Heat kernels on manifolds, graphs and fractals

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Abstract. We consider heat kernels on different spaces such as Riemannian manifolds, graphs, and abstract metric measure spaces including fractals. The talk is an overview of the relationships between the heat kernel upper and lower bounds and the geometric properties of the underlying space. As an application some estimate of higher eigenvalues of the Dirichlet problem is considered.

Contents

1. Introduction

This is a brief survey on heat kernel long time estimates on various underlying spaces such as Riemannian manifolds, graphs and fractals. We have selected generic results which, when properly modified, remain true in all the cases, although each particular result is presented here in one of the settings which is most convenient. In Section 2 we give necessary definitions. In Sections 3 and 4, we consider heat kernel on-diagonal estimates in relation with the first eigenvalue estimates. In Sections 5 and 6 we consider off-diagonal upper and lower estimates, of Gaussian and sub-Gaussian types, respectively. In Section 7, we discuss relations to isoperimetric properties of higher eigenvalues.

For simplicity, we restrict our consideration to *uniform* estimates of the heat kernel. For further results and for a detailed account of various related aspects of heat kernels and eigenvalues, we refer a reader to books and surveys [2], [6], [7], [9], [10], [13], [21], [29], [30], [31], [33], [34], [36], [40], [41].

Throughout the paper, C and c normally denote large and small positive constants, respectively, which may be different on different occurrences. A relation $f \simeq g$ means that the ratio of the functions f and g remains bounded between two positive constants for a specified range of their arguments.

2. The notion of heat kernel

2.1. Manifolds

Let M be a smooth connected Riemannian manifold, and let $d(x, y)$ be a geodesic distance on M. Assume that a Borel measure μ is defined on M, which has a smooth density m with respect a Riemannian measure (in particular, μ may be the Riemannian measure if $m = 1$). The couple (M, μ) is called *a weighted manifold*.

A natural Laplace operator Δ_μ is associated with (M, μ), namely,

$$\Delta_\mu = m^{-1} \operatorname{div} (m\nabla)$$

where ∇ and div are the Riemannian gradient and divergence, respectively. An energy form of Δ_μ is given by

$$\mathcal{E}_\mu(f) = \int_M |\nabla f|^2 \, d\mu.$$

It is well-known that the heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u,$$

(where $u(t,x)$ is defined for $t > 0$ and $x \in M$) has *a heat kernel*, which is denoted by $p_t(x,y)$ and can be defined in various alternative ways. Here are some equivalent definitions:

1. For any $y \in M$, the function $(t,x) \mapsto p_t(x,y)$ is the smallest positive fundamental solution to the heat equation with a source at y.
2. The function $p_t(x,y)$ is an integral kernel of *the heat semigroup* $P_t := e^{t\Delta_\mu}$ that is defined by using the spectral theorem (indeed, the operator Δ_μ with domain $C_0^\infty(M)$ is essentially self-adjoint in $L^2(M,\mu)$ and negative definite).
3. The function $p_t(x,y)$ is the transition density of *a Brownian motion* X_t on (M,μ), which is by definition a diffusion process generated by Δ_μ.

Here are some examples of exact heat kernels.

If $(M,\mu) = \mathbb{R}^N$ then the heat kernel is given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{d^2(x,y)}{4t}\right).$$

If $(M,\mu) = \mathbb{H}^3$ – the 3-dimensional hyperbolic space of constant negative curvature -1, then

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-t - \frac{d(x,y)^2}{4t}\right) \frac{d(x,y)}{\sinh d(x,y)}.$$

If M is a compact manifold then the operator Δ_μ is compact. Let $\{\varphi_k\}_{k=0}^\infty$ be an orthonormal basis in $L^2(M,\mu)$ of eigenfunctions of $-\Delta_\mu$ with eigenvalues

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le ...$$

(here $\varphi_0 = \text{const}$). Then the heat kernel on (M,μ) is determined by

$$p_t(x,y) = \sum_{k=0}^\infty e^{-\lambda_k t} \varphi_k(x)\varphi_k(y).$$

2.2. Graphs

Consider now a discrete version of the heat kernel. Let Γ be a connected graph and let $d(x,y)$ be a combinatorial distance on Γ, that is, $d(x,y)$ is the smallest number of edges in a path connecting $x, y \in \Gamma$.

