Blatt 6. Abgabe bis 28.11.2025

28. *(Product rule for divergence)* Let $(M, \mathbf{g})$ be a Riemannian manifold. Let $\nabla = \nabla_{\mathbf{g}}$ and $\operatorname{div} = \operatorname{div}_{\mathbf{g}}$ be the gradient and divergence associated with $\mathbf{g}$, respectively. Let u be any smooth function on M and v be any smooth vector field on M.

    (a) Prove the identity $\operatorname{div}(uv) = \langle \nabla u, v\rangle + u \operatorname{div} v$.

    Hint. Use the divergence theorem and the gradient product rule of Exercise 21a.

    (b) Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. Prove that the weighted divergence $\operatorname{div}_{\mathbf{g},\mu}$ satisfies the identity $\operatorname{div}_{\mathbf{g},\mu}(uv) = \langle \nabla u, v\rangle + u \operatorname{div}_{\mathbf{g},\mu} v$.

29. Recall that the Laplace-Beltrami operator $\Delta = \Delta_{\mathbf{g}}$ on a Riemannian manifold $(M, \mathbf{g})$ is defined for any function $u \in C^\infty(M)$ by $\Delta u = \operatorname{div}(\nabla u)$.

    (a) *(Product rule for the Laplacian)* Prove that, for smooth functions u and v on M,
    $$\Delta(uv) = u\Delta v + 2\langle \nabla u, \nabla v\rangle + (\Delta u) v.$$

    (b) *(Chain rule for the Laplacian)* Prove that, for functions $u \in C^\infty(M)$ and $f \in C^\infty(\mathbb{R})$,
    $$\Delta f(u) = f''(u) |\nabla u|^2 + f'(u) \Delta u.$$

30. Let $(M, \mathbf{g}, \mu)$ be a weighted manifold. Prove the following identities.

    (a) (*The divergence theorem*) If u is a smooth function on M and v is a smooth vector field, such that either u or v has a compact support then
    $$\int_M (\operatorname{div}_{\mathbf{g},\mu} v)\, u \, d\mu = -\int_M \langle v, \nabla u\rangle \, d\mu. \tag{11}$$

    (b) (*The Green formula*) If u, v are smooth functions on M and one of them has a compact support then
    $$\int_M u\, \Delta_{\mathbf{g},\mu} v \, d\mu = -\int_M \langle \nabla u, \nabla v\rangle \, d\mu = \int_M v\, \Delta_{\mathbf{g},\mu} u \, d\mu. \tag{12}$$

31. (*Change of metric and measure*) Let $(M, \mathbf{g}, \mu)$ be a weighted manifold.

    (a) Let $a(x)$, $b(x)$ be smooth positive functions on M. Define new metric $\widetilde{\mathbf{g}}$ and measure $\widetilde{\mu}$ by
    $$\widetilde{\mathbf{g}} = a\, \mathbf{g} \quad \text{and} \quad d\widetilde{\mu} = b\, d\mu,$$
    where the first identity means that $\langle \xi, \eta\rangle_{\widetilde{\mathbf{g}}} = a(x) \langle \xi, \eta\rangle_{\mathbf{g}}$ for all $\xi, \eta \in T_x M$. Prove that the Laplace operator $\Delta_{\widetilde{\mathbf{g}},\widetilde{\mu}}$ of the weighted manifold $(M, \widetilde{\mathbf{g}}, \widetilde{\mu})$ is given by the formula
    $$\Delta_{\widetilde{\mathbf{g}},\widetilde{\mu}} u = \frac{1}{b} \operatorname{div}_{\mathbf{g},\mu}\left(\frac{b}{a} \nabla_{\mathbf{g}} u\right) \quad \text{for any } u \in C^\infty(M).$$

    Hint. Use the Green formula (13).

(b) Consider the following operator L

$$Lu = \Delta_{\mathbf{g},\mu} u + \langle \nabla v, \nabla u \rangle_{\mathbf{g}},$$

acting on functions $u \in C^\infty(M)$, where $v \in C^\infty(M)$ is a given fixed function. Prove that $L = \Delta_{\mathbf{g},\widetilde{\mu}}$ for some measure $\widetilde{\mu}$, and determine this measure.

32. * Consider in $\mathbb{R}^n$ the following differential operator

$$L = \frac{1}{b(x)} \frac{\partial}{\partial x^i} \left(a^{ij}(x) \frac{\partial}{\partial x^j} \right),$$

where $(a^{ij}(x))$ is a symmetric positive definite matrix smoothly depending on $x \in \mathbb{R}^n$, and $b(x)$ is a smooth positive function. Find in $\mathbb{R}^n$ a Riemannian metric $\mathbf{g}$ and a measure μ such that the weighted Laplace operator $\Delta_{\mathbf{g},\mu}$ coincides with L.

33. * Fix n reals $a_1, ..., a_n$ and consider the matrix

$$B = \begin{pmatrix} 1+a_1^2 & a_1a_2 & a_1a_3 & \dots & a_1a_n \\ a_2a_1 & 1+a_2^2 & a_2a_3 & \dots & a_2a_n \\ a_3a_1 & a_3a_2 & 1+a_3^2 & \dots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \dots & 1+a_n^2 \end{pmatrix}$$

that is, $B = (b_{ij})$ where $b_{ii} = 1 + a_i^2$ and $b_{ij} = a_i a_j$ for $i \neq j$. The purpose of this question is to prove the identity

$$\det B = 1 + a_1^2 + ... + a_n^2. \tag{13}$$

(a) Consider an auxiliary $(n+1) \times (n+1)$ matrix

$$A = \begin{pmatrix} 1 & -a_1 & -a_2 & \dots & \dots & -a_n \\ a_1 & 1 & & & & \\ a_2 & & 1 & & \mathbf{0} & \\ \vdots & & & \ddots & & \\ \vdots & & \mathbf{0} & & \ddots & \\ a_n & & & & & 1 \end{pmatrix},$$

where all the entries of the matrix outside the first column, the first row and the main diagonal are zeros. Prove that $\det A = 1 + a_1^2 + ... + a_n^2$.

(b) Prove the identity (21).

Hint. Prove first that the matrix AA^T has the block diagonal form

$$AA^T = \begin{pmatrix} c & \mathbf{0} \\ \mathbf{0} & \boxed{B} \end{pmatrix},$$

where B is the above matrix and $c = 1 + a_1^2 + ... + a_n^2$.

Remark. The identity (21) will be used in one of the problems in the next problem sheet in order to compute Riemannian measure on certain submanifolds.