Regularity of jump-type Dirichlet forms on metric measure spaces

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Abstract

Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the volume doubling condition. Given a *jump measure* $J(x, dy)$, consider the symmetric bilinear form $\mathcal{E}$ defined by its quadratic part

$$\mathcal{E}(u,u) := \int_{\mathcal{X}} \int_{\mathcal{X}} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x),$$

where u is in the natural domain $\mathcal{F} := \{u \in L^2(\mathcal{X}, \mu) : \mathcal{E}(u,u) < \infty\}$. The purpose of this paper is to provide conditions that ensure that $(\mathcal{E}, \mathcal{F})$ is a *regular* Dirichlet form. Our main result - Theorem 2.9, says that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form provided the jump measure satisfies the following three hypotheses: the *Andres-Barlow condition* $(\mathrm{AB})_W$, the *Poincaré inequality* $(\mathrm{PI})_W$ and the *tail estimate* $(\mathrm{TJ})_W$, where $W = W(x, r)$ $(x \in \mathcal{X}, r > 0)$ is a certain *scaling function*.

Combining with the known heat kernel estimates, we obtain the following result stated in Theorem 2.15: the conjunction of the hypotheses $(\mathrm{AB})_W$, $(\mathrm{PI})_W$, and $(\mathrm{TJ})_W$ is *equivalent* to the fact that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form and its heat kernel satisfies certain upper and lower estimates.

For example, let measure μ be α-regular, $W(x, r) = r^\beta$ (where $\beta > 0$) and the jump measure be given by the jump kernel $J(x, y) \simeq d(x, y)^{-(\alpha+\beta)}$. In this case the corresponding bilinear form is denoted by $(\mathcal{E}_\beta, \mathcal{F}_\beta)$. Then the Poincaré inequality and the tail estimate are satisfied automatically, and we conclude that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form provided the Andres-Barlow condition $(\mathrm{AB})_\beta$ is satisfied. The latter condition holds trivially if $\beta < 2$, and is highly non-trivial if $\beta \geq 2$.

Moreover, by Theorem 2.22, the condition $(\mathrm{AB})_\beta$ is *equivalent* to the fact that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form and its heat kernel $p_t(x, y)$ satisfies the following two-sided *stable-like* estimate: $p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}$.

Contents

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1 Introduction

1.1 Dirichlet forms and their regularity

Let $(\mathcal{X}, d)$ be a separable metric space such that every closed ball is compact. Let μ be a Radon measure on $\mathcal{X}$ with full support. Such a triple $(\mathcal{X}, d, \mu)$ will be referred to as a *metric measure space*.

A symmetric bilinear form $\mathcal{E}$ with domain $\mathcal{F}$ is called a *Dirichlet form* on $L^2(\mathcal{X}) := L^2(\mathcal{X}, \mu)$ (cf. [19]) if $\mathcal{F}$ is a dense subspace of $L^2(\mathcal{X})$ and $(\mathcal{E}, \mathcal{F})$ satisfies the following properties:

- $(\mathcal{E}, \mathcal{F})$ is *closed*, that is, $\mathcal{F}$ is complete with respect to the norm

$$\|u\|_{\mathcal{E}_1} := \sqrt{\|u\|^2_{L^2(\mathcal{X})} + \mathcal{E}(u, u)}.$$

- $(\mathcal{E}, \mathcal{F})$ is *Markovian*, that is, for any $u \in \mathcal{F}$, also the function $v = u \vee 0 \wedge 1$ belongs to $\mathcal{F}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

Any Dirichlet form has the generator that is a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(\mathcal{X})$ with a maximal domain $\text{Dom}\,(\mathcal{L}) \subset \mathcal{F}$ such that

$$(\mathcal{L} f, g) = \mathcal{E}(f, g) \quad \text{for all } f \in \text{Dom}\,(\mathcal{L}) \text{ and } g \in \mathcal{F}.$$

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_c(\mathcal{X})$ is dense both in $\mathcal{F}$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$ and in $C_c(\mathcal{X})$ with respect to the sup-norm.

Here are two examples of regular Dirichlet forms in $\mathbb{R}^n$: a *local* form

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x)\, dx$$

with the domain $\mathcal{F} = W_2^1(\mathbb{R}^n)$, and a *non-local* one: for any $\beta \in (0,2)$,

$$\mathcal{E}_\beta(u,v) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\beta}}\, dy\, dx \tag{1.1}$$

with the domain $\mathcal{F} = B_{2,2}^{\beta/2}(\mathbb{R}^n)$. The former one has the generator $\mathcal{L} = -\Delta = -\sum_{i=1}^n \partial_{x_i}^2$, while the generator of the latter one is the fractional Laplace operator $\mathcal{L} = (-\Delta)^{\beta/2}$.

The existence of a Dirichlet form provides a certain differential structure on the underlying metric measure space and can serve as a starting point for development of analysis on such spaces (see, e.g. [1, 3, 4, 8, 17, 19, 21, 31, 32, 30, 33, 34]).

Much of the theory and applications of Dirichlet forms concerns with regular forms. The regularity of a Dirichlet form, in particular, guarantees the existence of the associated Hunt process $\{X_t\}$ on $\mathcal{X}$ (see [19]), whose transition probability is determined by the *heat semigroup* $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ as follows: for any Borel set $A \subseteq \mathcal{X}$,

$$\mathbb{P}_x(X_t \in A) = e^{-t\mathcal{L}}\mathbf{1}_A(x).$$

Therefore, it is extremely important to have tools for deciding whether a given symmetric bilinear form $(\mathcal{E},\mathcal{F})$ is a *regular* Dirichlet form.

In this paper we deal with non-local bilinear forms of the type

$$\mathcal{E}(u,v) := \int_{\mathcal{X}} \int_{\mathcal{X}} (u(x)-u(y))(v(x)-v(y)) J(x,dy)\, d\mu(x), \tag{1.2}$$

where $J(x,dy)$ is a jump measure, and the domain of $\mathcal{E}$ is $\mathcal{F} := \{u \in L^2(\mathcal{X}) : \ \mathcal{E}(u,u) < \infty\}$.

Our main result – Theorem 2.9, provides the following sufficient condition for $(\mathcal{E},\mathcal{F})$ to be a regular Dirichlet form: if the measure μ is doubling and if $(\mathcal{E},\mathcal{F})$ satisfies the following three conditions:

- the *Andres-Barlow condition* $(\mathrm{AB})_W$,
- the *Poincaré inequality* $(\mathrm{PI})_W$,
- the *tail estimate* $(\mathrm{TJ})_W$,

where $W = W(x,r)$ $(x \in \mathcal{X}, r > 0)$ is a certain scaling function, then $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. These three conditions are stated in details in the next section.