Let μ_{xy} be a weight on edges of Γ. More precisely, if vertices $x, y \in \Gamma$ are connected by an edge then we write $x \sim y$, denote the edge by $\overline{xy}$ and assign to it a positive number μ_{xy}. In particular, it may happen that $\mu_{xy} = 1$ for all edges, in which case we say that μ_{xy} is *a standard weight*. It is convenient to extend μ_{xy} by zero to those x, y which are not neighbors.

Any weight μ_{xy} gives rise to a measure on vertices by

$$\mu(x) = \sum_{y \sim x} \mu_{xy}$$

and then to a measure on all finite sets $\Omega \subset \Gamma$ by

$$\mu(\Omega) = \sum_{x \in \Omega} \mu(x).$$

The couple (Γ, μ) is called a weighted graph (here μ refers both to the weight μ_{xy} and to the measure). There is a natural Laplace operator Δ_μ on (Γ,μ) which acts on functions on Γ by

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y \in \Gamma} (\nabla_{xy} f)\, \mu_{xy}$$

where

$$\nabla_{xy} f = f(y) - f(x).$$

It is easy to verify that Δ_μ is a bounded self-adjoint operator in $L^2(\Gamma, \mu)$. Its energy form is given by

$$\mathcal{E}_\mu(f) = \sum_{x,y \in \Gamma} |\nabla_{xy} f|^2 \mu_{xy}.$$

The heat kernel $p_n(x,y)$ is defined for non-negative integers n and for all $x, y \in \Gamma$ as a kernel with respect to μ of the operator $(I + \Delta)^n$. An operator $P = I + \Delta$ acts by

$$Pf(x) = \frac{1}{\mu(x)} \sum_{y \in \Gamma} f(y) \mu_{xy} = \sum_{y \in \Gamma} P(x,y) f(y)$$

where

$$P(x,y) = \frac{\mu_{xy}}{\mu(x)}.$$

The operator P is Markov and defines a nearest neighborhood random walk X_n on Γ by the rule

$$\mathbb{P}\left(X_{n+1} = y \mid X_n = x\right) = P(x,y).$$

Denote by $P_n(x,y)$ the transition function of X_n, that is

$$P_n(x,y) = \mathbb{P}\left(X_n = y \mid X_0 = x\right).$$

Then the heat kernel p_n is a density of P_n with respect to measure μ:

$$p_n(x,y) = \frac{P_n(x,y)}{\mu(y)}.$$

There are practically no explicit formulas for heat kernels on graphs, even in simple situations. If $\Gamma = \mathbb{Z}^N$ and μ is a standard weight then $p_n(x,y)$ admits the following Gaussian estimates[1]

$$p_n(x,y) \simeq \frac{1}{n^{N/2}} \exp\left(-\frac{d^2(x,y)}{cn}\right) \tag{1}$$

provided

$$n \geq d(x,y) \quad \text{and} \quad n \equiv d(x,y) \bmod 2.$$

If $n < d(x,y)$ then always $p_n(x,y) = 0$. If n and $d(x,y)$ have different parities then $p_n(x,y) = 0$ for all bipartite graphs, in particular for $\mathbb{Z}^N$.

2.3. Metric-measure-energy spaces

Let (M, d) be a locally compact separable metric space. Suppose that a Borel measure μ is defined on M, which is finite on bounded sets. Assume also that (M, μ) admits an energy form $\mathcal{E}$ which is a regular Dirichlet form on $L^2(M, \mu)$.

In general, any energy form $\mathcal{E}$ has a generator $-\Delta$ which is a self-adjoint operator on $L^2(M, \mu)$ with a dense domain, so that

$$\mathcal{E}(f) = -\int_M f \, \Delta f \, d\mu,$$

for all $f \in dom(\Delta)$ (see for example [14, Theorem 4.4.2], [16]). It is natural to say that Δ is a Laplace operator of the space $(M, \mu, \mathcal{E})$.

The heat semigroup P_t is defined as an one-parameter family of operators $\left\{e^{t\Delta}\right\}_{t \geq 0}$ in $L^2(M, \mu)$. If P_t has an integral kernel $p_t(x,y)$ with respect to measure μ then it is called a heat kernel of Δ (or $\mathcal{E}$). There are various conditions for existence of a heat kernel and its continuity. With $\mathcal{E}$ one associates a Hunt process X_t on M with generator Δ and transition density $p_t(x,y)$.

