On biparabolicity of Riemannian manifolds

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1 Introduction

The notion of parabolicity of a Riemannian manifold has been studied by many authors for more than a half-century. A Riemannian manifold M is called parabolic if any positive superharmonic function on M is identically constant. For example, it is well known that $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$.

The term "parabolic" comes from the Classification Theory of Riemann surfaces. By the famous Uniformization Theorem of Koebe-Poincaré, any simply connected Riemann surface S is conformally equivalent to either the sphere $\mathbb{S}^2$ or the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$. In the first case S is called *elliptic*, in the second case *parabolic*, and in the third case *hyperbolic*. It is easy to prove that for non-compact S the parabolicity of S is equivalent to the property, that any positive superharmonic function is constant. Hence, the latter is taken as definition of parabolicity for a Riemannian manifold of arbitrary dimension.

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It is well-known that the parabolicity of a Riemannian manifold M is equivalent to the absence of a positive Green function as well as to the recurrence of Brownian motion on M (see, for example, [4], [8]).

A famous theorem of Cheng and Yau [1] provides a sufficient condition for parabolicity in terms of the volume growth. Let $V(x,r)$ denote the Riemannian volume of the geodesic ball on M of radius r centered at $x \in M$. The theorem of Cheng and Yau says that, if M is geodesically complete and, for some $x_0 \in M$ and constant C,

$$V(x_0,r) \le Cr^2 \quad \text{for } r \to \infty,$$

then M is parabolic.

The purpose of this work is to investigate a similar notion of biparabolicity of a Riemannian manifold M. Let Δ denote the *Laplace-Beltrami* operator on M. A function $u \in C^4(M)$ is called *bi-superharmonic* if $\Delta u \le 0$ and $\Delta^2 u \ge 0$. The manifold M is called *biparabolic*, if any positive bi-superharmonic function is harmonic, that is $\Delta u = 0$.

Note that the notion of parabolicity admits a similar equivalent definition: M is parabolic if and only if any positive superharmonic function is harmonic.

The main result of this work is the following sufficient condition for biparabolicity: if M is a geodesically complete manifold and for some $x_0 \in M$

$$V(x_0,r) \le C\frac{r^4}{\log r} \quad \text{for } r \to \infty, \tag{1.1}$$

then M is biparabolic (Theorem 3.5).

We also show that the condition (1.1) is nearly optimal in the following sense: for any $\beta > 1$ there exists a geodesically complete manifold M with

$$V(x_0,r) \le C\, r^4 \log^\beta r \quad \text{for } r \to \infty$$

such that M is not biparabolic (Section 4).

2 Weighted manifolds

In fact, we state and prove the main result in a more general setting of weighted manifolds. A weighted manifold is a couple (M,μ) where M is a connected Riemannian manifold and μ is a measure on M with a positive smooth density with respect to the Riemannian measure ν. Denote this density by h^2, that is, $d\mu = h^2 d\nu$, where h is a smooth positive function on M. The weighted Laplace operator of (M,μ) is defined by

$$\Delta_\mu = \frac{1}{h^2}\operatorname{div}(h^2\nabla),$$

where div and ∇ are the Riemannian divergence and gradient, respectively. Of course, in the case $h \equiv 1$ the operator Δ_μ coincides with the Laplace-Beltrami operator Δ. For convenience we will use the notation

$$\mathcal{L} = -\Delta_\mu.$$

A C^2 function u on M is called superharmonic if $\mathcal{L}u \ge 0$. The weighted manifold (M,μ) is called parabolic if any positive superharmonic function is constant.