As far as we know, the result of this kind is entirely new and has no previous analogue.

As an example, consider the following specific bilinear form $\mathcal{E}_\beta$ that frequently occurs in applications. Denote by $B(x,r)$ an open metric ball on $\mathcal{X}$ of radius r centered at x, that is,

$$B(x,r) := \{y \in \mathcal{X} : \ d(y,x) < r\}.$$

For any $x,y \in \mathcal{X}$ and $r > 0$, set

$$V(x,r) := \mu(B(x,r)) \quad \text{and} \quad V(x,y) := V(x,d(x,y)) + V(y,d(x,y)). \tag{1.3}$$

Fix a parameter $\beta \in (0, \infty)$ and define the following symmetric bilinear form on $\mathcal{X}$:

$$\mathcal{E}_\beta(u, v) := \int_X \int_X \frac{(u(x) - u(y))(v(x) - v(y))}{V(x, y)\, d(x, y)^\beta}\, d\mu(y)\, d\mu(x), \tag{1.4}$$

with the domain

$$\mathcal{F}_\beta := \left\{ u \in L^2(\mathcal{X}) : \mathcal{E}_\beta(u, u) < \infty \right\}. \tag{1.5}$$

That is, the jump measure $J(x, dy)$ of $\mathcal{E}_\beta$ is determined by the *jump kernel*

$$J_\beta(x, y) = \frac{1}{V(x, y)\, d(x, y)^\beta}.$$

In particular, if the space $(\mathcal{X}, d, \mu)$ is α-regular, that is, for some $\alpha > 0$

$$V(x, r) \simeq r^\alpha \quad \text{for all } x \in \mathcal{X},\ r \in (0, \operatorname{diam}(\mathcal{X}))$$

then

$$J_\beta(x, y) \simeq d(x, y)^{-(\alpha+\beta)}$$

(as in (1.1) in the case of $\mathbb{R}^n$). The sign $\simeq$ means that the ratio of the both sides is bounded by positive constants from above and below.

The following natural question arises:

for which values $\beta \in (0, \infty)$ the bilinear form $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form?

If $\mathcal{F}_\beta$ is dense in $L^2(\mathcal{X})$ then $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a Dirichlet form. However, for large enough β, the domain $\mathcal{F}_\beta$ as in (1.5) may not be dense in $L^2(\mathcal{X})$ (for example, this happens in $\mathbb{R}^n$ for $\beta > 2$), or even if it is dense, $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ may not be regular.

Denote by $(\mathrm{AB})_\beta$, $(\mathrm{PI})_\beta$, $(\mathrm{TJ})_\beta$ the above conditions with respect to $(\mathcal{E}_\beta, \mathcal{F}_\beta)$, which corresponds to the scaling function $W(x, r) = r^\beta$. It turns out that the hypotheses $(\mathrm{PI})_\beta$ and $(\mathrm{TJ})_\beta$ are in this case satisfied automatically (Lemma 3.1). It follows from Theorem 2.9 that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form provided $(\mathrm{AB})_\beta$ holds.

The remaining question about the validity of $(\mathrm{AB})_\beta$ is rather complicated and is left for the future research. It is easy to check that $(\mathrm{AB})_\beta$ is always satisfied for $\beta < 2$ (Lemma 3.2) but so far there are no practical tools for verification of $(\mathrm{AB})_\beta$ for $\beta \geq 2$.

1.2 Heat kernels

Let us now discuss the connection between the conditions $(\mathrm{AB})_W$, $(\mathrm{PI})_W$, $(\mathrm{TJ})_W$ and the heat kernel estimates.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form. If, for any $t > 0$, the operator $e^{-t\mathcal{L}}$ is an integral operator in $L^2(\mathcal{X})$ then its integral kernel is referred to as the *heat kernel* of $\mathcal{L}$ (or that of $(\mathcal{E}, \mathcal{F})$) and is denoted by $p_t(x, y)$. The heat kernel also serves as the transition density of the associated Hunt process.

There is a vast literature devoted to the existence and estimates of heat kernels of regular Dirichlet forms (see, e.g. [5, 6, 7, 9, 17, 21, 27, 13, 11, 14, 15, 16]). Recall that the heat kernel satisfies the following properties for all (or almost all) values of the variables involved:

(P1) *Markov property:* for any $t > 0$, $p_t(x, y)$ is a measurable nonnegative function of x, y, and

$$\int_X p_t(x, y)\, d\mu(y) \leq 1; \tag{1.6}$$

(P2) *symmetry:* $p_t(x, y) = p_t(y, x)$;

(P3) *semigroup property*:

$$\int_{\mathcal{X}} p_t(x, z) p_s(z, y)\, d\mu(z) = p_{t+s}(x, y);$$

(P4) *approximation of identity*: for any $f \in L^2(\mathcal{X})$,

$$\int_{\mathcal{X}} p_t(x, y) f(y)\, d\mu(y) \to f(x) \;\text{ as }\; t \to 0,$$

where the convergence is in $L^2(\mathcal{X})$.

The heat kernel is called *stochastically complete* if the integral in (1.6) is identically equal to 1.

Conversely, any function $p_t(x, y)$ satisfying (P1)-(P4) gives rise to a heat semigroup

$$P_t f(x) := \int_{\mathcal{X}} p_t(x, y) f(y)\, d\mu(y)$$

acting in $L^2(\mathcal{X})$, and the heat semigroup determines a Dirichlet form in a standard way (see [19]).

Let us first discuss heat kernel estimates on α-regular spaces. The dichotomy property of heat kernel estimates (see [28]) states that there are only two kinds of *self-similar* estimates for heat kernel $\{p_t\}_{t>0}$ on an α-regular metric measure space $(\mathcal{X}, d, \mu)$.

The first kind is the *sub-Gaussian estimate* $(\mathrm{SG})_{\alpha, d_w}$ of the form

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/d_w}} \exp\left(-c\left(\frac{d(x, y)}{t^{1/d_w}}\right)^{\frac{d_w}{d_w-1}}\right) \qquad (\mathrm{SG})_{\alpha, d_w}$$

for all $x, y \in \mathcal{X}$ and $t > 0$, where the sign $\asymp$ means that both $\le$ and $\ge$ hold but with different values of positive constants C, c. Here d_w is a parameter from $[2, \infty)$ that is called the *walk dimension* of the heat kernel. Besides, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is in this case local.

The second kind is the *stable-like estimate* $(\mathrm{ULE})_{\alpha,\beta}$ of the form

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \qquad (\mathrm{ULE})_{\alpha,\beta}$$

for all $x, y \in \mathcal{X}$ and $t > 0$, where the parameter $\beta \in (0, \infty)$ is called the *index* of the heat kernel. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is in this case non-local.