Weighted manifolds are simple examples of metric-measure-energy spaces. Graphs fit also apart from the fact that on graphs it is more natural to consider the heat kernel with discrete time, which arises from the semi-group $(I + \Delta)^n$ rather than $e^{t\Delta}$, although the latter can also be considered.

An example of different nature arises from fractal sets. Let M be a fractal set in $\mathbb{R}^N$ such as a Sierpinski gasket or a Sierpinski carpet (see [2], [35]). A distance d on M is inherited from the ambient space. A measure μ is a Hausdorff measure of a proper dimension α, namely, α is just the Hausdorff dimension of M. Definition of an energy form is highly non-trivial. This is done by using the self-similarity structure of M and a limiting procedure, and we refer the reader to [2]. In any case, $\mathcal{E}$ can be defined and, moreover, the corresponding heat kernel $p_t(x,y)$ is jointly continuous in $x, y \in M$ and $t > 0$.

[1] The constant c in the exponential term of (1) may be different for the upper and lower bounds. This remark applies to all Gaussian estimates to be considered below.

The heat kernel estimate which is described below is due to Barlow – Perkins [4] for a Sierpinski gasket and Barlow – Bass [3] for a Sierpinski carpet. This estimate requires two parameters which describe the geometry of the underlying space. The first one is α which is the Hausdorff dimension of M, and the second one is "a walk dimension" which we denote by β. It can be defined, for example, as follows. Denote by $B(x,r)$ a metric ball

$$B(x,r) = \{y \in M : d(x,y) < r\} \tag{2}$$

and by $T_{x,r}$ the first exit time from the ball $B(x,r)$, that is

$$T_{x,r} = \inf\{t \geq 0 : X_t \notin B(x,r)\}. \tag{3}$$

Then β is defined by the relation

$$\mathbb{E}_x T_{x,r} \simeq r^\beta,$$

for all $x \in M$ and $r < r_0$ (here r_0 is either finite or infinite depending on whether M is bounded or not). It has been proved that $\beta > 2$ on the fractals, and the heat kernel admits the following *sub-Gaussian estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{ct}\right)^{\frac{1}{\beta-1}}\right), \tag{4}$$

for all $x, y \in M$ and $t < t_0 = t_0(r_0)$.

Observe that the Gaussian heat kernel in $\mathbb{R}^N$ satisfies (4) with $\alpha = N$ and $\beta = 2$. Hence, we can say that fractals extend the family of Euclidean spaces in two ways: first, allowing fractional values of α, and second, introducing a second parameter β so that the potential theory on such spaces is determined by *two* parameters.

3. On-diagonal upper bounds and Faber-Krahn inequalities

We will distinguish various types of heat kernel estimates, and start with an *on-diagonal upper bound*

$$p_t(x,x) \leq \frac{C}{f(ct)}, \tag{5}$$

where $f(t)$ is an increasing function of t. For simplicity, let us restrict to the case when the underlying space is a complete non-compact weighted manifold. Then (5) is supposed to hold for all $t > 0$ and $x \in M$.

The necessary and sufficient condition for (5) can be stated in terms of *a Faber-Krahn inequality*. For any precompact region $\Omega \subset M$, let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of $-\Delta_\mu$, that is

$$\lambda_1(\Omega) = \inf_{\varphi \in C_0^\infty(\Omega), \varphi \not\equiv 0} \frac{\int_\Omega |\nabla\varphi|^2 d\mu}{\int_\Omega \varphi^2 d\mu}.$$

Given a decreasing non-negative function Λ on $(0,\infty)$, we say that (M,μ) admits a Faber-Krahn inequality with function Λ if, for any precompact region Ω,

$$\lambda_1(\Omega) \geq \Lambda(\mu(\Omega)). \tag{6}$$

For example, $\mathbb{R}^N$ admits a Faber-Krahn inequality with function $\Lambda(v) = cv^{-2/N}$.

Theorem 3.1. ([18]) *If (M,μ) admits a Faber-Krahn inequality with function Λ then the heat kernel upper bound (5) holds with function f defined by*

$$t = \int_0^{f(t)} \frac{dv}{v\Lambda(v)}, \tag{7}$$

assuming that the integral converges at 0.

Conversely, if (5) holds with a function f satisfying certain regularity condition then (M,μ) admits a Faber-Krahn inequality with function $c\Lambda(C\cdot)$ where Λ is defined by (7).

