Blatt 4. Abgabe bis 14.11.2025

19. Let M be a smooth manifold of dimension n, $F \in C^\infty(M)$ and S be a non-singular null set of F, that is,

$$S = \{x \in M : F(x) = 0\} \text{ and } \nabla F \neq 0 \text{ on } S.$$

Consequently, S is a submanifold of M of dimension $n-1$. Fix $x_0 \in S$. Every tangent vector $\xi \in T_{x_0}S$ can be regarded as an element of $T_{x_0}M$ by using the identity

$$\xi(f) := \xi(f|_S) \quad \text{for any } f \in C^\infty(M),$$

as the restriction $f|_S$ on S is a smooth function on S. Hence, the tangent space $T_{x_0}S$ is a subspace of $T_{x_0}M$. Prove that $T_{x_0}S$ as a subspace of $T_{x_0}M$ is given by the equation

$$T_{x_0}S = \{\xi \in T_{x_0}M : \langle dF, \xi \rangle = 0\}. \tag{2}$$

Hint. Verify first that every $\xi \in T_{x_0}S$ satisfies as an element of $T_{x_0}M$ the equation $\langle dF, \xi \rangle = 0$.

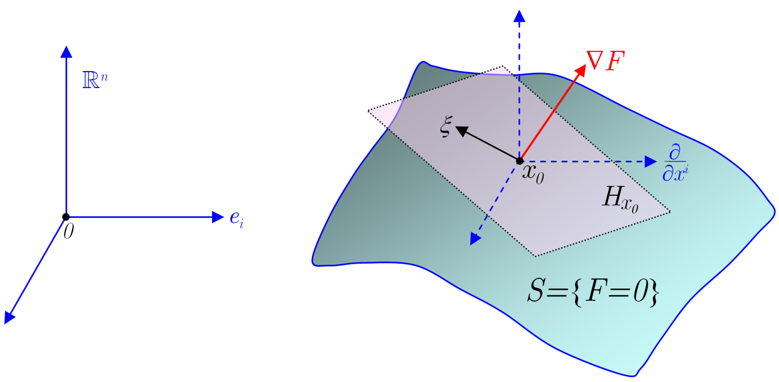
20. * In the setting of Exercise 19, let $M = \mathbb{R}^n$. Let us identify the tangent space $T_{x_0}M$ with $\mathbb{R}^n$ by using the isomorphism $I : T_{x_0}M \to \mathbb{R}^n$ defined by

$$I\left(\frac{\partial}{\partial x^i}\right) = e_i,$$

where $\{e_i\}_{i=1}^n$ is the canonical basis in $\mathbb{R}^n$. Prove that the set

$$x_0 + I(T_{x_0}S)$$

is the hyperplane H_{x_0} in $\mathbb{R}^n$ that goes through x_0 and has the normal $\nabla F(x_0)$, where $\nabla F = \left(\frac{\partial F}{\partial x^1}, ..., \frac{\partial F}{\partial x^n}\right)$.



Remark. This result means that the tangent space $T_{x_0}S$ can be naturally identified with the tangent hyperplane H_{x_0} in $\mathbb{R}^n$ to the hypersurface S at the point x_0.

21. Let M be a Riemannian manifold.

(a) Prove the product rule for the operators d and ∇ on M:

$$d\,(uv) = udv + vdu \tag{3}$$

and

$$\nabla\,(uv) = u\nabla v + v\nabla u, \tag{4}$$

where u and v are smooth function on M.

(b) Prove the chain rule for the operators d and ∇ on M:

$$df\,(u) = f'\,(u)\,du$$

and

$$\nabla f\,(u) = f'\,(u)\,\nabla u,$$

where u and f are smooth functions on M and $\mathbb{R}$, respectively.

22. Let $(M, \mathbf{g})$ be a Riemannian manifold. Let U and V be charts on M with the local coordinates $x^1, ..., x^n$ and $y^1, ..., y^n$, respectively. Denote by g^x and g^y the matrices of the metric $\mathbf{g}$ in U and V, respectively. Let $J = \left(J_i^k\right)_{k,i=1}^n$ be the Jacobian matrix of the change $y = y\,(x)$ defined in $U \cap V$ by

$$J_i^k = \frac{\partial y^k}{\partial x^i}, \tag{5}$$

where k is the row index and i is the column index. Prove the following identity in $U \cap V$:

$$g^x = J^T g^y J, \tag{6}$$

where J^T denotes the transposed matrix.

23. Let $\mathbf{g}$, $\widetilde{\mathbf{g}}$ be two Riemannian metrics on a smooth manifold M and let g^x and $\widetilde{g}^x$ be the matrices of $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, in some local coordinate system $x^1, ..., x^n$. Prove that the ratio

$$\frac{\det \widetilde{g}^x}{\det g^x}$$

does not depend on the choice of the coordinate system (although separately $\det g^x$ and $\det \widetilde{g}^x$ do depend on the coordinate system).

Hint. Use the formula (6) from Exercise 22.