In the α-regular space the jump kernel J_β of the Dirichlet form $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ from (1.4) admits the estimate

$$J_\beta(x, y) \simeq\; d(x, y)^{-(\alpha+\beta)},$$

and this estimate is a necessary condition for the heat kernel bounds $(\mathrm{ULE})_{\alpha,\beta}$. If $\beta \in (0, 2)$, then all Lipschitz functions on $\mathcal{X}$ with compact supports belong to $\mathcal{F}_\beta$, which implies that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form. Moreover, in this case the heat kernel $\{p_t\}_{t>0}$ of $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ exists and satisfies $(\mathrm{ULE})_{\alpha,\beta}$ (see [12, 17, 21]).

Let $\beta \in [2, \infty)$ and assume a priori that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form (it is known that this situation can actually occur on fractal spaces). In order to obtain the heat kernel estimate $(\mathrm{ULE})_{\alpha,\beta}$, Chen-Kumagai-Wang [17] and Grigor'yan-Hu-Hu [21] introduced independently some analytic condition (called a *cut-off Sobolev inequality* or a *generalized capacity condition*) and proved that this condition is equivalent to $(\mathrm{ULE})_{\alpha,\beta}$. We use this condition in the form of $(\mathrm{AB})_\beta$.

Combining these results with ours we conclude that (AB)$_\beta$ is *equivalent* to the fact that $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ is a regular Dirichlet form and its heat kernel satisfies (ULE)$_{\alpha,\beta}$.

In our second main result - Theorem 2.22, we prove a similar equivalence in a higher generality, for the bilinear form $(\mathcal{E}_\beta, \mathcal{F}_\beta)$ as given in (1.4)-(1.5) assuming that measure μ satisfies the volume doubling and the reverse volume doubling conditions.

Moreover, our most general Theorem 2.15 says that a non-local bilinear form $(\mathcal{E}, \mathcal{F})$ with an arbitrary jump measure $J(x, dy)$ (as in (1.2)) is a regular Dirichlet form and its heat kernel satisfies certain upper and lower estimates if and only if all the hypotheses (AB)$_\mathrm{W}$, (PI)$_\mathrm{W}$, and (TJ)$_\mathrm{W}$ are fulfilled.

1.3 Notation

We use the following notation throughout the paper.

- $\mathbb{N} = \{0, 1, 2, \ldots, \}$.
- For any set $E \subseteq \mathcal{X}$, $\overline{E}$ denotes the closure of E, and $E^{\complement} = \mathcal{X} \setminus E$.
- For any function $f : \mathcal{X} \to \mathbb{R}$, its support $\operatorname{supp} f$ is the complement of the largest open set where $f = 0$ μ-a.e.
- For any μ-measurable set $E \subseteq \mathcal{X}$ with $\mu(E) > 0$ and any μ-integrable function $f : E \to \mathbb{R}$, set
$$\fint_E f\, d\mu = \frac{1}{\mu(E)} \int_E f(x)\, d\mu(x).$$
- $C(\mathcal{X})$ denotes the space of all continuous functions on $\mathcal{X}$; $C_c(\mathcal{X})$ is the subspace of $C(\mathcal{X})$ that consists of functions with compact supports and is endowed with the sup-norm.
- The letters C and c are used to denote positive constants that are independent of the variables in question, but may vary at each occurrence. The relation $u \lesssim v$ (resp., $u \gtrsim v$) between functions u and v means that $u \le Cv$ (resp., $u \ge Cv$) for a positive constant C and for a specified range of the variables. We write $u \simeq v$ if $u \lesssim v \lesssim u$.
- For any $a, b \in \mathbb{R}$, set $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

2 Statement of the main results

2.1 Conditions (TJ)$_\mathrm{W}$, (PI)$_\mathrm{W}$, (AB)$_\mathrm{W}$

We begin with the following setup of a bilinear form $(\mathcal{E}, \mathcal{F})$ and a jump measure $J(x, dy)\, d\mu(x)$. Let $\mathcal{B}(\mathcal{X})$ denote the family of Borel sets on $\mathcal{X}$.

Definition 2.1. Let $J(\cdot, \cdot) : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \mapsto \mathbb{R}_+ := [0, \infty)$ be a function such that

- for each $x \in \mathcal{X}$, $A \mapsto J(x, A)$ is a measure on $\mathcal{B}(\mathcal{X})$;
- for each $A \in \mathcal{B}(\mathcal{X})$, $x \mapsto J(x, A)$ is a nonnegative measurable function on $\mathcal{X}$.

Let J satisfy also the following two conditions:

(J1) for any $r > 0$, $J(x, B(x, r)^{\complement})$ is, as a function of $x \in \mathcal{X}$, locally integrable with respect to μ;

(J2) for all nonnegative Borel measurable functions u, v on $\mathcal{X}$,

$$\int_{\mathcal{X}} u(x) \int_{\mathcal{X}} v(y)\, J(x, dy)\, d\mu(x) = \int_{\mathcal{X}} v(x) \int_{\mathcal{X}} u(y)\, J(x, dy)\, d\mu(x).$$

Then we refer to J as a *jump measure*.

It was shown in [19, Example 1.2.4, p. 14] that any jump measure $J(x, dy)$ determines a symmetric Radon measure $j(dx, dy) = J(x, dy)\, d\mu(x)$ on $\mathcal{B}(\mathcal{X} \times \mathcal{X})$ and, for all $f \in C_c(\mathcal{X} \times \mathcal{X})$,

$$\iint_{\mathcal{X}\times\mathcal{X}} f(x, y)\, J(x, dy)\, d\mu(x) = \iint_{\mathcal{X}\times\mathcal{X}} f(y, x)\, J(x, dy)\, d\mu(x).$$

Note that the term "jump measure" refers usually to the measure j. By slightly abusing the terminology, we use this term with respect to the function J.

Any jump measure J gives rise to the following symmetric bilinear form $(\mathcal{E}, \mathcal{F})$:

$$\begin{cases} \mathcal{E}(u, v) = \displaystyle\iint_{\mathcal{X}\times\mathcal{X}} (u(x) - u(y))(v(x) - v(y))\, J(x, dy)\, d\mu(x); \\ \mathcal{F} = \{u \in L^2(\mathcal{X}) : \ u \ \text{ is Borel measurable on } \mathcal{X},\ \mathcal{E}(u, u) < \infty\}. \end{cases} \tag{2.1}$$

It was also shown in [19, Example 1.2.4, p. 14] that:

(a) If u is Borel measurable on $\mathcal{X}$ and $u = 0$ μ-a.e. on $\mathcal{X}$, then $\mathcal{E}(u, u) = 0$.