Consider some examples. Let (M,μ) be a N-dimensional Riemannian manifold of bounded geometry. Then it admits a Faber-Krahn inequality with function

$$\Lambda(v) = c\begin{cases} v^{-2/N}, & v < 1, \\ v^{-2}, & v \geq 1, \end{cases}$$

which implies the estimate

$$p_t(x,x) \le \frac{C}{\min\left(t^{N/2}, t^{1/2}\right)}$$

(see [8], [19]). This estimate is sharp if $M = \mathbb{R} \times K$ where K is an $(N-1)$-dimensional compact manifold.

Let (M,μ) admit a discrete group G of isometries with a compact fundamental domain. Then, by [12] (see also [21, Section 7.6]) a Faber-Krahn function is determined by the volume function $V(x,r) = \mu(B(x,r))$ as follows: fix a point x_0, denote $V(r) = V(x_0,r)$ and define

$$\Lambda(v) = \left(\frac{c}{V^{-1}(Cv)}\right)^2. \tag{8}$$

For example, if $V(r) \simeq e^{cr}$, for large r, then we obtain, for large v,

$$\Lambda(v) \simeq \frac{1}{\log^2 v}$$

whence, for large t,

$$p_t(x,x) \le C \exp\left(-ct^{1/3}\right). \tag{9}$$

Similar relation between a volume growth and a Faber-Krahn function Λ takes place on manifolds of non-negative Ricci curvature. Indeed, if M is such a manifold and in addition $V(x,r) \ge cr^\alpha$ for all $r \ge 1$ and $x \in M$, then M admits a Faber-Krahn inequality with function Λ defined by (8) for $V(r) = cr^\alpha$, which yields for large v

$$\Lambda(v) \simeq v^{-2/\alpha}$$

(see [22, p.198]). Therefore, we obtain by Theorem 3.1, for large t,

$$p_t(x,x) \le Ct^{-\alpha/2}.$$

This estimate follows also from [28].

4. On-diagonal lower bounds and anti-Faber-Krahn inequalities

Here we consider the case of graphs and lower bounds of the following type:

$$\sup_{x\in\Gamma} p_{2n}(x,x) \ge \frac{c}{f(Cn)}. \tag{10}$$

We say that a graph (Γ,μ) admits *anti-Faber-Krahn inequality* with a function Λ, if there exists a sequence $\{\Omega_k\}$ of non-empty finite sets in Γ and a numerical sequence $\{v_k\}$ such that

$$v_{k+1} \le Cv_k, \quad \lim_{k\to\infty} v_k = +\infty,$$

and

$$\mu(\Omega_k) \le v_k \quad \text{and} \quad \lambda_1(\Omega_k) \le \Lambda(v_k).$$

Theorem 4.1. ([11]) *If M admits anti-Faber-Krahn inequality with function Λ then the heat kernel lower estimate (10) holds with function f defined by (7) provided f possesses certain regularity.*

A particularly nice application occurs if Γ is a Cayley graph of a finitely generated group G. In this case we take μ to be a standard measure, which implies that $p_n(x,x)$ does not depend on x. On certain classes of groups, one can directly construct the sets $\{\Omega_k\}$ with controlled volumes and eigenvalues. To obtain an upper bound for $\lambda_1(\Omega_k)$, it suffices to have a smaller set $\Omega'_k \subset \Omega_k$ such that

$$\mu(\Omega'_k) \ge c\mu(\Omega_k) \quad \text{and} \quad d(\Omega'_k, \complement\Omega_k) \ge c\rho_k$$

with some numerical sequence $\{\rho_k\}$. Then it is easy to show that

$$\lambda_1(\Omega_k) \le C\rho_k^{-2}.$$

For example, if G is a polycyclic group then, using structure results, it is possible to construct sequences of sets Ω_k and Ω'_k so that $v_k = C^k$ and $\rho_k = k$ whence

$$\lambda_1(\Omega_k) \le \frac{C}{k^2} = \frac{C}{\log^2 v_k}$$

Therefore, we can take

$$\Lambda(v) = \frac{C}{\log^2 v},$$

which implies by (7) and (10)

$$p_{2n}(x,x) \geq c \exp\left(-Cn^{1/3}\right).$$

This estimate was first proved by Alexopoulos [1] by a different method. Together with a discrete version of the upper bound (9), it provides a sharp on-diagonal heat kernel decay on polycyclic groups of exponential volume growth.