It is easy to see that $\mathcal{L}$ satisfies the Green formula with respect to the measure μ, that is, for smooth functions u and v

$$\int_M u\mathcal{L}v\,d\mu = \int_M \langle\nabla u, \nabla v\rangle d\mu = \int_M v\mathcal{L}u\,d\mu, \tag{2.1}$$

provided u or v has a compact support. Consequently, the operator $\mathcal{L}$ with the domain $C_0^\infty(M)$ is a symmetric operator in $L^2(M,\mu)$. It extends canonically to a self-adjoint, non-negative definite operator in $L^2(M,\mu)$ that will be denoted also by $\mathcal{L}$. Hence, it determines a heat semigroup $P_t = e^{-t\mathcal{L}}$, $t \geq 0$, acting in $L^2(M,\mu)$. This semigroup has a smooth positive kernel $p_t(x,y)$ that is called the heat kernel of (M,μ). The Green function $g(x,y)$ is then defined by

$$g(x,y) = \int_0^\infty p_t(x,y)dt, \tag{2.2}$$

The parabolicity of (M,μ) is equivalent to $g(x,y) \equiv \infty$. We assume in what follows that the Green function $g(x,y)$ is finite (which means that $g(x,y) < \infty$ for all $x \neq y$).

Define the Green operator G on all non-negative measurable functions f on M by

$$Gf(x) = \int_M g(x,y)f(y)d\mu(y) = \int_0^\infty P_tf\ (x)\,dt. \tag{2.3}$$

For any $k \in \mathbb{N}$, let G^k be the k-th operator power of G, that is,

$$G^kf(x) = \int_{M^k} g(x,x_1)\,g(x_1,x_2)\,...g(x_{k-1},x_k)\,f(x_k)\,d\mu(x_1)\,...d\mu(x_k)\,. \tag{2.4}$$

It is easy to prove that

$$G^kf(x) = \int_0^\infty \frac{t^{k-1}}{(k-1)!}P_tf(x)\,dt. \tag{2.5}$$

If the integral in (2.4) (or, equivalently, in (2.5)) diverges for any non-zero, non-negative function $f \in C_0^\infty(M)$ then, we write $G^k \equiv \infty$, which is equivalent to

$$\int_{M^{k-1}} g(x,x_1)\,g(x_1,x_2)\,...g(x_{k-1},y)\,d\mu(x_1)\,...d\mu(x_{k-1}) \equiv \infty$$

for all $x, y \in M$.

Let $d(x,y)$ be the geodesic distance on M and $B(x,r)$ denote open geodesic balls in M.

3 Sufficient conditions for biparabolicity

Let (M,μ) be a weighted manifold as above. A function $u \in C^4(M)$ is called bi-superharmonic if $\mathcal{L}u \geq 0$ and $\mathcal{L}^2u \geq 0$. The manifold M is called biparabolic, if any positive bi-superharmonic function is harmonic, that is $\mathcal{L}u = 0$.

3.1 Biparabolicity and Green operator

Theorem 3.1 *A weighted manifold (M, μ) is biparabolic if and only if $G^2 \equiv \infty$.*

We will prove this Theorem after the following lemmas.

Lemma 3.2 *If $G^2 \equiv \infty$ then, for any positive harmonic function h on M, we have $Gh \equiv \infty$.*

Proof. Assume from the contrary that, for some positive harmonic function h on M, $Gh < \infty$, and let us prove that $G^2 < \infty$.

The proof is split into a series of claims. Fix a point $y \in M$ and some ball B centered at y.

Claim 1. *There exists a constant c depending on y and such that for all $z \in B^c$*

$$g(z, y) \leq ch(z).$$

Indeed, since $h > 0$ on ∂B and $g(z, y) < \infty$ on ∂B, then there exists c such that $g(z, y) \leq ch(z)$ for all $z \in \partial B$. It follows from the minimality property of the Green function and from the maximum principle that this inequality holds also in B^c.

Claim 2. *For all $x \in M$ we have*

$$\int_{B^c} g(x, z)g(z, y)d\mu\left(z\right) < \infty.$$

Indeed, it follows from Claim 1 that

$$\int_{B^c} g(x, z)g(z, y)d\mu\left(z\right) \leq c \int_{B^c} g(x, z)h\left(z\right) d\mu\left(z\right) \leq cGh\left(x\right) < \infty.$$

Claim 3. *For all $x \neq y$ we have*

$$\int_{B} g(x, z)g(z, y)d\mu(z) < \infty.$$

Indeed, the Green function $z \mapsto g(z, y)$ behaves in a small neighborhood of y as the Green function in $\mathbb{R}^n$ where $n = \dim M$; in particular, the Green function is locally integrable, whence the claim follows.