(b) If $u \in \mathcal{F}$ and v is a Borel measurable function on $\mathcal{X}$ satisfying

$$|v(x)| \le |u(x)| \quad \text{and} \quad |v(x) - v(y)| \le |u(x) - u(y)| \qquad \text{for all } x, y \in \mathcal{X},$$

then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \le \mathcal{E}(u, u)$. In particular, $(\mathcal{E}, \mathcal{F})$ satisfies Markov property.

(c) $(\mathcal{E}, \mathcal{F})$ is closed.

In other words, the bilinear form $(\mathcal{E}, \mathcal{F})$ defined in (2.1) is a Dirichlet form on $L^2(\mathcal{X})$ provided $\mathcal{F}$ is dense in $L^2(\mathcal{X})$.

Throughout the whole paper, we always assume that J is a jump measure as defined above, and $(\mathcal{E}, \mathcal{F})$ is the bilinear form as defined in (2.1). We will investigate sufficient (and/or necessary) conditions of $(\mathcal{E}, \mathcal{F})$ to be a *regular* Dirichlet form.

If the measure $J(x, dy)\, d\mu(x)$ has a density with respect to $d\mu(x)\, d\mu(y)$ then the density function will be denoted by $J(x, y)$ and referred to as a *jump kernel*. Clearly, any nonnegative symmetric Borel function $J(x, y)$ on $\mathcal{X} \times \mathcal{X}$ determines a jump measure

$$J(x, dy)\, d\mu(x) = J(x, y)\, d\mu(y)\, d\mu(x)$$

provided the function

$$x \mapsto \int_{B(x,r)^{\complement}} J(x, y)\, d\mu(y)$$

is locally integrable for any $r > 0$.

Definition 2.2. A function $W : \mathcal{X} \times [0, \infty) \to [0, \infty)$ is called *a space/time scaling function* if

- for any $x \in \mathcal{X}$, the function $r \mapsto W(x, \cdot)$ is continuous and strictly increasing on $[0, \infty)$;
- $W(x, 0) = 0$ and $\lim_{r\to\infty} W(x, r) = \infty$;

- there exist three positive numbers C_W, β_1, β_2 where $\beta_1 \le \beta_2$, such that, for all $0 < r \le R < \infty$ and $x, y \in \mathcal{X}$ with $d(x,y) \le R$,

$$C_W^{-1} \left(\frac{R}{r}\right)^{\beta_1} \le \frac{W(x,R)}{W(y,r)} \le C_W \left(\frac{R}{r}\right)^{\beta_2}. \tag{2.2}$$

It follows that, for any $x \in \mathcal{X}$, the inverse function $W^{-1}(x,\cdot)$ of $W(x,\cdot)$ exists and satisfies the following inequalities, for all $0 < r \le R < \infty$ and all $x \in \mathcal{X}$,

$$C_W^{-1/\beta_2} \left(\frac{R}{r}\right)^{1/\beta_2} \le \frac{W^{-1}(x,R)}{W^{-1}(x,r)} \le C_W^{1/\beta_1} \left(\frac{R}{r}\right)^{1/\beta_1}.$$

An example of a space/time scaling function is $W(x,r) = r^\beta$ for all $(x,r) \in \mathcal{X} \times (0,\infty)$, where $\beta \in (0,\infty)$. In this example W is independent of the space variable x, but there exist other interesting examples of W that depend on x. We refer the reader to Section 3 below for more discussions.

Let us fix for now a scaling function $W(x,r)$ and a jump measure $J(x,dy)$.

Definition 2.3. (*Tail of jump measure*) We say that J satisfies condition $(\mathrm{TJ})_W$ if there exists a constant $C > 0$ such that, for all $x \in \mathcal{X}$ and $R > 0$,

$$J(x, B(x,R)^{\complement}) = \int_{B(x,R)^{\complement}} J(x,dy) \le \frac{C}{W(x,R)}.$$

Remark 2.4. Note that condition $(\mathrm{TJ})_W$ and (2.2) imply condition (J1).

Definition 2.5. (*Poincaré inequality*) We say that J satisfies the *Poincaré inequality* $(\mathrm{PI})_W$ if there exist constants $C > 0$ and $\kappa \in [1,\infty)$ such that, for any ball $B := B(x_0,R)$ with $x_0 \in \mathcal{X}$, $R \in (0,\infty)$ and for any function $u \in \mathcal{F} \cap L^\infty(\mathcal{X})$,

$$\int_B |u(x) - u_B|^2 \, d\mu(x) \le C W(x_0,R) \iint_{(\kappa B)\times(\kappa B)} |u(x) - u(y)|^2 \, J(x,dy) \, d\mu(x), \tag{2.3}$$

where $u_B := \frac{1}{\mu(B)} \int_B u(x) \, d\mu(x)$ denotes the arithmetic mean of u over B.

Definition 2.6. Let U be an open subset of $\mathcal{X}$ and A be any Borel subset of U. A function $\phi \in C_c(\mathcal{X})$ is called a *cutoff function* of the pair (A,U) if it satisfies the following properties:

(i) $0 \le \phi \le 1$ on $\mathcal{X}$;

(ii) $\phi \equiv 1$ in A;

(iii) $\phi \equiv 0$ on $U^{\complement}$.

Denote by $\mathrm{cutoff}(A,U)$ the collection of all cutoff functions of the pair (A,U).

We define below a condition $(\mathrm{AB})_W$. It is named after Andres and Barlow because they first introduced in [2] a similar condition for *local* Dirichlet forms (which was referred to in [2] as (CSA) – *a cutoff Sobolev inequality in annuli*). For jump-type Dirichlet forms and with the scaling function $W(x,r) = r^\beta$, this condition was introduced in [21] where it was used to characterize the two-sided stable-like estimates of the heat kernel.

