and assume that, for any point $x \in M$, the Jacobi matrix $F'(x)$ has the rank m. Prove that M is a C-manifold of dimension n.

   Hint. Use the implicit function theorem.

8. Let K be a compact subset of a smooth manifold M and $\{U_j\}_{j=1}^k$ be a finite family of open sets covering K. Prove that there exist non-negative functions $\varphi_j \in C_0^\infty(U_j)$ such that $\sum_{j=1}^k \varphi_j \equiv 1$ in an open neighbourhood of K and $\sum_{j=1}^k \varphi_j \leq 1$ in M.

   Remark. The family $\{\varphi_j\}$ is called a partition of unity at K subordinate to $\{U_j\}$. If all U_j are charts then the existence of the partition of unity was proved in lectures.

   Hint. Choose first a finite family $\{W_i\}$ of charts covering K and such that each W_i is contained in one of the sets U_j. By a theorem from lectures, there exists a partition of unity $\{\psi_i\}$ of K subordinate to $\{W_i\}$. Use functions ψ_i to construct functions φ_j.