Combining Claims 2 and 3, we obtain $G^2 < \infty$, which finishes the proof of Theorem 3.1. ■

Lemma 3.3 ([5, Theorem 13.1]) *Let f be a non-negative function from $L^2_{loc}(M)$ such that $Gf \in L^2_{loc}(M)$. Then the function $u = Gf$ is the minimal non-negative solution in $L^2_{loc}(M)$ of the equation $\mathcal{L}u = f$ considered in the distributional sense. If in addition $f \in C^\infty$, then also $u \in C^\infty$.*

Lemma 3.4 *Let $f \in C^\infty\left(M\right)$ be a non-negative function such that $Gf(x) < \infty$, for some $x \in M$. Then $Gf \in C^\infty\left(M\right)$ and $\mathcal{L}(Gf) = f$.*

Proof. Let $\{\Omega_n\}$ be an exhaustion of M by a sequence of precompact, connected domains. Since f is bounded and smooth in Ω_n, by Lemma 3.3 we conclude that $G_{\Omega_n}f \in C^\infty(\Omega_n)$ and $\mathcal{L}(G_{\Omega_n}f) = f$ in Ω_n.

It is well know that $G_{\Omega_n}f \to Gf$ pointwise in M as $n \to \infty$. Let us verify that $Gf \in C^\infty(M)$. Fix some $m \in \mathbb{N}$ and consider for any $n > m$ the function

$$u_n = G_{\Omega_n}f - G_{\Omega_m}f.$$

Since $\mathcal{L}u_n = 0$ in Ω_m, the sequence $\{u_n\}$ is a monotone increasing sequence of harmonic functions that is bounded by $Gf(x)$ for any $x \in \Omega_m$. Therefore, the limit $\lim_{n\to\infty} u_n$ is a harmonic function in Ω_m. It follows that in Ω_m

$$Gf = G_{\Omega_m}f + \text{a harmonic function}.$$

Consequently, Gf is a locally bounded function on M, which implies by Lemma 3.3 that $Gf \in C^\infty(M)$ and $\mathcal{L}(Gf) = f$. ■

Now we can prove the Theorem 3.1.

Proof of Theorem 3.1. If $G^2 \not\equiv \infty$ then there exists a non-trivial, non-negative function $\varphi \in C_0^\infty(M)$ such that $G^2\varphi < \infty$ at least at one point. Then also $G\varphi$ is finite at least at one point. Applying twice Lemma 3.4 we obtain that $u := G^2\varphi \in C^\infty(M)$ as well as $\mathcal{L}u = G\varphi \geq 0$ and $\mathcal{L}^2u = \varphi \geq 0$. Hence, u is bi-superharmonic, but not harmonic (not even biharmonic).

Let now $G^2 \equiv \infty$. If u is a positive bi-superharmonic function, then set $v = \mathcal{L}u \geq 0$ and $w = \mathcal{L}v \geq 0$. Using the minimality property of the Green function, we obtain $v \geq Gw$ and $u \geq Gv$, whence $u \geq G^2w$. However, $G^2 \equiv \infty$, which implies $w \equiv 0$. Hence, $\mathcal{L}v = 0$ so that v is harmonic. We still have $u \geq Gv$, while by Lemma 3.2 $Gv \equiv \infty$, unless $v = 0$. Hence, $v = 0$ and u is harmonic. ■

3.2 Volume growth and biparabolicity

Our main result is as follows.

Theorem 3.5 *Let M be a geodesically complete weighted manifold. If, for some $x_0 \in M$ and for all $r \gg 1$,*

$$V(x_0, r) \leq C\frac{r^4}{\log r}, \tag{3.1}$$

where C is a positive constant, then M is biparabolic.

