Mock exam (Probeklausur) "Analysis on manifolds", WS 2025/26

Duration of the exam 120 minutes. Each problem is worth of 25 points.
Full mark (Note 1,0) $\approx$ 90-95 points, pass mark (Note 4,0) $\approx$ 45-50 points.
No scripts, books, calculators, computers etc. are allowed.

Problem 1

Let M be a smooth manifold.

(a) Give the definitions of a tangent vector and a tangent space T_xM.

(b) State and prove a theorem about the dimension of T_xM.

Problem 2

Let $(M, \mathbf{g})$ be a Riemannian manifold.

(a) Give the definitions of the differential df and gradient ∇f for a smooth function f on M. State without proof the divergence theorem giving the definition of the divergence $\operatorname{div} v$ of a smooth vector field v on M.

(b) Prove the product rule for the operators d and ∇: for any two smooth functions u and v on M,

$$d\,(uv) = udv + vdu \tag{1}$$

and

$$\nabla\,(uv) = u\nabla v + v\nabla u. \tag{2}$$

(c) Prove the product rule for divergence: for any smooth function u and any smooth vector field v on M,

$$\operatorname{div}\,(uv) = \langle \nabla u, v\rangle + u \operatorname{div} v. \tag{3}$$

Problem 3

Let $(M, \mathbf{g})$ be a Riemannian manifold.

(a) Let S be a smooth submanifold of M of dimension m. Give the definition of the induced metric $\mathbf{g}_S$ on S. State and prove the formula for the induced metric in the local coordinates.

(b) Let Γ be the graph of a smooth function $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{n+1}$ is an open set. Let $y^1, ..., y^n$ be the Cartesian coordinates in U. Considering Γ as a submanifold of $\mathbb{R}^{n+1}$, prove that the components of the induced metric $\mathbf{g}_\Gamma$ are given by the formula

$$(g_\Gamma)_{ij} = \delta_{ij} + \frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

(c) Prove that the induced measure ν_Γ on Γ is given in the coordinates $y^1, ..., y^n$ by the formula

$$d\nu_\Gamma = \sqrt{1 + \left(\frac{\partial f}{\partial y^1}\right)^2 + ... + \left(\frac{\partial f}{\partial y^n}\right)^2} dy. \tag{4}$$

Hint. You can use without proof the identity

$$\det\left(\delta_{ij} + a_i a_j\right)_{i,j=1}^n = 1 + a_1^2 + ... + a_n^2. \tag{5}$$

Problem 4

Let $(M, \mathbf{g})$ be a Riemannian manifold.

(a) Give the definition of the Laplace-Beltrami operator $\Delta_\mathbf{g}$ acting on smooth functions on M. State and prove the representation of $\Delta_\mathbf{g}$ in the local coordinates.

(b) Define the notion of a model manifold $(M, \mathbf{g})$ with the profile function $\psi(r)$. State and prove the representation of the Laplace-Beltrami operator $\Delta_\mathbf{g}$ in the polar coordinates (r, θ) of M. Write down the Laplace-Beltrami operator of the hyperbolic space $\mathbb{H}^n$ in the polar coordinates using the profile function $\psi(r) = \sinh r$ of the hyperbolic space.

Problem 5

Let $(M, \mathbf{g}, \mu)$ be a weighted manifold and $\Delta = \Delta_{\mathbf{g},\mu}$. Let Ω be an open subset of M.

(a) Define a weak Dirichlet problem in Ω. State without proof a theorem about existence and uniqueness of solution and about the properties of the resolvent operator R_α for $\alpha > 0$.

(b) Let Ω be precompact. State without proof a theorem about existence of the eigenfunctions and the eigenvalues of Δ in Ω.

(c) $\{v_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ that consists of the eigenfunctions of Δ in Ω, and $\{\lambda_k\}_{k=1}^\infty$ be the corresponding eigenvalues. Let a function $f \in L^2(\Omega)$ have an eigenfunction expansion $f = \sum_{k=1}^\infty a_k v_k$ that converges in $L^2(\Omega)$. Prove that, for any $\alpha > 0$,

$$R_\alpha f = \sum_{k=1}^\infty \frac{1}{\alpha + \lambda_k} a_k v_k. \tag{6}$$

Using (6) prove the following *resolvent identity*: for all $\alpha, \beta > 0$

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta. \tag{7}$$