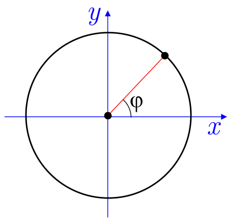
Blatt 5. Abgabe bis 21.11.2025

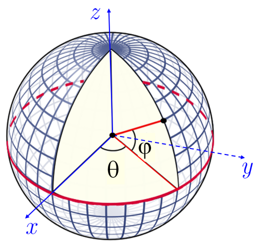
24. Let M be a smooth manifold, S be a submanifold, and $x \in S$. Prove that, for any $f \in C^\infty(M)$,
$$d\left(f|_S\right) = (df)\,|_{T_xS}, \tag{7}$$
where d in the left hand side is differential on S, while d in the right hand side is differential on M, and $(df)\,|_{T_sS}$ means the restriction of df to the tangent space T_xS.

25. For any submanifold S of $\mathbb{R}^n$, denote by $\mathbf{g}_S$ the Riemannian metric on S that is induced by the canonical Euclidean metric
$$\mathbf{g}_{\mathbb{R}^n} = \left(dx^1\right)^2 + ... + \left(dx^n\right)^2. \tag{8}$$

(a) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$. Express the induced metric $\mathbf{g}_{\mathbb{S}^1}$ using the polar angle φ on $\mathbb{S}^1$ as a local coordinate.

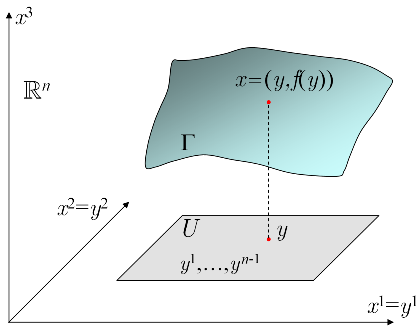


(b) Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Express the induced metric $\mathbf{g}_{\mathbb{S}^2}$ on $\mathbb{S}^2$ in terms of the local coordinates θ, φ where θ is the longitude on $\mathbb{S}^2$ and φ is the latitude.



Hint. Express the Cartesian coordinates in terms of the polar coordinates and use the representation (8) of the metric in the Cartesian coordinates.

26. Let Γ be the graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n-1}$ is an open set. Let $\mathbf{g}$ be the canonical metric in $\mathbb{R}^n$. Denote by $\mathbf{g}_\Gamma$ the induced Riemannian metric on Γ considering Γ as a submanifold of $\mathbb{R}^n$. Let $y^1, ..., y^{n-1}$ be the Cartesian coordinates in U; consider them as local coordinates in Γ. Prove that the components of the metric $\mathbf{g}_\Gamma$ in the coordinates $y^1, ..., y^{n-1}$ are as follows:



$$\left(g_\Gamma\right)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{9}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Hint. Use the following result from lectures: if S is a submanifold of a Riemannian manifold $(M, \mathbf{g})$ then the induced metric $\mathbf{g}_S$ is given in the local coordinates $x^1, ..., x^n$ on M and $y^1, ..., y^m$ on S by the formula

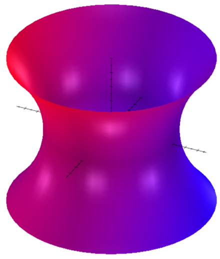
$$(g_S)_{ij} = g_{kl} \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j}. \tag{10}$$

27. A catenoid Cat is a surface in $\mathbb{R}^3$ that is given by the parametric equations

$$x^1 = \cosh \rho \cos \theta, \quad x^2 = \cosh \rho \sin \theta, \quad x^3 = \rho,$$

where $\rho \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$.

Express the induced Riemannian metric on Cat in terms of the coordinates ρ, θ.



Catenoid

Remark. The catenoid Cat is the image of the mapping $\mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ given by the above equations. By using Exercise 17, it is possible to show that Cat is a submanifold of $\mathbb{R}^3$ of dimension 2.