Blatt 3. Abgabe bis 07.11.2025

13. Let $\{V_\alpha\}$ be a family of charts covering a smooth manifold M. Prove that if a function $f: M \to \mathbb{R}$ belongs to $C^\infty(V_\alpha)$ for any α then $f \in C^\infty(M)$.

    Remark. By definition, $f \in C^\infty(M)$ if $f \in C^\infty(U)$ for *any* chart U in M.

14. Prove that a smooth hypersurface in $\mathbb{R}^{n+1}$ is a smooth n-dimensional manifold.

    Remark. Recall that a smooth hypersurface is a subset M of $\mathbb{R}^{n+1}$ that is locally a graph of a smooth function. Each graph gives rise to a chart on M. You need to prove that the change of coordinates between any two of such charts is given by smooth functions.

15. (a) Let U be an open set in $\mathbb{R}^n$ and $\Psi: U \to \mathbb{R}^m$ be a smooth mapping. Let Γ be the graoh of Ψ, that is,
    $$\Gamma = \left\{(x, \Psi(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\right\}.$$
    Prove that Γ is a submanifold of $\mathbb{R}^{n+m}$ of dimension n.

    (b) Prove that any smooth hypersurface in $\mathbb{R}^{n+1}$ in a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

    Hint. Use the definition of a submanifold.

16. Let M be a smooth manifold of dimension n and S be its submanifold of dimension m. Let $x^1, ..., x^n$ be local coordinates in a chart U in M and $y^1, ..., y^m$ be local coordinates in a chart V on S. Assume that $V \subset U$. Then, for any point in V, its x-coordinates can be expressed as functions of its y-coordinates:
    $$x^i = f^i(y^1, ..., y^m), \quad i = 1, ..., n,$$
    where f^i are some real-valued functions on V. Prove that $f^i \in C^\infty(V)$.

    Hint. Use the definition of a submanifold.

17. * Let X and Y be smooth manifolds of dimensions n and m, respectively, with $n \geq m$. A mapping $\Phi: Y \to X$ is called smooth if in local coordinates $x^1, ..., x^n$ in X and $y^1, ..., y^m$ in Y it is given by equations
    $$x^i = \Phi^i(y^1, ..., y^m), \ i = 1, ..., n,$$
    where Φ^i are smooth functions. Let Φ be a smooth mapping as above satisfying the following three properties:

    (1) the mapping $\Phi: Y \to X$ is injective;

    (2) the rank of the Jacobi matrix $J = \left(\frac{\partial \Phi^i}{\partial y^j}\right)$ of Φ is maximal at all points, that is, it is equal to m;

    (3) Φ is a homeomorphism of Y onto its image $S := \Phi(Y) \subset X$.

    Prove that S is a submanifold of X of dimension m.

18. * Give examples to show that any of the above conditions $(1), (2), (3)$ is essential for the statement of Exercise 17.