Blatt 9. Abgabe bis 19.12.2025

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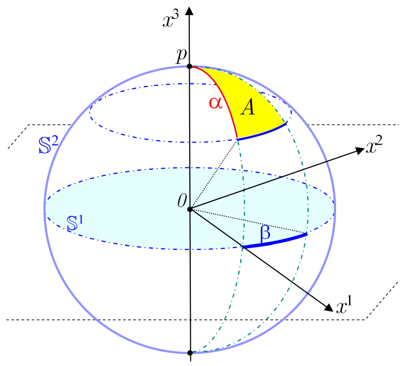
Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

47. Denote by σ_n the canonical Riemannian measure on the sphere $\mathbb{S}^n$ and by η_n the canonical Riemannian measure on the hyperbolic space $\mathbb{H}^n$.

   (a) Let (φ, θ) be the polar coordinates on $\mathbb{S}^2$, where $\varphi \in (0, \pi)$ is the polar radius and $\theta \in (0, 2\pi)$ is the polar angle. Compute $\sigma_2(A)$ for the following subset A of $\mathbb{S}^2$:

   $$A = \{(\varphi, \theta) : 0 < \varphi < \alpha, \quad 0 < \theta < \beta\},$$

   where $\alpha \in (0, \pi)$ and $\beta \in (0, 2\pi)$ are given.



   (b) Let (r, φ, θ) the *spherical coordinates* on $\mathbb{S}^3$, where $r \in (0, \pi)$ is the polar radius and (φ, θ) are the polar coordinates on $\mathbb{S}^2$ as in (a). Compute $\sigma_3(B)$ for the following subset B of $\mathbb{S}^3$:

   $$B = \{(r, \varphi, \theta) : 0 < r < R, \quad 0 < \varphi < \alpha, \quad 0 < \theta < \beta\},$$

   and $0 < R < \pi$, $\alpha \in (0, \pi)$, $\beta \in (0, 2\pi)$ are given.

   (c) Let (r, θ) be the polar coordinates in $\mathbb{H}^2$. Compute $\eta_2(C)$ for the following subset C of $\mathbb{H}^2$:

   $$C = \{(r, \theta) : 0 < r < R, \ 0 < \theta < \beta\},$$

   where $R > 0$ and $\beta \in (0, 2\pi)$ are given.

   (d) Let (r, φ, θ) be the *spherical coordinates* on $\mathbb{H}^3$, where $r > 0$ is the polar radius and (φ, θ) are the polar coordinates on $\mathbb{S}^2$ as in (a). Compute $\eta_3(D)$ for the following subset D of $\mathbb{H}^3$:

   $$D = \{(r, \varphi, \theta) : 0 < r < R, \ 0 < \varphi < \alpha, \ 0 < \theta < \beta\},$$

   where $R > 0$, $\alpha \in (0, \pi)$, $\beta \in (0, 2\pi)$ are given.

48. Let $(M, \mathbf{g})$ be a Riemannian manifold. A smooth function u in an open set $\Omega \subset M$ is called *harmonic* in Ω if $\Delta_{\mathbf{g}} u = 0$ in Ω. Suppose that M is a model manifold of radius r_0 with the area function $S(r)$. A function u in $M \setminus \{o\}$ is called *radial* if it depends only on the polar radius r (and does not depend on the polar angle θ).

(a) Prove that a smooth radial function u in $M \setminus \{o\}$is harmonic if and only if

$$u(r) = C_1 \int_{r_1}^{r} \frac{dt}{S(t)} + C_2, \tag{27}$$

where C_1, C_2 are arbitrary real constants and $r_1 \in (0, r_0)$ is arbitrary.
Hint. Use the representation of the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ in polar coordinates using the area function.

(b) With help of (27) find all radial harmonic functions in $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ for $n = 2, 3$.

49. (Continuation of Exercises 27 and 34). A catenoid Cat is a surface in $\mathbb{R}^3$ that is given by the parametric equations

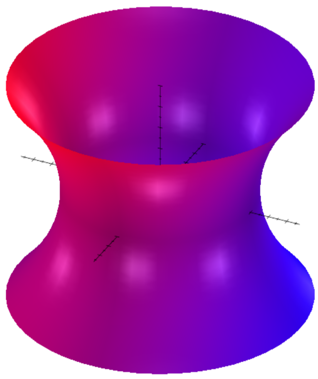
$$x^1 = \cosh \rho \cos \theta, \quad x^2 = \cosh \rho \sin \theta, \quad x^3 = \rho,$$

where $\rho \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$.

(a) Write down the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ on Cat in the coordinates ρ, θ.

(b) Considering the Cartesian coordinates x^1, x^2, x^3 as functions on the catenoid, prove that they are harmonic, that is,

$$\Delta_{\mathbf{g}} x^1 = \Delta_{\mathbf{g}} x^2 = \Delta_{\mathbf{g}} x^3 = 0.$$


Catenoid

Hint. Use the Riemannian metric on Cat stated in Exercise 34.

