Blatt 10. Abgabe bis 09.01.2026

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Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

54. Let $(M, \mathbf{g})$ be a Riemannian manifold of dimension n. Let $F : M \to \mathbb{R}$ be a smooth function on M such that F is non-singular[1] on the null set $S = \{x \in M : F(x) = 0\}$. In particular, S is a submanifold of dimension $n-1$.

   (a) Prove that, at any point $p \in S$, the gradient $\nabla F(p)$ is orthogonal to T_pS in the tangent space T_pM.
   Hint. Use Exercise 19.

   (b) Consider the set
   $$\Omega := \{x \in M : F(x) < 0\} \tag{29}$$
   and prove that $S = \partial\Omega$.
   Remark. An open set $\Omega \subset M$ is called a *region* if it can be represented in the form (29), where F is a smooth function on M that is non-singular on its null set.

55. Let H be the semi-hyperbola
$$H = \left\{(x_1, x_2) \in \mathbb{R}^2 : \ x_2^2 - x_1^2 = 1, \ x_2 > 0\right\}.$$
For any $s > 0$, consider the following subset of H:
$$H_s = \{(x_1, x_2) \in H : \ 0 < x_1 < s\}.$$
Let ν be the Riemannian measure of $(H, \mathbf{g}_H)$, where $\mathbf{g}_H$ is the hyperbolic metric on H. Prove that
$$\nu(H_s) = \ln\left(s + \sqrt{s^2+1}\right).$$
Remark. Note that the function $\ln\left(s + \sqrt{s^2+1}\right)$ is the inverse to sinh.
Hint. Use the chart on H with the coordinate y from Exercise 36.

56. For any two-dimensional Riemannian manifold $(M, \mathbf{g})$, the Gauss curvature $K_{\mathbf{g}}(x)$ is defined in a certain way as a function on M. It is known that if the metric $\mathbf{g}$ has in coordinates x^1, x^2 the form
$$\mathbf{g} = \frac{(dx^1)^2 + (dx^2)^2}{f^2(x)}, \tag{30}$$
where f is a smooth positive function, then the Gauss curvature can be computed in this chart as follows
$$K_{\mathbf{g}} = f^2 \Delta \ln f, \tag{31}$$
where $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$ is the Euclidean two-dimensional Laplace operator in the coordinates x^1, x^2.

[1] Recall that F is non-singular on a set S if $dF(x) \neq 0$ at any point $x \in S$.

(a) Compute the Gauss curvature of $\mathbb{R}^2$ and the catenoid Cat (see Exercise 34).

(b) Let $(M, \mathbf{g})$ be a two-dimensional model manifold with the profile function ψ, so that in the polar coordinates (r, θ)

$$\mathbf{g} = dr^2 + \psi^2(r)\, d\theta^2. \tag{32}$$

Prove that

$$K_{\mathbf{g}} = -\frac{\psi''(r)}{\psi(r)}. \tag{33}$$

Hint. Find other coordinates (ρ, θ) on M where the metric (32) has the form

$$\mathbf{g} = \frac{d\rho^2 + d\theta^2}{f^2(\rho)},$$

and then use (31).

(c) Using (33), compute the Gauss curvature of the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2$, and the two-dimensional pseudosphere PS from Exercise 44b.

57. Let $\mathbf{g}$ be the metric (30) on a two-dimensional manifold M. Consider the metric $\widetilde{\mathbf{g}} = \frac{1}{h^2}\mathbf{g}$ where h is a smooth positive function on M. Prove that

$$K_{\widetilde{\mathbf{g}}} = (K_{\mathbf{g}} + \Delta_{\mathbf{g}} \log h)\, h^2,$$

where $\Delta_{\mathbf{g}}$ is the Laplace-Beltrami operator of the metric $\mathbf{g}$.

58. * Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metric tensors on a smooth n-dimensional manifold M. Assume that, for some constant C,

$$\widetilde{\mathbf{g}} \leq C\mathbf{g}, \tag{34}$$

that is, for all $x \in M$ and $\xi \in T_xM$,

$$\widetilde{\mathbf{g}}(x)(\xi, \xi) \leq C\mathbf{g}(x)(\xi, \xi). \tag{35}$$

(a) Prove that if ν and $\widetilde{\nu}$ are the Riemannian measures of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, then

$$\frac{d\widetilde{\nu}}{d\nu} \leq C^{n/2}.$$

(b) Prove that, for any smooth function f on M,

$$|\nabla f|^2_{\mathbf{g}} \leq C\, |\nabla f|^2_{\widetilde{\mathbf{g}}}.$$

Hint. Fix $x_0 \in M$ and consider $T_{x_0}M$ as a Euclidean space with the inner product $\mathbf{g}$. Since $\widetilde{\mathbf{g}}$ is a symmetric bilinear form in this space, there exists a $\mathbf{g}$-orthonormal basis $\{e_1, ..., e_n\}$ in $T_{x_0}M$ in which $\widetilde{\mathbf{g}}$ has a diagonal form, that is, $(\widetilde{g_{ij}}) = \operatorname{diag}\{\alpha_1, ..., \alpha_n\}$ with some reals α_i. By a linear change of coordinates in a neighborhood of x_0, you can assume that $\frac{\partial}{\partial x^i} = e_i$. For (a) note also that, by Exercise 23, the ratio $\frac{\det \widetilde{g}(x_0)}{\det g(x_0)}$ does not depend on the choice of local coordinates.