Definition 2.7. (*Andres-Barlow condition*) Set

$$\mathcal{F}' := \mathcal{F} + \{\text{const}\} = \{u + c : u \in \mathcal{F} \text{ and } c \text{ is a constant}\}.$$

We say that a jump measure J satisfies condition $(AB)_W$, if there exist positive constants ζ and C such that, for any $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$ and for any three concentric balls

$$\begin{cases} B_0 = B(x_0, R); \\ B = B(x_0, R + r); \\ \Omega = B(x_0, R'), \end{cases} \tag{2.4}$$

with $x_0 \in \mathcal{X}$ and $0 < R < R + r < R' < \infty$, there exists a function $\phi \in \text{cutoff}(B_0, B)$ such that

$$\begin{aligned} &\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\le \zeta \iint_{B\times B} |\phi(x)|^2 |u(x) - u(y)|^2 \, J(x, dy)\, d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x). \end{aligned} \tag{2.5}$$

Definition 2.8. (*Volume doubling condition*) We say that a measure μ on a metric space $(\mathcal{X}, d)$ satisfies the *volume doubling* condition, denoted by (VD), if there exists a constant $C_D \ge 1$ such that, for all $x \in \mathcal{X}$ and all $r > 0$,

$$V(x, 2r) \le C_D V(x, r). \tag{2.6}$$

Note that (2.6) holds if and only if there exists constants $C'_D \in (1, \infty)$ and $\alpha_+ > 0$ such that, for all x, $y \in \mathcal{X}$ and $0 < r \le R$,

$$\frac{V(x, R)}{V(y, r)} \le C'_D \left(\frac{d(x, y) + R}{r}\right)^{\alpha_+}. \tag{2.7}$$

Condition (VD) also implies that, for all x, $y \in \mathcal{X}$,

$$V(x, y) \le (C_D + 1) V(x, d(x, y)), \tag{2.8}$$

where $V(x, y)$ is defined by (1.3).

The main result of this paper is the following theorem.

Theorem 2.9. **(Main theorem)** *For any bilinear form $(\mathcal{E}, \mathcal{F})$ (as defined in (2.1)) with a jump measure $J(x, dy)$ and any scaling function W, the following implication holds:*

$$(VD) + (TJ)_W + (AB)_W + (PI)_W \Rightarrow (\mathcal{E}, \mathcal{F}) \text{ is a regular Dirichlet form on } L^2(\mathcal{X}). \tag{2.9}$$

2.2 Relation to heat kernel bounds

Next, we combine Theorem 2.9 with the previously known results about heat kernel estimate to obtain some interesting consequences. In particular, in our next result, we replace the implication sign in (2.9) by the equivalence sign, at expense of adding a certain heat kernel estimate in the right hand side. For that we need some more definitions.

Definition 2.10. The measure μ is said to satisfy the *reverse volume doubling condition* (RVD) if there exist constants $C_{RD} \in (0, \infty)$ and $\alpha_- > 0$ such that

$$\frac{V(x, R)}{V(x, r)} \ge C_{RD} \left(\frac{R}{r}\right)^{\alpha_-} \quad \text{for all } x \in \mathcal{X} \text{ and } 0 < r \le R < \text{diam}(\mathcal{X}).$$

The reverse volume doubling condition (RVD) is a rather mild assumption. Indeed, if $(\mathcal{X}, d, \mu)$ is connected and the exterior of any ball is non-empty then (VD) implies (RVD) (see [26, Proposition 5.2]). Clearly, if both (VD) and (RVD) are satisfied then $0 < \alpha_- \le \alpha_+$, and if (RVD) is satisfied then $\mu(\{x\}) = 0$ for any $x \in \mathcal{X}$, so that $(\mathcal{X}, d, \mu)$ is non-atomic.

Definition 2.11. We say that the condition $(\mathrm{AB}')_W$ holds, if there exist $\zeta > 0$ and $C > 0$ such that for any $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$ and for any three concentric balls B_0, B, Ω given in (2.4), there exists a function $\phi \in \mathrm{cutoff}(B_0, B)$ satisfying

$$\begin{aligned}&\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\quad \le \zeta \iint_{(B\setminus B_0)\times(B\setminus B_0)} |\phi(x)|^2 |u(x) - u(y)|^2 \, J(x, dy)\, d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x). \qquad (2.10)\end{aligned}$$

The difference between (2.5) and (2.10) is that for the latter the integration in the middle term is done over a smaller annulus $B \setminus B_0$. Hence, we have $(\mathrm{AB}')_W \Rightarrow (\mathrm{AB})_W$. The converse implication $(\mathrm{AB})_W \Rightarrow (\mathrm{AB}')_W$ is true under some additional assumptions (see Corollary 2.16).

Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. For any open set $\Omega \subset \mathcal{X}$, set

$$\mathcal{F}(\Omega) = \overline{\mathcal{F} \cap C_c(\Omega)}, \qquad (2.11)$$

where the closure is taken with respect to $\mathcal{E}_1$-norm. By [19, Theorem 4.4.3], $\mathcal{F}(\Omega)$ is a dense subspace of $L^2(\Omega)$ and $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form on $L^2(\Omega)$. If it has the heat kernel then the latter is called the *Dirichlet heat kernel* in Ω and is denoted by $p_t^\Omega(x, y)$.

Definition 2.12. (*Localized lower estimate*) We say that a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies condition $(\mathrm{LLE})_W$ if the following two properties are satisfied:

(i) for any bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^\Omega(x, y)$ exists and is locally Hölder continuous in $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$;

(ii) there exist $C > 0$ and $\delta \in (0, 1)$ such that, for any ball $B := B(x_0, R)$ with $R > 0$, for any $t \le W(x_0, \delta R)$ and for all $x, y \in B(x_0, \delta W^{-1}(x_0, t))$,

$$p_t^B(x, y) \ge \frac{C^{-1}}{V(x_0, W^{-1}(x_0, t))}.$$

Our second main result is as follows.

Theorem 2.13. *Assume that* (VD), (RVD), $(\mathrm{TJ})_W$ *are satisfied. Then, the following three conditions are equivalent:*

(i) $(\mathrm{AB}')_W + (\mathrm{PI})_W$

(ii) $(\mathrm{AB})_W + (\mathrm{PI})_W$

(iii) $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form satisfying* $(\mathrm{LLE})_W$.

Moreover, if any of the conditions (i), (ii), (iii) *holds, then the heat kernel* $\{p_t\}_{t>0}$ *of* $(\mathcal{E}, \mathcal{F})$ *exists and is stochastically complete, that is, for any* $t \in (0, \infty)$ *and* $x \in M$,

$$\int_{\mathcal{X}} p_t(x, y)\, d\mu(y) = 1.$$

Our next Theorem 2.15 is a modification of Theorem 2.13 where $(\mathrm{TJ})_W$ is removed from the list of standing assumptions and included into the statements (i) and (ii). In order to state it, we need one more definition.

Definition 2.14. (*Tail estimate of heat semigroup*) We say that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies condition $(\mathrm{TP})_W$ if, for any ball $B = B(x, R)$ with $x \in \mathcal{X}$, $R \in (0, \infty)$ and any $t \in (0, \infty)$,

$$P_t \mathbf{1}_{B^\complement} \le \frac{Ct}{W(x,R)} \quad \text{in } \frac{1}{4}B$$

for a positive constant C independent of B, t.