5. Gaussian estimates

Let us return to the setting of manifolds.

Theorem 5.1. ([39], [20]) *Assume that, for two points x, y on a arbitrary weighted manifold (M, μ),*

$$p_t(x,x) \leq \frac{1}{f(t)} \quad \text{and} \quad p_t(y,y) \leq \frac{1}{g(t)}, \quad \text{for all } 0 < t < t_0,$$

where f and g are monotone increasing functions possessing certain regularity. Then, for all $0 < t < t_0$,

$$p_t(x,y) \leq \frac{C}{\sqrt{f(ct)g(ct)}} \exp\left(-\frac{d^2(x,y)}{Ct}\right). \tag{11}$$

Here t_0 may be either finite or infinite. In particular, the Faber-Krahn inequality (6) implies (11) for all $x, y \in M$ and $t > 0$ with functions $f = g$ defined by (7). As we see, the Gaussian upper bound does *not* require any additional geometric assumptions on top of those which already provide the on-diagonal estimates.

The question of Gaussian lower bound is less understood. However, for two-sided Gaussian estimates there is the following result. Denote

$$V(x,r) = \mu(B(x,r)).$$

Also, let $\lambda_1^{(N)}(\Omega)$ be the first non-zero Neumann eigenvalue of $-\Delta_\mu$ in a region Ω. The following result can be extracted from [32], [17].

Theorem 5.2. *Let (M, μ) be a complete non-compact weighted manifold, and let $f(t)$ be a positive increasing function on $(0, \infty)$ satisfying the doubling property*

$$f(2t) \leq Cf(t), \quad \forall t > 0.$$

Then a two-sided heat kernel bound

$$p_t(x,y) \simeq \frac{1}{f(t)} \exp\left(-\frac{d^2(x,y)}{ct}\right), \tag{12}$$

for all $x, y \in M$ and $0 < t < t_0$, is equivalent to the following two conditions (valid for all $x \in M$ and $0 < r < c\sqrt{t_0}$):

1. *the volume growth condition*

$$V(x,r) \simeq f(r^2)$$

2. *and the Poincaré inequality*

$$\lambda_1^{(N)}(B(x,r)) \geq cr^{-2}.$$

Here t_0 may be either finite or infinite. A similar result holds on graphs [15] and on local Dirichlet spaces [37]. For example, a uniform volume growth $V(x,r) \simeq r^\alpha$ together with the Poincaré inequality is equivalent on any of those spaces to the estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/2}} \exp\left(-\frac{d^2(x,y)}{ct}\right).$$

A discrete version of Theorem 5.2 applies on a Cayley graph (Γ, μ) of any finitely generated group G with polynomial volume growth (see [26]), in which case α is exactly the exponent of the volume growth of G.

Note also that the Poincaré inequality holds on any complete Riemannian manifold of non-negative Ricci curvature (see [5]). For such manifolds, the estimate (12) was first proved by Li and Yau [28].

6. Sub-Gaussian estimates

We state the result of this section for graphs although with certain modifications it holds also on manifolds and fractals. We would like to find conditions for a weighted graph (Γ, μ) which would ensure the heat kernel sub-Gaussian estimate similar to (4). By sub-Gaussian heat kernel estimates on graphs we mean the following inequalities:

$$p_n(x,y) \le Cn^{-\alpha/\beta} \exp\left(-\left(\frac{d(x,y)^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right) \tag{13}$$

and

$$p_n(x,y) + p_{n+1}(x,y) \ge cn^{-\alpha/\beta} \exp\left(-\left(\frac{d(x,y)^\beta}{cn}\right)^{\frac{1}{\beta-1}}\right), \quad n \ge d(x,y), \tag{14}$$

where x, y are arbitrary points on Γ and n is a positive integer. The necessity of considering $p_n + p_{n+1}$ for the lower bound arrises from a possible parity problem (in general, either p_n or p_{n+1} may vanish or be very small but a priori we cannot say which one is small).

To state the result, we need the volume function

$$V(x,r) = \mu(B(x,r)) = \mu\left\{y \in \Gamma : d(x,y) < r\right\},$$

as well as the Green function

$$G(x,y) = \sum_{n=0}^{\infty} P_n(x,y),$$

which, alternatively, is the infimum of all positive fundamental solutions to Δ_μ. It may happen $G \equiv \infty$ in which case the random walk X_n is recurrent.

