Blatt 8. Abgabe bis 12.12.2025

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Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

40. Prove that if a Riemannian manifold $(M, \mathbf{g})$ is connected then $d(x,y) < \infty$ for all $x, y \in M$, where d is the geodesic distance function.

    Hint: Show that, for any $x \in M$, the set $N := \{y \in M : d(x,y) < \infty\}$ is open and closed.

41. Let $(M, \mathbf{g})$ be a Riemannian model, and let x', x'' be two points in $M \setminus \{o\}$ with the polar coordinates (r', θ') and (r'', θ''), respectively.

    (a) Prove that, for any piecewise C^1 path γ on M connecting the points x' and x'',
    $$\ell_{\mathbf{g}}(\gamma) \geq |r' - r''|.$$
    Deduce that $d(x', x'') \geq |r' - r''|$, where d is the geodesic distance on $(M, \mathbf{g})$.
    Hint. Use the metric $\mathbf{g}$ in the polar coordinates on M.

    (b) Prove that if $\theta' = \theta''$ then $d(x', x'') = |r' - r''|$.

    (c) Prove that, for any point $x = (r, \theta)$, we have $d(o, x) = r$.

    (d) Conclude that in $(\mathbb{R}^n, \mathbf{g}_{\mathbb{R}^n})$ the geodesic distance $d(x,y)$ is equal to $|x - y|$ for all $x, y \in \mathbb{R}^n$.

42. Let $\gamma(t) : (a,b) \to M$ be a parametric C^1 curve on a Riemannian manifold $(M, \mathbf{g})$.

    (a) Consider a *time change* $\tau : (\alpha, \beta) \to (a,b)$ where the function τ is bijective and C^1 smooth. Then τ determines a new parametric curve
    $$\begin{aligned} \widetilde{\gamma} &: (\alpha, \beta) \to M \\ \widetilde{\gamma}(s) &= \gamma(\tau(s)). \end{aligned}$$
    Prove that $\ell_{\mathbf{g}}(\widetilde{\gamma}) = \ell_{\mathbf{g}}(\gamma)$.
    Remark. This identity means that the length of the parametric curve does not depend on a specific parametrization.

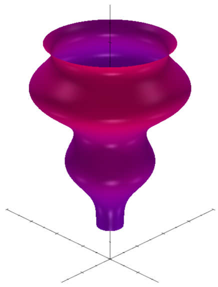
    (b) Assume in additon that γ is C^∞ smooth, injective, $\dot{\gamma}(t) \neq 0$ for all $t \in (a,b)$ and that γ is a homeomorphism of (a,b) onto the image $S = \gamma(a.b)$. Then, by Exercise 17, S is a submanifold of dimension 1. Let ν_S be the induced metric on S. Prove that
    $$\ell_{\mathbf{g}}(\gamma) = \nu_S(S).$$
    Hint. Write down the induced metric $\mathbf{g}_S$ using the local coordinate t.

43. Let I be an open interval in $\mathbb{R}$ and S be a *surface of revolution* in $\mathbb{R}^{n+1}$ around I that is given by the equation

$$|x'| = \varphi\left(x^{n+1}\right), \quad x^{n+1} \in I,$$

where $x' = (x^1, ..., x^n)$ and $\varphi(t)$ is a smooth positive function on I.

Here is an example of a surface of revolution:



(a) Prove that S is a submanifold of $\mathbb{R}^{n+1}$ of dimension n.

(b) Let us introduce on S the *prepolar coordinates* (t, θ) as follows: for any point $(x', x^{n+1}) \in S$, set

$$t = x^{n+1} \in I \quad \text{and} \quad \theta = \frac{x'}{|x'|} \in \mathbb{S}^{n-1}.$$

Prove that in the coordinates (t, θ) the induced metric $\mathbf{g}_S := \mathbf{g}_{\mathbb{R}^{n+1}}|_S$ has the form

$$\mathbf{g}_S = \left(1 + \varphi'(t)^2\right) dt^2 + \varphi^2(t)\, \mathbf{g}_{\mathbb{S}^{n-1}}.$$

Hint. Express all x^i in terms of t and the Cartesian coordinates $f^i(\theta)$ of θ.

(c) Define the *polar coordinates* (r, θ) on S as follows: θ is as above, while $r = r(t)$ is defined by

$$r = \int_{t_0}^{t} \sqrt{1 + \varphi'(\xi)^2}\, d\xi, \tag{23}$$

where t_0 is any fixed point from I. Prove that the metric $\mathbf{g}_S$ has in the coordinates (r, θ) the *model* form

$$\mathbf{g}_S = dr^2 + \psi^2(r)\, \mathbf{g}_{\mathbb{S}^{n-1}}, \tag{24}$$

where the function ψ is defined by the identity $\psi(r(t)) = \varphi(t)$.

Hint. Use (23) to express dr via dt.

Remark. The manifold $(S, \mathbf{g}_S)$ is called a *cylindrical* model, which refers the fact that S is homeomorphic to a cylinder $I \times \mathbb{S}^{n-1}$ (rather than to a ball).

(d) Represent in the model form (24) the induced metric of the cone

$$Cone = \left\{x \in \mathbb{R}^{n+1} : |x'| = \alpha x^{n+1} + \beta, \quad x^{n+1} > 0\right\},$$

where $\alpha > 0$ and $\beta \geq 0$.

44. * The purpose of this question is to compute the induced metric $\mathbf{g}_S$ on surfaces of revolution given in parametric form.

