Blatt 13. Abgabe bis 30.01.2026

Die mit *markierten Aufgaben sind zusätzlich und werden korrigiert

In all questions, $(M, \mathbf{g}, \mu)$ is a weighted manifold, Ω is a precompact open subset of M, $\{\lambda_k(\Omega)\}_{k=1}^{\infty}$ is the sequence of the Dirichlet eigenvalues of $\Delta = \Delta_{\mathbf{g},\mu}$ in Ω in the increasing order, and $\{v_k\}$ is the sequence of the corresponding eigenfunctions that forms an orthonormal basis in $L^2(\Omega)$.

82. Let M be a compact connected manifold. Set $\Omega = M$. Prove that $\lambda_1(\Omega) = 0$ and $\lambda_2(\Omega) > 0$. In other words, 0 is a simple eigenvalue of Δ in Ω.

    Hint. You need to prove that if v is an eigenfunction of Δ in Ω with the eigenvalue 0 then $v = \text{const}$.

83. Recall that the *Rayleigh quotient* of a non-zero function $u \in W^1(\Omega)$ is defined by

$$\mathcal{R}(u) := \frac{\|\nabla u\|_{\vec{L}^2}^2}{\|u\|_{L^2}^2}.$$

Assume that, for a function $f \in W_0^1(\Omega) \setminus \{0\}$,

$$\mathcal{R}(f) = \lambda_1(\Omega). \tag{59}$$

Prove that f is the eigenfunction of Δ in Ω with the eigenvalue $\lambda_1(\Omega)$.

Remark. We know that if v is an eigenfunction of Δ in Ω with the eigenvalue λ then $\mathcal{R}(v) = \lambda$. The above claim says that the converse statement is also true provided $\lambda = \lambda_1(\Omega)$ (but it is not true for higher eigenvalues).

Hint. Set $\lambda = \lambda_1(\Omega)$ and recall that by Exercise 74,

$$\lambda = \inf_{u \in W_0^1(\Omega)\setminus\{0\}} \mathcal{R}(u).$$

Hence, for any $\varphi \in W_0^1(\Omega)$ and any $t \in \mathbb{R}$, we have

$$\mathcal{R}(f + t\varphi) \geq \lambda = \mathcal{R}(f).$$

Use this inequality with $t \to 0$ to deduce that $(\nabla f, \nabla \varphi) - \lambda(f, \varphi) = 0$, which will imply the claim.

84. Let M be a compact connected manifold and $\Omega = M$. Let u be a solution of the mixed problem for the heat equation in $\mathbb{R}_+ \times \Omega$ with the initial function $f \in L^2(\Omega)$. Prove that, for any $t > 0$,

$$\int_\Omega u(t,\cdot)d\mu = \int_\Omega f d\mu. \tag{60}$$

Hint. Use Exercise 82 and expansions of f and $u(t,\cdot)$ in the eigenfunction basis $\{v_k\}_{k=1}^{\infty}$.

85. *(Product rule for L^2-derivatives)* Let I be an interval in $\mathbb{R}$ and $u(t), v(t) : I \to L^2(\Omega)$ be differentiable paths. Prove that

$$\frac{d}{dt}(u, v) = (u, \frac{dv}{dt}) + (\frac{du}{dt}, v),$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$.

86. * (*Chain rule for L^2-derivatives*) Let $u(t) : I \to L^2(\Omega)$ be a differentiable path. Consider a function $\psi \in C^1(\mathbb{R})$ such that

$$\psi(0) = 0 \text{ and } \sup |\psi'| < \infty. \tag{61}$$

Prove that the path $\psi(u(t))$ is also differentiable in $t \in I$ and

$$\frac{d\psi(u)}{dt} = \psi'(u) \frac{du}{dt}.$$