Theorem 6.1. ([23]) *Let (Γ, μ) be a weighted graph, and assume that, for some positive constant p_0,*

$$P(x,y) \ge p_0 \quad \forall x \sim y. \tag{15}$$

Given two numbers $\alpha > \beta > 1$, the sub-Gaussian estimates (13) and (14) are equivalent to the following two conditions:

1. *the polynomial volume growth, for all $x \in M$ and $r \ge 1$,*

$$V(x,r) \simeq r^\alpha \tag{16}$$

2. *and the polynomial Green function decay, for all $x \ne y$,*

$$G(x,y) \simeq d(x,y)^{-(\alpha-\beta)}. \tag{17}$$

The Green kernel uniform decay (17) implies a *uniform Harnack inequality*

$$\begin{array}{c} \sup_{B(x,r)} u \le C \inf_{B(x,r)} u\,, \\ \text{provided } \Delta_\mu u = 0 \text{ and } u \ge 0 \text{ in } B(x,2r), \end{array} \tag{18}$$

for all $x \in \Gamma$ and $r > 0$. Conversely, assuming the Harnack inequality (18) and the volume growth (16), one can deduce the Green function estimate (17) from one of the following conditions:

1. The estimate of the first Dirichlet eigenvalue of $-\Delta_\mu$ in a ball

$$\lambda_1(B(x,r)) \simeq r^{-\beta} \tag{19}$$

2. The capacity estimate

$$\operatorname{cap}(B(x,r), B(x,2r)) \simeq r^{\alpha-\beta} \tag{20}$$

3. The mean exit time estimate

$$\mathbb{E}_x T_{x,r} \simeq r^\beta. \tag{21}$$

All conditions are assumed to be true for all $x \in \Gamma$ and $r \ge 1$. The capacity is defined to be the infimum of $\mathcal{E}(\varphi)$ over test functions φ which are equal to 1 on $B(x,r)$ and vanish outside $B(x,2r)$. The first exit time $T_{x,r}$ is defined by (3).

The following theorem covers also the recurrent case $G \equiv \infty$ when Theorem 6.1 is not applicable.

Theorem 6.2. ([24]) *Assume that (15) holds on (Γ, μ). Given two numbers $\alpha > 0$ and $\beta > 1$, the sub-Gaussian estimates (13) and (14) are equivalent to the following three conditions:*

1. *the Harnack inequality (18)* (which provides the homogeneity of the graph in question)

2. *the volume growth (16)* (which determines the parameter α)
3. *and any one of the conditions (19), (20), (21)* (which determines the second parameter β).

Although a priori we assume only $\alpha > 0$ and $\beta > 1$, the hypothesis (16) and any of the conditions (19), (20), (21) imply $2 \leq \beta \leq \alpha + 1$ (see [38]).

Note that (19) contains a Faber-Krahn inequality for balls:

$$\lambda_1(B(x,r)) \geq cV(x,r)^{-\beta/\alpha}$$

which in the presence of the Harnack inequality can be extended to arbitrary non-empty finite sets Ω as follows:

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}. \tag{22}$$

By a discrete version of Theorem 3.1, one obtains from (22) an on-diagonal heat kernel upper bound

$$p_n(x,x) \leq \frac{C}{n^{a/\beta}},$$

which is the first step in the proof of Theorem 6.2.

A geometric background for the Harnack inequality (18) is yet to be understood.

7. Higher eigenvalues

Let (M,μ) be a weighted manifold. For any precompact domain $\Omega \subset M$, denote by $\lambda_k(\Omega)$ the k-th Dirichlet eigenvalue of $-\Delta_\mu$ in Ω.