Theorem 2.15. *Assume that* (VD) *and* (RVD) *are satisfied. Then, the following three conditions are equivalent:*

(i) $(\mathrm{TJ})_W + (\mathrm{AB}')_W + (\mathrm{PI})_W$

(ii) $(\mathrm{TJ})_W + (\mathrm{AB})_W + (\mathrm{PI})_W$

(iii) $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form satisfying* $(\mathrm{LLE})_W + (\mathrm{TP})_W$.

Finally, the next statement is a direct consequence of Theorem 2.13 or 2.15.

Corollary 2.16. *Under* (VD), (RVD), $(\mathrm{TJ})_W$ *and* $(\mathrm{PI})_W$*, the following equivalence holds:*

$$(\mathrm{AB})_W \iff (\mathrm{AB}')_W.$$

2.3 Special scaling function

Fix some $\beta \in (0, \infty)$. As an application of the above theorems, let us consider a special case when the scaling function is

$$W(x, r) = r^\beta \quad \text{for all } x \in \mathcal{X} \text{ and } r \in [0, \infty), \tag{2.12}$$

and the *jump kernel* $J(x, y) := \frac{J(x,dy)\, d\mu(x)}{d\mu(y)\, d\mu(x)}$ exists and satisfies the following condition $(\mathrm{J})_\beta$.

Definition 2.17. We say that the jump kernel $J(x, y)$ satisfies condition $(\mathrm{J})_\beta$ if

$$J(x, y) \simeq \frac{1}{V(x, y) d(x, y)^\beta} \quad \text{for all distinct } x, y \in \mathcal{X}.$$

Let us introduce the following family of Besov function spaces.

Definition 2.18. For any $s \in (0, \infty)$, define the *homogeneous Besov space* $\dot{\Lambda}^s_{2,2}(\mathcal{X})$ as the collection of all locally integrable functions f on $\mathcal{X}$ such that

$$\|f\|_{\dot{\Lambda}^s_{2,2}(\mathcal{X})} := \left(\int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|f(x) - f(y)|^2}{V(x,y) d(x,y)^{2s}} \, d\mu(y)\, d\mu(x) \right)^{\frac{1}{2}} < \infty.$$

Define the *inhomogeneous Besov space* by

$$\Lambda^s_{2,2}(\mathcal{X}) := \left\{ f \in L^2(\mathcal{X}) : \|f\|_{\dot{\Lambda}^s_{2,2}(\mathcal{X})} < \infty \right\}.$$

For the scaling function (2.12) and the jump kernel $J(x,y)$ satisfying $(\mathrm{J})_\beta$, the domain $\mathcal{F}$ of the associated bilinear form $\mathcal{E}$ in Definition 2.1 coincides with the inhomogeneous Besov space $\Lambda^{\beta/2}_{2,2}(\mathcal{X})$. In this setting, we rename the conditions $(\mathrm{AB})_{\mathrm{W}}$ and $(\mathrm{AB}')_{\mathrm{W}}$ to $(\mathrm{AB})_\beta$ and $(\mathrm{AB}')_\beta$, respectively. For convenience of the reader, we state here the following independent definitions of $(\mathrm{AB})_\beta$ and $(\mathrm{AB}')_\beta$.

Definition 2.19. We say that a metric measure space $(\mathcal{X}, d, \mu)$ satisfies condition $(\mathrm{AB})_\beta$ if there exist $\zeta > 0$ and $C > 0$ such that, for any function

$$u \in \left(\Lambda^{\beta/2}_{2,2}(\mathcal{X}) + \{\mathrm{const}\}\right) \cap L^\infty(\mathcal{X})$$

and for any three concentric balls B_0, B, Ω given in (2.4), there exists $\phi \in \mathrm{cutoff}(B_0, B)$ such that

$$\begin{aligned}\iint_{\Omega\times\Omega} &\frac{|u(x)|^2|\phi(x)-\phi(y)|^2}{d(x,y)^\beta}\frac{d\mu(y)\,d\mu(x)}{V(x,y)}\\ &\leq \zeta \iint_{B\times B}\frac{|\phi(x)|^2|u(x)-u(y)|^2}{d(x,y)^\beta}\frac{d\mu(y)\,d\mu(x)}{V(x,y)} + \frac{C}{r^\beta}\int_\Omega |u(x)|^2\,d\mu(x).\end{aligned} \tag{2.13}$$

Condition $(\mathrm{AB}')_\beta$ is defined similarly and is obtained from $(\mathrm{AB})_\beta$ by replacing the integration area $B \times B$ in the first integral on the right hand side of (2.13) by $(B \setminus B_0) \times (B \setminus B_0)$.

Definition 2.20. *(Upper and lower estimates of the heat kernel)* We say that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the condition $(\mathrm{ULE})_\beta$ if its heat kernel $\{p_t\}_{t>0}$ exists and satisfies the following estimate: for all $x, y \in \mathcal{X}$ and $t > 0$,

$$p_t(x,y) \simeq \frac{1}{V(x, t^{1/\beta} + d(x,y))}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \tag{ULE$_\beta$}$$

Remark 2.21. It is easy to see that if the measure μ is α-regular then the condition $(\mathrm{ULE})_\beta$ coincides with $(\mathrm{ULE})_{\alpha,\beta}$. Recall also that the heat kernel is related to the jump kernel by the identity

$$J(x,y) = \lim_{t\to 0}\frac{p_t(x,y)}{2t},$$

which easily yields that $(\mathrm{ULE})_\beta$ implies $(\mathrm{J})_\beta$.

The next result (that is essentially a consequence of Theorems 2.9, 2.13 together with the previously known heat kernel estimates) shows that, for a bilinear form $(\mathcal{E}, \mathcal{F})$ satisfying $(\mathrm{J})_\beta$, any of the conditions $(\mathrm{AB})_\beta$ or $(\mathrm{AB}')_\beta$ can be used to prove that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, as well to obtain its heat kernel bounds.

Theorem 2.22. *Assume that* (VD) *and* (RVD) *are satisfied. Then, for any $\beta \in (0, \infty)$, the following conditions are equivalent:*

(i) $(\mathrm{AB}')_\beta$

(ii) $(\mathrm{AB})_\beta$

(iii) *For any jump kernel J satisfying $(\mathrm{J})_\beta$, the following bilinear form*

$$\mathcal{E}(u,v) := \int_X\int_X (u(x)-u(y))(v(x)-v(y))J(x,y)\,d\mu(y)\,d\mu(x) \tag{2.14}$$

with domain

$$\mathcal{F} := \left\{f \in L^2(\mathcal{X}) : \ \mathcal{E}(f,f) < \infty\right\} \tag{2.15}$$

is a regular Dirichlet form satisfying $(\mathrm{ULE})_\beta$.

