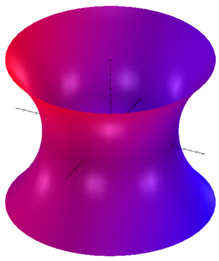
Blatt 7. Abgabe bis 05.12.2025

34. (Continuation of Exercise 27). A catenoid Cat is a surface in $\mathbb{R}^3$ that is given by the parametric equations

$$x^1 = \cosh\rho\cos\theta, \quad x^2 = \cosh\rho\sin\theta, \quad x^3 = \rho,$$

where $\rho \in (-\infty, +\infty)$ and $\theta \in (-\pi, \pi)$.



Catenoid

By Exercise 27, the Riemannian metric of Cat is given by

$$\mathbf{g}_{Cat} = \cosh^2\rho\left(d\rho^2 + d\theta^2\right).$$

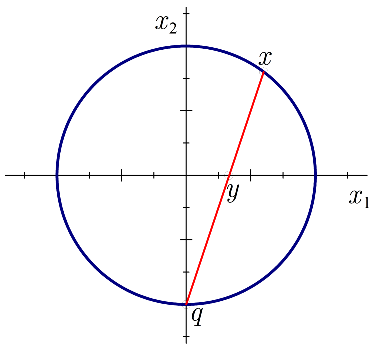
Evaluate the integral

$$\int_{Cat} \frac{1}{\cosh^4\rho} d\nu,$$

where ν is the induced Riemannian measure on Cat.

35. Consider the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ and set $U = \mathbb{S}^1 \setminus \{q\}$ where $q = (0, -1) \in \mathbb{S}^1$.

For any point $x \in U$ define its *stereographic projection* onto $\mathbb{R}^1$ as the point $y \in \mathbb{R}^1$ such that $(y, 0) \in \mathbb{R}^2$ lies on the straight line that goes through x and q.



(a) Prove that the stereographic projection is a homeomorphism between U and $\mathbb{R}^1$, and that it is given by

$$x_1 = \frac{2y}{1+y^2}, \quad x_2 = \frac{1-y^2}{1+y^2},$$

where $(x_1, x_2) \in U$ and $y \in \mathbb{R}^1$. Hence, U is a chart on $\mathbb{S}^1$ with the coordinate y.

(b) Prove that the canonical spherical metric $\mathbf{g}_{\mathbb{S}^1} := \mathbf{g}_{\mathbb{R}^2}|_{\mathbb{S}^1}$ has in the coordinate y the form

$$\mathbf{g}_{\mathbb{S}^1} = \frac{4}{(1+y^2)^2} dy^2.$$

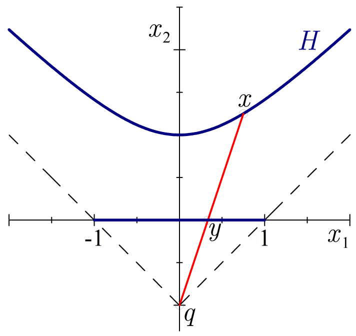
(c) Evaluate $\sigma(\mathbb{S}^1)$, where σ the Riemannian measure of $(\mathbb{S}^1, \mathbf{g}_{\mathbb{S}^1})$.

36. Consider in $\mathbb{R}^2$ a semi-hyperbola

$$H := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2^2 - x_1^2 = 1, \quad x_2 > 0\right\}$$

that is a submanifold of $\mathbb{R}^2$ of dimension 1.

For any point $x \in H$, define its *stereographic projection* onto $\mathbb{R}^1$ as the point $y \in \mathbb{R}^1$ such that $(y, 0) \in \mathbb{R}^2$ lies on the straight line that goes through x and $q = (0, -1)$.



(a) Prove that the stereographic projection is a homeomorphism between H and the unit interval $I = \{y \in \mathbb{R}^1 : -1 < y < 1\}$, and that it is given by

$$x_1 = \frac{2y}{1 - y^2}, \quad x_2 = \frac{1 + y^2}{1 - y^2}, \tag{14}$$

where $(x_1, x_2) \in H$ and $y \in I$. Hence, H itself is a chart with the coordinate y.

(b) Consider in $\mathbb{R}^2$ the *Minkowski metric tensor*

$$\mathbf{g}_{Mink} := dx_1^2 - dx_2^2.$$

Prove that its restriction $\mathbf{g}_H := \mathbf{g}_{Mink}|_H$ is given in the coordinate y by

$$\mathbf{g}_H = \frac{4}{(1 - y^2)^2} dy^2.$$

(c) Denoting by ν the Riemannian measure of $(H, \mathbf{g}_H)$, evaluate the integral

$$\int_H \frac{1}{x_2} d\nu,$$

where x_2 is the second coordinate in $\mathbb{R}^2$ of a point $x \in H$ (as in (14)).

37. Let Γ be the graph in $\mathbb{R}^{n+1}$ of a smooth function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$. Let $\mathbf{g}_\Gamma$ be the Riemannian metric on Γ that is induced by the canonical Euclidean metric in $\mathbb{R}^{n+1}$. Let $y^1, ..., y^n$ be the Cartesian coordinates in U that can be regarded as local coordinates on Γ. Denote by ν_Γ the Riemannian measure of $(\Gamma, \mathbf{g}_\Gamma)$.