Theorem 7.1. ([18]) *Assume that (M,μ) admits a Faber-Krahn inequality with function Λ, that is, for any precompact Ω,*

$$\lambda_1(\Omega) \geq \Lambda(\mu(\Omega)).$$

If Λ possesses certain regularity property then, for all integers $k > 1$ and precompact Ω,

$$\lambda_k(\Omega) \geq c\Lambda\left(C\frac{\mu(\Omega)}{k}\right).$$

The proof goes through the heat kernel upper bound given by Theorem 3.1. In particular, we have

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\delta} \quad \Longrightarrow \quad \lambda_k(\Omega) \geq c\left(\frac{k}{\mu(\Omega)}\right)^{\delta}.$$

This results admits a generalization as follows. Assume that, apart from the measure μ, there is another Radon measure σ on M, and consider the following quadratic form associated with σ:

$$\mathcal{E}_\sigma(f) = \int_M |\nabla f|^2 \, d\sigma.$$

Assuming that in any precompact region $\Omega \subset M$ the form $\mathcal{E}_\sigma$ with domain $C_0^\infty(\Omega) \subset L^2(\Omega,\mu)$ is closable and has a discrete spectrum, we denote its k-th eigenvalue by $\lambda_k(\Omega,\mathcal{E}_\sigma)$. Note that the associated Rayleigh quotient is

$$\frac{\int_\Omega |\nabla f|^2 \, d\sigma}{\int_\Omega f^2 d\mu}.$$

The proof of the following theorem is based on the heat kernel techniques as well as on ideas from [27].

Theorem 7.2. ([25]) *Assume that there exists a Radon measure ν on M such that, for any precompact domain Ω and some $\delta > 0$,*

$$\lambda_1(\Omega,\mathcal{E}_\sigma) \geq \nu(\Omega)^{-\delta}.$$

Then, for all integers $k > 1$ and precompact Ω,

$$\lambda_k(\Omega,\mathcal{E}_\sigma) \geq c\left(\frac{k}{\nu(\Omega)}\right)^{\delta}.$$

References

[1] **Alexopoulos G.K.**, A lower estimate for central probabilities on polycyclic groups, *Can. J. Math.*, **44** (1992) no.5, 897-910.

[2] **Barlow M.T.**, Diffusions on fractals, *in:* "Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités de Saint-Flour XXV - 1995", Lecture Notes Math. 1690, Springer, 1998. 1-121.

[3] **Barlow M.T., Bass R.F.**, Brownian motion and harmonic analysis on Sierpinski carpets, *Canad. J. Math.*, **54** (1999) 673-744.

[4] **Barlow M.T., Perkins E.A.**, Brownian motion on the Sierpinski gasket, *Probab. Th. Rel. Fields*, **79** (1988) 543–623.

[5] **Buser P.**, A note on the isoperimetric constant, *Ann. Sci. Ecole Norm. Sup.*,, **15** (1982) 213-230.

[6] **Chavel I.**, "Eigenvalues in Riemannian geometry", Academic Press, New York, 1984.

[7] **Chavel I.**, "Isoperimetric inequalities: differential geometric and analytic perspectives", Cambridge Tracts in Mathematics 145, Cambridge University Press, 2001.

[8] **Chavel I., Feldman E.A.**, Modified isoperimetric constants, and large time heat diffusion in Riemannian manifolds, *Duke Math. J.*, **64** (1991) no.3, 473-499.

[9] **Chung F.R.K.**, "Spectral Graph Theory", CBMS Regional Conference Series in Mathematics 92, AMS publications, 1996.

[10] **Coulhon T.**, Heat kernels on non-compact Riemannian manifolds: a partial survey, *Séminaire de théorie spectrale et géométrie, 1996-1997, Institut Fourier, Grenoble*, **15** (1998) 167-187.

[11] **Coulhon T., Grigor'yan A., Pittet Ch.**, A geometric approach to on-diagonal heat kernel lower bounds on groups, *Ann. Inst. Fourier, Grenoble*, **51** (2001) no.6, 1763-1827.

[12] **Coulhon T., Saloff-Coste L.**, Isopérimétrie pour les groupes et les variétés, *Revista Matemática Iberoamericana*, **9** (1993) no.2, 293-314.

[13] **Davies E.B.**, "Heat kernels and spectral theory", Cambridge University Press, 1989.

[14] **Davies E.B.**, "Spectral theory and differential operators", Cambridge University Press, 1995.

[15] **Delmotte T.**, Parabolic Harnack inequality and estimates of Markov chains on graphs, *Revista Matemática Iberoamericana*, **15** (1999) no.1, 181-232.

[16] **Fukushima M., Oshima Y., Takeda M.**, "Dirichlet forms and symmetric Markov processes", Studies in Mathematics 19, De Gruyter, 1994.

[17] **Grigor'yan A.**, The heat equation on non-compact Riemannian manifolds, (in Russian) *Matem. Sbornik*, **182** (1991) no.1, 55-87. Engl. transl. *Math. USSR Sb.*, **72** (1992) no.1, 47-77.