59. * Consider two Riemannian manifolds $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$. Let us define a Riemannian metric tensor $\mathbf{g}$ on the product manifold $M = X \times Y$ as follows

$$\mathbf{g} = \mathbf{g}_X + \psi^2(x)\,\mathbf{g}_Y, \tag{36}$$

where ψ is a smooth positive function on X. The Riemannian manifold $(M, \mathbf{g})$ with this metric is called a *warped product* of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$ with profile ψ.

(a) Prove that the Riemannian measure $\nu_{\mathbf{g}}$ of the metric (36) is given by

$$d\nu_{\mathbf{g}} = \psi^m(x)\,d\nu_X d\nu_Y, \tag{37}$$

where ν_X and ν_Y are the Riemannian measures of $(X, \mathbf{g}_X)$ and $(Y, \mathbf{g}_Y)$, respectively, and $m = \dim Y$.

(b) Prove that the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ of the metric (36) is given by

$$\Delta_{\mathbf{g}} f = \Delta_X f + m\langle \nabla_X \ln \psi, \nabla_X f\rangle_{\mathbf{g}_X} + \frac{1}{\psi^2(x)}\Delta_Y f, \tag{38}$$

where ∇_X is gradient on X and Δ_X, Δ_Y are the Laplace-Beltrami operators on X and Y, respectively.

60. ** Let X, Y be smooth manifolds of the same dimension n and let $\Phi : Y \to X$ be a diffeomorphism. Let S be a submanifold of Y, and set $R = \Phi(S)$.

(a) Prove that R is a submanifold of X and that $\Psi := \Phi|_S$ is a diffeomorphism of S onto R.

(b) Prove that, for any $y \in S$,

$$d\Phi|_{T_yS} = d\Psi$$

(that is, for any $\xi \in T_yS$, we have $d\Phi\xi = d\Psi\xi$).

(c) Let $\mathbf{g}(x)$ be a bilinear form on any space T_xX (for example, a Riemannian metric) and consider the induced form $\mathbf{g}_R := \mathbf{g}|_R$. Prove that

$$(\Phi_*\mathbf{g})_S = \Psi_*(\mathbf{g}_R).$$

(d) Prove that if X and Y are Riemannian manifolds and Φ is a Riemannian isometry of Y and X then Ψ is a Riemannian isometry of the Riemannian manifolds S and R with the induced metrics.

61. ** Fix a point a on a Riemannian manifold $(M, \mathbf{g})$ and consider on M the function $\rho(x) = d(x, a)$. Assume that ρ is finite and smooth in a neighborhood of a point $b \in M \setminus \{a\}$. The purpose of this Exercise is to prove that

$$|\nabla \rho(b)|_{\mathbf{g}} \le 1. \tag{39}$$

(a) Let $\gamma : [0, \varepsilon] \to M$ be a smooth path on M such that $\gamma(0) = b$ and $\dot\gamma(0) = \xi \in T_bM$. Prove that

$$\frac{d}{dt}(\rho(\gamma(t)))\bigg|_{t=0} \le |\xi|_{\mathbf{g}}. \tag{40}$$

Hint. Use the definition of the geodesic distance d and the triangle inequality.

(b) Prove (39).

Hint. It suffices to prove that, for any $\xi \in T_b M$,

$$\langle \nabla \rho(b), \xi \rangle_{\mathbf{g}} \leq |\xi|_{\mathbf{g}}. \tag{41}$$

Use (40) to prove (41).

62. ** Consider the Riemannian manifold $(\mathbb{R}^n_+, \mathbf{g})$ where

$$\mathbf{g} = \frac{(dx^1)^2 + ... + (dx^n)^2}{(x^n)^2}.$$

Prove that $(\mathbb{R}^n_+, \mathbf{g})$ is isometric to the hyperbolic space $\mathbb{H}^n$.

Remark. This manifold $(\mathbb{R}^n_+, \mathbf{g})$ is called the *Poincaré half-space model* of the hyperbolic space.