(a) Prove that in the coordinates $y^1, ..., y^n$

$$d\nu_\Gamma = \sqrt{1 + \left(\frac{\partial f}{\partial y^1}\right)^2 + ... + \left(\frac{\partial f}{\partial y^n}\right)^2} dy. \tag{15}$$

Hint. Use the result of Exercise 26 that

$$(g_\Gamma)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{16}$$

and then the formula (13) of Exercise 33.

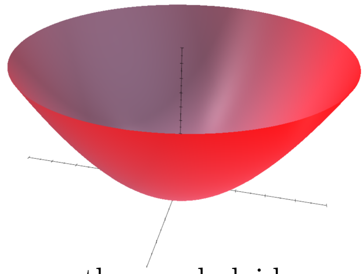
(b) Using (15), evaluate the area (=the Riemannian measure) of the paraboloid that is the graph in $\mathbb{R}^3$ of the function

$$f(x,y) = \frac{1}{2}(x^2 + y^2)$$

in a disc

$$U = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

Hint. Compute $\nu_\Gamma(\Gamma)$ using integration in the polar coordinates in $\mathbb{R}^2$.



the paraboloid

38. * Let q be the south pole of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, that is,

$$q = (\underbrace{0, ..., 0}_{n \text{ zeros}}, -1). \tag{17}$$

For any point $x \in U := \mathbb{S}^n \setminus \{q\}$, its stereographic projection is the point $y \in \mathbb{R}^n$ such that the point $(y, 0) \in \mathbb{R}^{n+1}$ belongs to the straight line that goes through x and q.

(a) Prove the following relations between $x \in U$ and $y \in \mathbb{R}^n$:

$$x_i = (1 + x_{n+1})\, y_i, \quad i = 1, ..., n \tag{18}$$

and

$$|y|^2 = \frac{2}{1 + x_{n+1}} - 1. \tag{19}$$

Show that the stereographic projection is a homeomorphism between U and $\mathbb{R}^n$. Hence, U is a chart on $\mathbb{S}^n$ with coordinates $y_1, ..., y_n$.

(b) Prove that the canonical spherical metric $\mathbf{g}_{\mathbb{S}^n} := \mathbf{g}_{\mathbb{R}^{n+1}}|_{\mathbb{S}^n}$ has in the coordinates $y_1, ..., y_n$ the form

$$\mathbf{g}_{\mathbb{S}^n} = \frac{4}{\left(1 + |y|^2\right)^2} \left(dy_1^2 + ... + dy_n^2\right).$$

Hint. Express the Euclidean metric $\mathbf{g}_{\mathbb{R}^{n+1}} = dx_1^2 + ...dx_n^2 + dx_{n+1}^2$ via dy_i using the relations (18) and (19).

39. * Define the n-dimensional hyperboloid $\mathbb{H}^n$ as the following submanifold of $\mathbb{R}^{n+1}$:

$$\mathbb{H}^n = \left\{x \in \mathbb{R}^{n+1} : x_{n+1}^2 - x_1^2 - ... - x_n^2 = 1,\ x_{n+1} > 0\right\}.$$

For any point $x \in \mathbb{H}^n$, its stereographic projection is the point $y \in \mathbb{R}^n$ such that the point $(y, 0) \in \mathbb{R}^{n+1}$ belongs to the straight line that goes through x and q (where q is given by (17)).

(a) Prove that the stereographic projection is a homeomorphism of $\mathbb{H}^n$ onto the unit ball $\mathbb{B}^n = \{y \in \mathbb{R}^n : |y| < 1\}$. Prove also the following relations between $x \in \mathbb{H}^n$ and $y \in \mathbb{B}^n$:

$$x_i = (1 + x_{n+1})\, y_i, \ i = 1, ..., n \tag{20}$$

and

$$|y|^2 = 1 - \frac{2}{1 + x_{n+1}}. \tag{21}$$

(b) Define the Minkowski metric tensor $\mathbf{g}_{Mink}$ in $\mathbb{R}^{n+1}$ by

$$\mathbf{g}_{Mink} = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2.$$

The induced metric $\mathbf{g}_{\mathbb{H}^n} = \mathbf{g}_{Mink}|_{\mathbb{H}^n}$ is called the *hyperbolic metric* on $\mathbb{H}^n$. Prove that the hyperbolic metric has in the coordinates $y_1, ..., y_n$ the form

$$\mathbf{g}_{\mathbb{H}^n} = \frac{4}{\left(1 - |y|^2\right)^2} \left(dy_1^2 + ... + dy_n^2\right). \tag{22}$$

Remark. Observe that the metric $\mathbf{g}_{\mathbb{H}^n}$ is positive definite and, hence, is Riemannian, although the Minkowski metric in $\mathbb{R}^{n+1}$ is not positive definite (it is called *pseudo-Riemannian*). The Riemannian manifold $(\mathbb{H}^n, \mathbf{g}_{\mathbb{H}^n})$ is called the *hyperbolic space*. The ball $\mathbb{B}^n$ with the metric (22) is called the *Poincaré model* of the hyperbolic space.