50. A non-zero smooth function v on a Riemannian manifold $(M, \mathbf{g})$ is called an *eigenfunction* of the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ if, for some constant λ,

$$\Delta_{\mathbf{g}} v + \lambda v = 0,$$

where the constant λ is called an *eigenvalue* of $\Delta_{\mathbf{g}}$. The *multiplicity* of the eigenvalue λ is defined as the dimension of the *eigenspace*

$$E_\lambda = \{v \in C^\infty(M) : \Delta_{\mathbf{g}} v + \lambda v = 0\}.$$

Prove that all the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^1}$ on the unit circle $\mathbb{S}^1$ are given by the sequence $\{m^2\}_{m=0}^{\infty}$, where the eigenvalue 0 has the multiplicity 1, and each eigenvalue m^2 with $m \geq 1$ has the multiplicity 2.

Hint. Write down the equation $\Delta_{\mathbb{S}^1} v + \lambda v = 0$ using the angle $\theta \in (-\pi, \pi)$ as a local coordinate on $\mathbb{S}^1$, and find solutions $v(\theta)$ that are 2π-periodic in θ.

51. * Consider in $\mathbb{H}^3$ a function u given in the polar coordinates (r, θ) by

$$u = \frac{r}{\sinh r}.$$

(a) Prove that, away from the pole of $\mathbb{H}^3$, the function u satisfies the equation

$$\Delta_{\mathbb{H}^3} u + u = 0. \tag{28}$$

Hint. Use the representation of $\Delta_{\mathbb{H}^3}$ in the polar coordinates.

(b) Prove that the function u extends to a smooth function on the entire space $\mathbb{H}^3$ and, hence, satisfies (28) on $\mathbb{H}^3$.

Hint. Show first that the function $v = r^2$ is a smooth function on $\mathbb{H}^3$ (as well as on any model manifold). Then represent u as a smooth function of r^2.

52. ** Consider the Riemannian manifold $(M, \mathbf{g})$, where

$$M = \mathbb{R}^n_+ := \left\{ \left(x^1, ..., x^n\right) \in \mathbb{R}^n : x^n > 0 \right\}$$

and

$$\mathbf{g} = \frac{\left(dx^1\right)^2 + ... + \left(dx^n\right)^2}{\left(x^n\right)^2}.$$

Consider on $\mathbb{R}^n_+$ the following function

$$u(x) = \left(x^n\right)^s$$

where s is any real parameter. Prove that $\Delta_{\mathbf{g}} u = \lambda u$, where $\lambda = s\left(s - n + 1\right)$.

53. ** A function $P : \mathbb{R}^N \to \mathbb{R}$ is called a *polynomial* if $P\left(x\right)$ is a finite $\mathbb{R}$-linear combination of the *monomials* $x_1^{m_1} ... x_N^{m_N}$ where $m_1, ..., m_N$ are non-negative integers. The sum $m_1 + ... + m_N$ is called the *degree* of the monomial. A polynomial P is called *homogeneous* of degree m if all non-zero monomials of P have the same degree m.

(a) Let P be a homogeneous polynomial of degree m on $\mathbb{R}^{n+1}$, where m is a non-negative integer. Assume that P is harmonic, that is, P satisfies the equation

$$\Delta_{\mathbb{R}^{n+1}} P = 0 \text{ in } \mathbb{R}^{n+1}.$$

Prove that the function $v = P|_{\mathbb{S}^n}$ is an eigenfunction of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^n}$ with the eigenvalue $\lambda = m\left(m + n - 1\right)$, that is,

$$\Delta_{\mathbb{S}^n} v + \lambda v = 0 \text{ in } \mathbb{S}^n.$$

Hint. Use the identity $P\left(x\right) = \alpha^m P\left(\frac{x}{\alpha}\right)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ and all $x \in \mathbb{R}^{n+1}$ that follows from the homogeneity of P, and represent $\Delta_{\mathbb{R}^{n+1}}$ in the polar coordinates.

(b) Prove that a polynomial in $\mathbb{R}^3$

$$P\left(x\right) = C_1 x_1^3 x_2 x_3 + C_2 x_1 x_2^3 x_3 + C_3 x_1 x_2 x_3^3$$

is harmonic for some non-zero coefficients C_1, C_2, C_3. Hence, prove that $\lambda = 30$ is an eigenvalue of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^2}$.

Remark. It is possible to prove that all the eigenvalues of $\Delta_{\mathbb{S}^n}$ are given by the sequence $\left\{m\left(m + n - 1\right)\right\}_{m=0}^{\infty}$, and the eigenvalue $m\left(m + n - 1\right)$ has the multiplicity

$$\frac{\left(n + m - 2\right)! \left(n + 2m - 1\right)}{(n-1)! m!}$$

if $m \geq 1$, and 1 if $m = 0$.