# Blatt 2. Abgabe bis 31.10.2025

In all exercises, $M$ is a smooth manifold of dimension $n$.

9. A *path* on $M$ is any smooth mapping $\gamma : [0, a] \to M$, where $a > 0$. Set $x = \gamma(0)$. For any function $f \in C^\infty(M)$, define the derivative of $f$ along the path $\gamma$ at the point $x$ by
$$\frac{\partial f}{\partial \gamma} := \frac{d}{dt} f(\gamma(t))\Big|_{t=0}.$$
   (a) Prove that $\frac{\partial}{\partial\gamma}$ is an $\mathbb{R}$-differentiation at $x$, that is, $\frac{\partial}{\partial\gamma} \in T_xM$.
   (b) Prove that any tangent vector $\xi \in T_xM$ can be represented in the form $\xi = \frac{\partial}{\partial\gamma}$ for some path $\gamma$.

10. A smooth *vector field* on $M$ is a mapping $X : C^\infty(M) \to C^\infty(M)$ such that, for any $x \in M$, the mapping
$$\begin{aligned} C^\infty(M) &\to \mathbb{R} \\ f &\mapsto X(f)(x) \end{aligned}$$
is a $\mathbb{R}$-differentiation at $x$. Prove that, in any chart $U$ with the local coordinates $x^1, ..., x^n$, there are functions $a^1, ..., a^n \in C^\infty(U)$ such that
$$X(f) = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} \text{ for any } f \in C^\infty(M).$$
*Hint.* Use the fact that any $\mathbb{R}$-differentiation $\xi$ can be represented in the form
$$\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$$
for some $\xi^i \in \mathbb{R}$.

11. Let $X$ and $Y$ be two smooth vector fields on $M$ (as in Exercise 10). Define the *Lie bracket* $[X, Y]$ of $X, Y$ as a mapping of $C^\infty(M)$ into itself by
$$[X, Y] := XY - YX,$$
that is, $[X, Y](f) = X(Y(f)) - Y(X(f))$ for any $f \in C^\infty(M)$.
Prove that $[X, Y]$ is a smooth vector field on $M$.
*Hint.* In the local coordinates, $X(f)$ is a combination of the first partial derivatives $\frac{\partial f}{\partial x^i}$ (by Exercise 10). Hence, $XY(f)$ and $YX(f)$ contain the second derivatives of $f$. The point of the present claim is that the difference $XY(f) - YX(f)$ depends on the first derivatives of $f$ only, that is, the second derivatives cancel out.

12. *(The Jacobi identity)* Prove the following identity for three smooth vector fields $X, Y, Z$ on a smooth manifold $M$:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \tag{1}$$

where $[\cdot, \cdot]$ is the Lie bracket defined in Exercise 11,

*Hint.* By linearity, it suffices to consider the case when $X, Y, Z$ are given in the local coordinates $x^1, ..., x^n$ by

$$X = a\frac{\partial}{\partial x^i}, \quad Y = b\frac{\partial}{\partial x^j}, \quad Z = c\frac{\partial}{\partial x^k},$$

where $a, b, c$ are smooth functions of $x^1, ..., x^n$ and $i, j, k$ are some indices from $1, ..., n$.