(iv) *There exists a jump kernel J satisfying* $(\mathrm{J})_\beta$ *such that the bilinear form* $(\mathcal{E},\mathcal{F})$ *given by* (2.14)-(2.15) *is a regular Dirichlet form satisfying* $(\mathrm{ULE})_\beta$.

Moreover, if any one of the above statements (i)-(iv) *holds, then the heat kernel* $\{p_t\}_{t>0}$ *of* $(\mathcal{E},\mathcal{F})$ *is jointly continuous on* $\mathcal{X}\times\mathcal{X}$ *and stochastically complete.*

According to Remark 3.3 below, if $0<\beta<2$, then both $(\mathrm{AB})_\beta$ and $(\mathrm{AB}')_\beta$ hold; hence, all conclusions of Theorem 2.22 are true when $0<\beta<2$.

2.4 Organization of the paper

In Sections 3.1 and 3.2 we present some examples of (absolutely continuous and singular) jump measures satisfying the conditions $(\mathrm{TJ})_W$, $(\mathrm{PI})_W$ and $(\mathrm{AB})_W$.

In Section 3.3, we discuss a product jump measure on a product $\mathcal{X}$ of m spaces $\mathcal{X}_i$, $i=1,\dots,m$, where each metric measure space $\mathcal{X}_i$ is endowed with a jump-type regular Dirichlet form that satisfies the conditions $(\mathrm{TJ})_W$, $(\mathrm{PI})_W$, $(\mathrm{AB})_W$. Note that the product jump measure is always singular. We show in Theorem 3.7 that the corresponding product Dirichlet form also satisfies the condition $(\mathrm{TJ})_W$, $(\mathrm{PI})_W$, $(\mathrm{AB})_W$ and, hence, it is regular and its heat kernel satisfies $(\mathrm{LLE})_W$ and $(\mathrm{TP})_W$.

Section 4 contains the proof of our main Theorem 2.9. In Section 4.1, we establish a relation between $(\mathrm{AB})_W$ and $(\mathrm{AB}')_W$. In Section 4.2 we prove a self-improvement property of condition $(\mathrm{AB})_W$. In Section 4.3 we construct a partition of unity on $\mathcal{X}$ by using the cutoff functions from the condition $(\mathrm{AB})_W$ or $(\mathrm{AB}')_W$. Finally, Theorem 2.9 is proved in Section 4.4.

In Section 5, we apply Theorem 2.9 to prove Theorems 2.13, 2.15 and 2.22. We first establish the implications $(\mathrm{S})_W \Rightarrow (\mathrm{AB}')_W$ and $(\mathrm{LLE})_W \Rightarrow (\mathrm{S})_W + (\mathrm{PI})_W$ in Sections 5.1 and 5.2, respectively. Based on them, we give the proofs of Theorems 2.13, 2.15 and 2.22 in Sections 5.3, 5.4 and 5.5, respectively.

3 Examples of jump measures

In this section, we provide various examples of non-singular and singular jump measures (see Sections 3.1 and 3.2). In Section 3.3, the conditions $(\mathrm{TJ})_W$, $(\mathrm{PI})_W$ and $(\mathrm{AB})_W$ are proved on product spaces.

3.1 Jump measures with density

Let the scaling function W be as in Definition 2.2. Let a jump measure J admit a jump kernel, that is,

$$J(x,dy)\,d\mu(x) = J(x,y)\,d\mu(y)\,d\mu(x),$$

where $J(x,y)$ is a symmetric jointly measurable function satisfying

$$J(x,y) \simeq \frac{1}{W(x,\ d(x,y))\,V(x,y)} \quad \text{for all distinct } x,\ y\in\mathcal{X}. \tag{3.1}$$

Note that by (2.2)

$$W(x,\ d(x,y)) \simeq W(y,\ d(x,y)), \tag{3.2}$$

so that the right hand side of (3.1) is "almost" symmetric in x,y.

Lemma 3.1. *Assume that* (VD) *is satisfied and that the jump kernel* J *satisfies* (3.1). *Then both* $(\mathrm{TJ})_W$ *and* $(\mathrm{PI})_W$ *are satisfied.*

Proof. Let us verify first the condition $(\mathrm{TJ})_W$. Using (VD) and (2.2), we obtain, for any $x \in \mathcal{X}$ and any $R > 0$,

$$\begin{aligned} J(x, B(x,R)^{\complement}) &= \int_{B(x,R)^{\complement}} J(x,dy) \simeq \int_{d(y,x)\geq R} \frac{d\mu(y)}{W(x,\ d(x,y))\, V(x,y)} \\ &\leq \sum_{j=0}^{\infty} \int_{2^j R \leq d(x,y) < 2^{j+1}R} \frac{d\mu(y)}{W(x,\ d(x,y))\, V(x, d(x,y))} \\ &\leq \sum_{j=0}^{\infty} \int_{2^j R \leq d(x,y) < 2^{j+1}R} \frac{d\mu(y)}{W(x, 2^j R)\, V(x, 2^j R)} \\ &\leq \frac{1}{W(x,R)} \sum_{j=0}^{\infty} \frac{W(x,R) V(x, 2^{j+1}R)}{W(x,2^j R)\, V(x, 2^j R)} \\ &\leq \frac{1}{W(x,R)} \sum_{j=0}^{\infty} C_D C_W 2^{-j\beta_1} \\ &\simeq \frac{1}{W(x,R)}, \end{aligned}$$

which proves $(\mathrm{TJ})_W$.

Let us now prove $(\mathrm{PI})_W$. For any ball B with center $x_0 \in \mathcal{X}$ and radius $R \in (0,\infty)$, and for any function $u \in \mathcal{F} \cap L^{\infty}(\mathcal{X})$, applying the Hölder inequality, we obtain

$$\begin{aligned} \int_B |u(x) - u_B|^2\, d\mu(x) &= \int_B \left| \frac{1}{\mu(B)} \int_B [u(x) - u(y)]\, d\mu(y) \right|^2 d\mu(x) \\ &\leq \int_B \left(\frac{1}{\mu(B)} \int_B |u(x) - u(y)|^2\ d\mu(y) \right) d\mu(x) \\ &= \int_B \int_B \frac{|u(x)-u(y)|^2}{W(x,\ d(x,y))} \frac{W(x,\ d(x,y))\, V(x,y)}{V(x_0,R)} \frac{d\mu(y)\, d\mu(x)}{V(x,y)}. \end{aligned}$$

