Blatt 0. Keine Abgabe

1. Let M be any topological space. Let K be a compact subset of M and F be a closed subset of M. Prove that if $F \subset K$ then F is compact.

2. A topological space M is called *Hausdorff* if, for any two disjoint points $x, y \in M$, there exist two disjoint open sets $U, V \subset M$ such that $x \in U$ and $y \in V$. Prove the following properties of a compact subset K of a Hausdorff topological space M.

   (a) For any $x \in K^c$ there exists an open set W_x containing x and disjoint from K.

   (b) K is a closed subset of M.

3. Let X, Y be two topological spaces and $f : X \to Y$ be a continuous mapping. Prove that if K is a compact subset of X then $f(K)$ is a compact subset of Y.

4. Prove that, on any C-manifold M, there exists a countable sequence $\{\Omega_k\}$ of relatively compact open sets such that $\Omega_k \Subset \Omega_{k+1}$ (that is, Ω_k is relatively compact and $\overline{\Omega}_k \subset \Omega_{k+1}$) and the union of all Ω_k is M. Prove also that if M is connected then the sets Ω_k can also be taken connected.

   Remark. An increasing sequence $\{\Omega_k\}$ of open subsets of M whose union is M, is called an *exhaustion sequence*. If in addition $\Omega_k \Subset \Omega_{k+1}$ then the sequence $\{\Omega_k\}$ is called a *compact exhaustion sequence*.

5. Let M be a C-manifold of dimension n. Let V be a chart on M and E be a subset of V. The compact inclusion $E \Subset V$ can be understood in two ways: in the sense of the topology of M as well as in the sense of the topology of $\mathbb{R}^n$, when identifying V with a subset of $\mathbb{R}^n$. Prove that these two meanings of $E \Subset V$ are equivalent.

6. Prove that, on any C-manifold M, there is a countable *locally finite* family of relatively compact charts covering M. (A family of sets is called locally finite if any compact subset of M intersects at most finitely many sets from this family).

7. Fix some positive integers n, m, let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a C^1-function. Consider the null set of F, that is, the set

$$M = \left\{ x \in \mathbb{R}^{n+m} : F(x) = 0 \right\},$$

   and assume that, for any point $x \in M$, the Jacobi matrix $F'(x)$ has the rank m. Prove that M is a C-manifold of dimension n.

   Hint. Use the implicit function theorem.

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In all exercises, M is a smooth manifold of dimension n.

8. A *path* on M is any smooth mapping $\gamma : [0, a] \to M$, where $a > 0$. Set $x = \gamma(0)$. For any function $f \in C^\infty(M)$, define the derivative of f along the path γ at the point x by
$$\frac{\partial f}{\partial \gamma} := \frac{d}{dt} f(\gamma(t)) \Big|_{t=0}.$$
   (a) Prove that $\frac{\partial}{\partial \gamma}$ is an $\mathbb{R}$-differentiation at x, that is, $\frac{\partial}{\partial \gamma} \in T_x M$.

   (b) Prove that any tangent vector $\xi \in T_x M$ can be represented in the form $\xi = \frac{\partial}{\partial \gamma}$ for some path γ.

9. A smooth *vector field* on M is a mapping $X : C^\infty(M) \to C^\infty(M)$ such that, for any $x \in M$, the mapping
$$\begin{aligned} C^\infty(M) &\to \mathbb{R} \\ f &\mapsto X(f)(x) \end{aligned}$$
is a $\mathbb{R}$-differentiation at x. Prove that, in any chart U with the local coordinates $x^1, ..., x^n$, there are functions $a^1, ..., a^n \in C^\infty(U)$ such that
$$X(f) = \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i} \text{ for any } f \in C^\infty(M).$$
Hint. Use the fact that any $\mathbb{R}$-differentiation ξ can be represented in the form
$$\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}$$
for some $\xi^i \in \mathbb{R}$.

10. Let X and Y be two smooth vector fields on M (as in Exercise 9). Define the *Lie bracket* $[X, Y]$ of X, Y as a mapping of $C^\infty(M)$ into itself by
$$[X, Y] := XY - YX,$$
that is, $[X, Y](f) = X(Y(f)) - Y(X(f))$ for any $f \in C^\infty(M)$.

    Prove that $[X, Y]$ is a smooth vector field on M.

    Hint. In the local coordinates, $X(f)$ is a combination of the first partial derivatives $\frac{\partial f}{\partial x^i}$ (by Exercise 9). Hence, $XY(f)$ and $YX(f)$ can contain the second derivatives of f. The point of the present claim is that the difference $XY(f) - YX(f)$ depends on the first derivatives of f only, that is, the second derivatives cancel out.

11. *(The Jacobi identity)* Prove the following identity for three smooth vector fields X, Y, Z on a smooth manifold M:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0. \tag{1}$$

Hint. By linearity, it suffices to consider the case when X, Y, Z are given in the local coordinates $x^1, ..., x^n$ by

$$X = a\frac{\partial}{\partial x^i}, \quad Y = b\frac{\partial}{\partial x^j}, \quad Z = c\frac{\partial}{\partial x^k},$$

where a, b, c are smooth functions of $x^1, ..., x^n$ and i, j, k are some indices from $1, ..., n$.