[18] **Grigor'yan A.**, Heat kernel upper bounds on a complete non-compact manifold, *Revista Matemática Iberoamericana*, **10** (1994) no.2, 395-452.

[19] **Grigor'yan A.**, Heat kernel on a manifold with a local Harnack inequality, *Comm. Anal. Geom.*, **2** (1994) no.1, 111-138.

[20] **Grigor'yan A.**, Gaussian upper bounds for the heat kernel on arbitrary manifolds, *J. Diff. Geom.*, **45** (1997) 33-52.

[21] **Grigor'yan A.**, Estimates of heat kernels on Riemannian manifolds, *in:* "Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh 1998", Ed. B.Davies and Yu.Safarov, London Math. Soc. Lecture Note Series 273, Cambridge Univ. Press, 1999. 140-225.

[22] **Grigor'yan A.**, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.*, **36** (1999) 135-249.

[23] **Grigor'yan A., Telcs A.**, Sub-Gaussian estimates of heat kernels on infinite graphs, *Duke Math. J.*, **109** (2001) no.3, 452-510.

[24] **Grigor'yan A., Telcs A.**, Harnack inequalities and sub-Gaussian estimates for random walks, *Math. Ann.*, **324** no.3, (2002) 521-556.

[25] **Grigor'yan A., Yau S.-T.**, Isoperimetric properties of higher eigenvalues of elliptic operator, to appear in *Amer. J. Math*

[26] **Hebisch W., Saloff-Coste, L.**, Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.*, **21** (1993) 673–709.

[27] **Li P., Yau S.-T.**, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.*, **88** (1983) 309–318.

[28] **Li P., Yau S.-T.**, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986) no.3-4, 153-201.

[29] **Maz'ya V.G.**, "Sobolev spaces", (in Russian) Izdat. Leningrad Gos. Univ. Leningrad, 1985. Engl. transl. Springer, 1985.

[30] **Pittet Ch., Saloff-Coste L.**, A survey on the relationship between volume growth, isoperimetry, and the behavior of simple random walk on Cayley graphs, with examples, preprint

[31] **Robinson D.W.**, "Elliptic operators and Lie groups", Oxford Math. Mono., Clarenton Press, Oxford New York Tokyo, 1991.

[32] **Saloff-Coste L.**, A note on Poincaré, Sobolev, and Harnack inequalities, *Duke Math J.*, **65** no.3, *Internat. Math. Res. Notices*, **2** (1992) 27-38.

[33] **Saloff-Coste L.**, Parabolic Harnack inequality for divergence form second order differential operators, *Potential Analysis*, **4** (1995) 429-467.

[34] **Schoen R., Yau S.-T.**, "Lectures on Differential Geometry", Conference Proceedings and Lecture Notes in Geometry and Topology 1, International Press, 1994.

[35] **Strichartz R.S.**, Analysis on fractals, *Notices of AMS*, **46** (1999) no.10, 1199-1208.

[36] **Stroock D.W.**, Estimates on the heat kernel for the second order divergence form operators, *in:* "Probability theory. Proceedings of the 1989 Singapore Probability Conference held at the National University of Singapore, June 8-16 1989", Ed. L.H.Y. Chen, K.P. Choi, K. Hu and J.H. Lou, Walter De Gruyter, 1992. 29-44.

[37] **Sturm K-Th.**, Analysis on local Dirichlet spaces III. The parabolic Harnack inequality, *Journal de Mathématiques Pures et Appliquées*, **75** (1996) no.3, 273-297.

[38] **Telcs A.**, Random walks on graphs, electric networks and fractals, *J. Prob. Theo. and Rel. Fields*, **82** (1989) 435-449.

[39] **Ushakov V.I.**, Stabilization of solutions of the third mixed problem for a second order parabolic equation in a non-cylindric domain, (in Russian) *Matem. Sbornik*, **111** (1980) 95-115. Engl. transl. *Math. USSR Sb.*, **39** (1981) 87-105.

[40] **Varopoulos N.Th., Saloff-Coste L., Coulhon T.**, "Analysis and geometry on groups", Cambridge University Press, Cambridge, 1992.

[41] **Woess W.**, "Random walks on infinite graphs and groups", Cambridge Tracts in Mathematics 138, Cambridge Univ. Press., 2000.

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