We use in the proof the following heat kernel estimate.

Lemma 3.6 ([2], [5, Theorem 16.5]) *Let M be a complete weighted manifold. Assume that, for some $x \in M$ and all $r \geq r_0$,*

$$V(x, r) \leq Cr^\nu,$$

where C, ν, r_0 are positive constant. Then, for all $t \geq t_0$,

$$p_t(x, x) \geq \frac{1/4}{V(x, \sqrt{Kt\log t})},$$

where $K = K(x, r_0, C, \nu)$ and $t_0 = max(r_0^2, 3)$.

We use also the following lemma that is a standard consequence of a local parabolic Harnack inequality for the heat equation (cf. [6], [7]).

Lemma 3.7 *For any ball $B_R(x_0)$, there is a constant $c = c(B_R(x_0)) > 0$ such that*

$$p_t(x, y) \geq cp_t(x_0, x_0),$$

for all $x, y \in B_{R/2}(x_0)$ and for all $t \geq t_0 = R^2$.

Proof of Theorem 3.5. By Theorem 3.1, it suffices to show that, for any non-negative non-zero function $\varphi \in C_0^\infty(M)$, we have $G^2\phi \equiv \infty$. By (2.5), we have, for any $x \in M$,

$$G^2\varphi(x) = \int_0^\infty tP_t\varphi(x)\, dt = \int_0^\infty \int_{\operatorname{supp}\varphi} tp_t(x, y)\varphi(y)d\mu(y)dt.$$

Fix arbitrary $x \in M$ and choose $R > 0$ so big that the ball $B_{R/2}(x_0)$ contains both $\operatorname{supp}\varphi$ and x. Applying Lemma 3.7 with this ball, we obtain

$$\begin{aligned} G^2\varphi(x) &= \int_0^\infty \int_{B_{R/2}(x_0)} tp_t(x, y)\varphi(y)d\mu(y)dt \\ &\geq \int_{t_0}^\infty \int_{B_{R/2}(x_0)} tp_t(x, y)\varphi(y)d\mu(y)dt \\ &\geq c||\varphi||_{L^1} \int_{t_0}^\infty tp_t(x_0, x_0)dt. \end{aligned}$$

By Lemma 3.6, we have, for large t,

$$p_t(x_0, x_0) \geq \frac{c}{v\left(\sqrt{t \log t}\right)},$$

where $v(r) = \frac{r^4}{\log r}$. Since for $t \to \infty$ we have

$$\frac{t}{v(\sqrt{t \log t})} = \frac{t}{\frac{(t \log t)^2}{\frac{1}{2}\log(t \log t)}} \sim \frac{2}{t \log t},$$

and hence

$$\int_{t_0}^\infty \frac{tdt}{v\left(\sqrt{t \log t}\right)} \asymp \int_{t_0}^\infty \frac{dt}{t \log t} = \infty,$$

we conclude that $G^2\varphi(x) = \infty$, which was to be proved. ■

In the next section, construct an example to show that under the volume growth

$$V(x_0, r) \leq Cr^4 \log^\beta r \tag{3.2}$$

with $\beta > 1$, one cannot claim biparabolicity.

Unfortunately, we were not able to fill in the gap between the positive result in the case of the volume growth (3.1) and the volume growth (3.2) in the counterexample.

4 Counter example

Fix $n \geq 2$ and consider a smooth manifold

$$M = \mathbb{R} \times \mathbb{S}^{n-1},$$

that is, any point $x \in M$ is represented in the polar form as (r, θ) where $r \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$. Define the Riemannian metric g on M by

$$\mathrm{g} = dr^2 + \psi^2(r) d\theta^2, \tag{4.1}$$

where $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and $\psi(r)$ is a smooth positive function on $\mathbb{R}$. Let μ be the Riemannian measure on M.