Hint. By Exercise 39, $\mathbb{H}^n$ is isometric to the *Poincaré ball*, that is, the unit ball

$$\mathbb{B}^n = \{y \in \mathbb{R}^n : |y| < 1\}$$

with the metric

$$\mathbf{g}_{\mathbb{B}^n} = 4\frac{(dy^1)^2 + ... + (dy^n)^2}{(1 - |y|^2)^2}.$$

Set $p = (0, ..., 0, 1) \in \mathbb{R}^n$ and consider the mapping $\Phi : \mathbb{R}^n \setminus \{-p\} \to \mathbb{R}^n$ given by

$$\Phi(y) = \frac{2(y+p)}{|y+p|^2} - p$$

(in fact, Φ is the inversion in the sphere of radius 2 centered at $-p$). Prove that Φ is a diffeomorphism of $\mathbb{R}^n \setminus \{-p\}$ onto itself, and that $y \in \mathbb{B}^n \Leftrightarrow x = \Phi(y) \in \mathbb{R}^n_+$. Conclude that Φ is a diffeomorphism of $\mathbb{B}^n$ onto $\mathbb{R}^n_+$. Then prove that Φ is isometry, that is, $\Phi_* \mathbf{g} = \mathbf{g}_{\mathbb{B}^n}$.

63. ** Fix a real α and consider the mapping $x = Q(y)$ of $\mathbb{R}^{n+1}$ onto itself given by

$$\begin{aligned} x^1 &= y^1 \\ &\vdots \\ x^{n-1} &= y^{n-1} \\ x^n &= y^n \cosh \alpha + y^{n+1} \sinh \alpha \\ x^{n+1} &= y^n \sinh \alpha + y^{n+1} \cosh \alpha. \end{aligned} \tag{42}$$

The mapping Q is called a *hyperbolic rotation* or the *Lorentz transformation*[2].

[2] Assuming $n = 1$ and denoting $x = x^1$, $t = x^2$, $x' = y^1, t' = y^2$, we obtain from (42)

$$x = \frac{x' + vt'}{\sqrt{1 - v^2}}, \quad t = \frac{t' + vx'}{\sqrt{1 - v^2}}$$

where $v = \tanh \alpha$. These are classical Lorentz transformations in the 2-dimensional space-time that describe in the Relativity Theory the change of coordinates in the inertial frame (x', t') moving at a speed v with respect to the frame (x, t). Note that $v < 1$ where 1 is the speed of light.

(a) Prove that Q is an isometry of $\mathbb{R}^{n+1}$ with respect to the Minkowski metric

$$\mathbf{g}_{Mink} = \left(dx^1\right)^2 + ... + \left(dx^n\right)^2 - \left(dx^{n+1}\right)^2.$$

(b) Prove that Q maps $\mathbb{H}^n$ onto itself. Prove that the restriction of Q to $\mathbb{H}^n$ is a Riemannian isometry of $(\mathbb{H}^n, \mathbf{g}_{\mathbb{H}^n})$.

Hint. Recall that the hyperbolic space $\mathbb{H}^n$ is defined as the hyperboloid

$$\left(y^1\right)^2 + ... + \left(y^n\right)^2 - \left(y^{n+1}\right)^2 = -1, \quad y^{n+1} > 0,$$

with the metric tensor $\mathbf{g}_{\mathbb{H}^n} = \mathbf{g}_{Mink}|_{\mathbb{H}^n}$.

64. ** We are concerned here with Riemannian isometries of $\mathbb{H}^n$.

(a) Prove that, for any point $a \in \mathbb{H}^n$, there exists a Riemannian isometry

$$\Phi : \mathbb{H}^n \to \mathbb{H}^n$$

such that $\Phi(a) = p$ where $p = (0, .., 0, 1) \in \mathbb{R}^{n+1}$ is the pole of $\mathbb{H}^n$.

(b) Prove that, for any four points $a, b, a', b' \in \mathbb{H}^n$ such that

$$d(a', b') = d(a, b), \tag{43}$$

there exists a Riemannian isometry Φ of $\mathbb{H}^n$ such that $\Phi(a') = a$ and $\Phi(b') = b$.

Hint. Use the hyperbolic rotation of Exercise 63.

65. ** Consider the weighted manifold $(\mathbb{R}, \mathbf{g}, \mu)$ where $\mathbf{g} = \mathbf{g}_{\mathbb{R}^n}$ is the canonical Euclidean metric and $d\mu = e^{-x^2}dx$. Consider also the corresponding weighted Laplace operator $\Delta_{\mathbf{g},\mu}$. Prove that the *Hermite polynomial*

$$h_k(x) = e^{x^2}\frac{d^k}{dx^k}e^{-x^2}$$

of degree k (where k is a non-negative integer) satisfies the equation

$$\Delta_{\mathbf{g},\boldsymbol{\mu}} h_k + 2k h_k = 0.$$

That is, h_k is an eigenfunction of $\Delta_{\mathbf{g},\mu}$.

Hint. Show first that the function $g(x) = e^{-x^2}$ satisfies the equation

$$\frac{d^{k+2}}{dx^{k+2}}g + 2x\frac{d^{k+1}}{dx^{k+1}}g + (2k+2)\frac{d^k}{dx^k}g = 0. \tag{44}$$