(a) Assume that a surface of revolution S in $\mathbb{R}^{n+1}$ is given by the parametric equations

$$x^{n+1} = a(s) \quad \text{and} \quad |x'| = b(s),$$

where a, b are smooth functions of s on some interval and $a'(s) > 0$. Prove that the polar radius r on S (see (23)) can be computed as a function of s by

$$r = \int_{s_0}^{s} \sqrt{\left(a'(\xi)\right)^2 + \left(b'(\xi)\right)^2}\, d\xi,$$

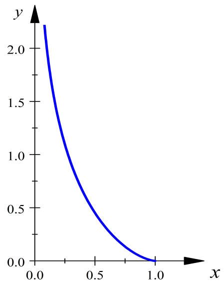
and the function ψ in (24) is determined by the equation $\psi(r(s)) = b(s)$.

(b) The *pseudo-sphere* PS in $\mathbb{R}^{n+1}$ is given by the parametric equations

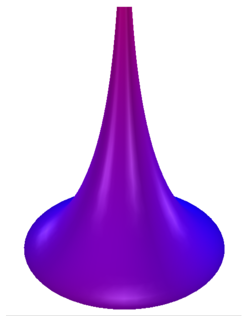
$$x^{n+1} = s - \tanh s \quad \text{and} \quad |x'| = \frac{1}{\cosh s}, \quad s > 0.$$

Prove that the induced metric on PS has in the polar coordinates the form

$$\mathbf{g}_{PS} = dr^2 + e^{-2r}\mathbf{g}_{\mathbb{S}^{n-1}}.$$



A tractrix $x = \frac{1}{\cosh s}$, $y = s - \tanh s$



A pseudosphere in $\mathbb{R}^3$

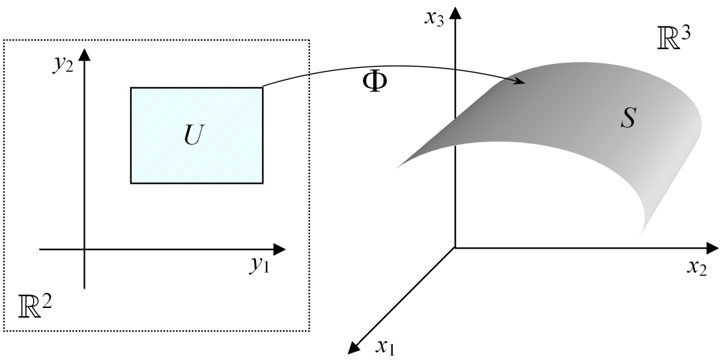
Remark. The pseudo-sphere is the surface of revolution of a *tractrix.*

45. * Let a surface S in $\mathbb{R}^3$ be given in a parametric form as follows:

$$S = \left\{x \in \mathbb{R}^3 : x = \Phi(y),\ y \in U\right\},$$

where U is an open subset of $\mathbb{R}^2$ and $\Phi : U \to \mathbb{R}^3$ is a smooth injective mapping.

Assume that the Jacobi matrix J of Φ has rank 2 at all points.



Assume also that Φ is a homeomorphism of U onto S. Then by Exercise 17 S is a 2-dimensional submanifold of $\mathbb{R}^3$.

Let the components of Φ be Φ^i, $i = 1, 2, 3$. Denoting by y^1, y^2 the Cartesian coordinates in U, consider at any point of U the following two 3-dimensional vectors:

$$u := \left(\frac{\partial\Phi^1}{\partial y^1}, \frac{\partial\Phi^2}{\partial y^1}, \frac{\partial\Phi^3}{\partial y^1}\right) \quad \text{and} \quad v := \left(\frac{\partial\Phi^1}{\partial y^2}, \frac{\partial\Phi^2}{\partial y^2}, \frac{\partial\Phi^3}{\partial y^2}\right).$$

(a) Prove that the induced metric $\mathbf{g}_S = \mathbf{g}_{\mathbb{R}^n}|_S$ is given in the local coordinates y^1, y^2 by the matrix

$$g_S = \begin{pmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{pmatrix}$$

where "$\cdot$" denotes the scalar product of vectors in $\mathbb{R}^3$. Prove also that

$$\det g_S = |u \times v|^2, \tag{25}$$

where "$\times$" denotes the cross product of vectors in $\mathbb{R}^3$.

(b) Using (25), compute the induced measure ν_S for the surface S that is given by the parametric equations

$$x^1 = \sin\varphi\cos\theta, \quad x^2 = \sin\varphi\sin\theta, \quad x^3 = \cos\varphi,$$

where $\varphi \in (0, \pi)$ and $\theta \in (-\pi, \pi)$.

46. ** Prove that, for any $n \geq 1$,

$$\omega_n = 2\frac{\pi^{n/2}}{\Gamma(n/2)}, \tag{26}$$

where ω_n is the surface area of $\mathbb{S}^{n-1}$ and Γ is the gamma function.

Hint. Consider the integrals

$$I_n = \int_0^\pi \sin^n r dr$$

and, using integration by parts, prove that

$$I_n = \frac{n-1}{n} I_{n-2}.$$

By induction obtain that

$$I_n = \sqrt{\pi}\frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)}.$$

Then prove (26) by means of the inductive relation $\omega_{n+1} = \omega_n I_{n-1}$ from lectures.

Remark. The gamma function is defined for all $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

It is known that $\Gamma(x) = (x-1)!$ for a positive integer x. The following identities are satisfied for all $x > -1$:

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1 \text{ and } \Gamma(1/2) = \sqrt{\pi}.$$