Define the area function $S(r)$, $r \in \mathbb{R}$, by

$$S(r) = \omega_n \psi(r)^{n-1},$$

where ω_n is the volume of $\mathbb{S}^{n-1}$. Then, for any domain of the form

$$\Omega_{a,b} = \left\{ (r, \theta) \in M : a < r < b, \theta \in \mathbb{S}^{n-1} \right\}$$

with $a < b$, we have

$$\mu(\Omega_{a,b}) = \int_a^b S(r)\, dr.$$

The Laplace-Beltrami operator Δ on M is represented in the polar coordinates as follows

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial}{\partial r} + \frac{1}{\psi^2(r)} \Delta_\theta, \tag{4.2}$$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. An easy consequence of (4.2) is that every radial harmonic function $v(r)$ in a domain $\Omega_{a,b}$ satisfies

$$v(r) = c_1 + c_2 \int_c^r \frac{dt}{S(t)}. \tag{4.3}$$

where $c \in [a, b]$ so that the integral converges, and c_1, c_2 are arbitrary constants.

Now let us choose ψ so that the area function $S(r)$ satisfies the identities

$$S(r) = \begin{cases} e^{-r^\alpha}, & r > r_0, \\ |r|^3 \log^\beta |r|, & r < -r_0, \end{cases} \tag{4.4}$$

with some $r_0 > 1$, where α, β are arbitrary real numbers such that

$$\alpha > 2 \text{ and } \beta > 1. \tag{4.5}$$

For $r \in [-r_0, r_0]$ the function $S(r)$ is defined arbitrary.

Proposition 4.1 *Under the hypotheses (4.4) and (4.5), the model manifold M is not biparabolic, and the volume growth function of M satisfies*

$$V(o, R) \leq C R^4 \log^\beta R. \tag{4.6}$$

Proof. Clearly, for large R, the volumes of the ball $B_R(o)$ and of the domain $\Omega_{-R,R}$ are comparable. It follows easily from (4.4), that

$$V(R) := \mu(\Omega_{-R,R}) = \int_{-R}^{R} S(r)dr \leq CR^4 \log^\beta R, \tag{4.7}$$

which implies (4.6).

In order to prove that M is not biparabolic, it suffices to construct a positive harmonic function h on M such that the function $u := Gh$ is finite at least at one point. Indeed, then we will conclude by Lemma 3.4 that $u \in C^\infty(M)$ and $\mathcal{L}u = h$. Hence, $\mathcal{L}u > 0$ and $\mathcal{L}^2 u = \mathcal{L}h = 0$ so that u is bi-superharmonic, but not harmonic, and M is not biparabolic.

The desired function $h(x)$ on M will depend only on the polar radius r of x, so define it as follows:

$$h(r) = \int_{-\infty}^{r} \frac{dt}{S(t)}. \tag{4.8}$$

The function h is harmonic on M because

$$\Delta h = h'' + \frac{S'}{S} h' = 0.$$

Before we can prove that $Gh < \infty$, let us discuss some properties of the Green function $g(x, y)$ on M.

For any $\theta \in \mathbb{S}^{n-1}$, set $y_\theta := (0, \theta) \in M$ and denote by Y the set of all points y_θ with arbitrary $\theta \in \mathbb{S}^{n-1}$. Define a function $\zeta : M \to \mathbb{R}$ by

$$\zeta(x) := \int_{\mathbb{S}^{n-1}} g(x, y_\theta) d\theta.$$

Since the model manifold M is invariant under rotations of $\mathbb{S}^{n-1}$, the Green function $g(x, y)$ is also invariant, which implies that $\zeta(x)$ depends only on the polar radius r of x. Hence, we will write also $\zeta(r) = \zeta(x)$.

Since the function $g(x, y_\theta)$ is harmonic in x in the domain $r \neq 0$, it follows that $\zeta(x)$ is also harmonic in the domains $\{r > 0\}$ and $\{r < 0\}$.

Fix some $x \in M$ with the polar radius $r \geq r_0$. Since the function $g(x, y)$ is harmonic in y in a neighborhood of Y, we obtain by the local Harnack inequality that

$$g(x, y) \asymp \zeta(r) \ \text{ for all } y \in Y. \tag{4.9}$$

It follows then from the well-known properties of the Green function that, for all $R > 0$,

$$\sup_{r > R} \zeta(r) < \infty \tag{4.10}$$

and

$$\inf_{r < -R} \zeta(r) = 0. \tag{4.11}$$

Before we continue the proof of Proposition 4.1, let us determine explicitly the function ζ as in the next statement.

Claim. *Function $\zeta(r)$ is constant in the domain $\{r > 0\}$, and it is of the form of*

$$\zeta(r) = c \int_{-\infty}^{r} \frac{dt}{S(t)} \tag{4.12}$$

in the domain $\{r<0\}$, where c is some positive constant.

Proof. Using the representation (4.3) of radial harmonic function on M, we obtain that, in the domain $\{r>0\}$, function $\zeta(r)$ has this form

$$\zeta(r)=c_1+c_2\int_0^r\frac{dt}{S(t)}.$$

By definition (4.4) of $S(r)$, the integral tends to $+\infty$ as $r\to+\infty$. Since by (4.10) the function $\zeta\,(r)$ is bounded for $r\to+\infty$, we see that $c_2=0$ and hence $\zeta(r)=\mathrm{const}$.

Since the integral

$$\int_{-\infty}^r\frac{dt}{S(t)}$$

converges due to (4.4), we obtain that the harmonic function $\zeta\,(r)$ admits in the domain $\{r<0\}$ the following representation:

$$\zeta(r)=c_1+c_2\int_{-\infty}^r\frac{dt}{S(t)}.$$

It follows from (4.4), that this integral tends to zero as $r\to-\infty$, whence.

$$\lim_{r\to-\infty}\zeta(r)=c_1.$$

It follows from (4.11) that $c_1=0$, which proves (4.12). ■

Returning to the proof of Proposition 4.1, consider the function $u=Gh$, where h is given by (4.8). Let us verify that $u\,(y)<\infty$ for any $y\in Y$, which will finish the proof. Indeed, for any $y\in Y$, we have

$$u(y)=Gh(y)=\int_{\Omega_{-\infty,-r_0}}g(\cdot,y)hd\mu+\int_{\Omega_{-r_0,r_0}}g(\cdot,y)hd\mu+\int_{\Omega_{r_0,+\infty}}g(\cdot,y)hd\mu.$$

Noticing that the middle integral is a constant, and estimating the other two integrals by (4.9), we obtain that

$$u(y)\asymp\int_{-\infty}^{-r_0}\zeta(r)h(r)S(r)dr+1+\int_{r_0}^{\infty}\zeta(r)h(r)S(r)dr. \tag{4.13}$$

To estimate the first integral, observe that, for $r<-r_0$, we have by (4.4), (4.8), (4.12)

$$h(r)\asymp\frac{1}{|r|^2\log^\beta|r|}\quad\text{and}\quad\zeta(r)\asymp\frac{1}{|r|^2\log^\beta|r|},$$

which implies that the first integral in (4.13) is comparable to

$$\int_{-\infty}^{-r_0}\frac{1}{|r|\log^\beta|r|}dr$$

which is finite since $\beta>1$. Similarly, for $r>r_0$, we have

$$h\,(r)=\mathrm{const}+\int_{r_0}^r e^{t^\alpha}dt\asymp\frac{e^{r^\alpha}}{r^{\alpha-1}}$$

and $\zeta\asymp1$, so that the third integral in (4.13) is comparable to

$$\int_{r_0}^\infty\frac{e^{r^\alpha}}{r^{\alpha-1}}e^{-r^\alpha}dr,$$

which is finite because $\alpha>2$. Hence, the right hand side of (4.13) is finite, which finishes the proof. ■

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