By means of (2.7) and (2.8) we obtain that, for all $x, y \in B$,

$$\frac{V(x,y)}{V(x_0,R)} \leq 2C_D \frac{V(x, d(x,y))}{V(x_0,R)} \leq 2C_D C_D' \left(\frac{d(x,y) + d(x,x_0)}{R} \right)^{\alpha_+} \leq 2C_D C_D' 3^{\alpha_+}.$$

Moreover, it follows from (2.2) that

$$\frac{W(x,\ d(x,y))}{W(x_0,R)} \leq \frac{W(x,2R)}{W(x_0,R)} \leq C_W 2^{\beta_2}.$$

Consequently, we obtain

$$\begin{aligned} \int_B |u(x) - u_B|^2\, d\mu(x) &\leq 2C_W C_D C_D' 3^{\alpha_+} 2^{\beta_2} W(x_0,R) \int_B \int_B \frac{|u(x)-u(y)|^2}{W(x,\ d(x,y))} \frac{d\mu(y)\, d\mu(x)}{V(x,y)} \\ &\simeq W(x_0,R) \iint_{B\times B} (u(x)-u(y))^2\, J(x,dy)\, d\mu(x), \end{aligned}$$

which proves $(\mathrm{PI})_W$. ◻

Lemma 3.2. *Assume that* (VD) *is satisfied and that the jump kernel J satisfies* (3.1). *If $\beta_2 < 2$ then both* $(\mathrm{AB})_{\mathrm{W}}$ *and* $(\mathrm{AB}')_{\mathrm{W}}$ *are satisfied (where β_2 is the exponent in* (2.2)*).*

Proof. Consider balls $B_0 = B(x_0, R)$, $B = B(x_0, R + r)$ and $\Omega = B(x_0, R')$, where $x_0 \in \mathcal{X}$, $R, r \in (0, \infty)$ and $R' > R + r$. Since d is a metric, by the classical Urysohn lemma, there exists a function $\phi \in \mathrm{cutoff}(B_0, B)$ such that

$$|\phi(x) - \phi(y)| \le Cr^{-1} d(x, y) \tag{3.3}$$

for all $x, y \in \mathcal{X}$ and for some constant $C \in (0, \infty)$. Using this and the fact that

$$\int_\Omega \int_\Omega |u(x)|^2 |\phi(x) - \phi(y)|^2 J(x, y)\, d\mu(y)\, d\mu(x) \simeq \int_\Omega \int_\Omega \frac{|u(x)|^2 |\phi(x) - \phi(y)|^2}{W(x, d(x, y))\, V(x, y)}\, d\mu(y)\, d\mu(x),$$

we see that both $(\mathrm{AB})_{\mathrm{W}}$ and $(\mathrm{AB}')_{\mathrm{W}}$ follow directly from the estimate

$$\int_{\mathcal{X}} \frac{|\phi(x) - \phi(y)|^2}{W(x, d(x, y))\, V(x, y)}\, d\mu(y) \lesssim \frac{1}{W(x, r)} \quad \text{for all } \ x \in \mathcal{X}. \tag{3.4}$$

To verify (3.4), we use the argument from the proof of $(\mathrm{TJ})_{\mathrm{W}}$ in Lemma 3.1 and the fact that $0 \le \phi \le 1$, which yields

$$\int_{d(x,y)\ge r} \frac{|\phi(x) - \phi(y)|^2}{W(x, d(x, y))\, V(x, y)}\, d\mu(y) \le \int_{d(x,y)\ge r} \frac{d\mu(y)}{W(x, d(x, y))\, V(x, y)} \lesssim \frac{1}{W(x, r)}.$$

Next, using (VD), (3.3) and the assumption $\beta_2 < 2$, we obtain

$$\begin{aligned}
\int_{d(x,y)<r} \frac{|\phi(x) - \phi(y)|^2}{W(x, d(x, y))\, V(x, y)} &\lesssim r^{-2} \int_{d(x,y)<r} \frac{d(x, y)^2}{W(x, d(x, y))\, V(x, y)}\, d\mu(y) \\
&\simeq r^{-2} \sum_{j=0}^{\infty} \int_{2^{-j-1}r \le d(x,y) < 2^{-j}r} \frac{d(x, y)^2}{W(x, d(x, y))\, V(x, y)}\, d\mu(y) \\
&\lesssim r^{-2} \sum_{j=0}^{\infty} \int_{2^{-j-1}r \le d(x,y) < 2^{-j}r} \frac{(2^{-j}r)^2}{W(x, 2^{-j-1}r)\, V(x, 2^{-j-1}r)}\, d\mu(y) \\
&\lesssim \frac{1}{W(x, r)} \sum_{j=0}^{\infty} 2^{-2j} \frac{W(x, r)}{W(x, 2^{-j-1}r)} \frac{V(x, 2^{-j}r)}{V(x, 2^{-j-1}r)} \\
&\lesssim \frac{1}{W(x, r)} \sum_{j=0}^{\infty} 2^{-j(2-\beta_2)} \\
&\simeq \frac{1}{W(x, r)},
\end{aligned}$$

which implies (3.4). □

Remark 3.3. Let $\beta \in (0, \infty)$ and $W(x, r) = r^\beta$ for all $(x, r) \in \mathcal{X} \times (0, \infty)$. In this case, the jump kernel $J(x, y)$ in (3.1) automatically satisfies $(\mathrm{J})_\beta$. Assume that (VD) holds. It follows then from Lemma 3.1 that both $(\mathrm{TJ})_{\mathrm{W}}$ and $(\mathrm{PI})_{\mathrm{W}}$ hold. Moreover, if $\beta < 2$, then by Lemma 3.2 both $(\mathrm{AB})_\beta$ and $(\mathrm{AB}')_\beta$ are satisfied.

3.2 Singular jump measures

Here we give examples of singular jump measures such that $(TJ)_W$, $(PI)_W$ and $(AB)_W$ are all satisfied.

Example 3.4. Fix $\alpha_1, \ \alpha_2 > 0$ and $\beta > 0$. Let $(\mathcal{X}_i, d_i, \mu_i)$, $i = 1,2$ be two ultrametric spaces satisfying the conditions

$$\mu_i(B_i(x,r)) \simeq r^{\alpha_i} \quad \text{for all } x \in \mathcal{X}_i \text{ and } r > 0,$$

where $B_i(x,r) = \{y \in \mathcal{X}_i : \ d_i(x,y) < r\}$. Let us consider the product space $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ with the metric d and product measure μ defined as follows:

$$d(x,y) := \max_{i=1,2}\{d_i(x_i,y_i)\}, \qquad \mu := \mu_1 \times \mu_2,$$

where $x = (x_1,x_2), \ y = (y_1,y_2) \in \mathcal{X}$. Clearly, $(\mathcal{X}, d)$ is an ultrametric space and μ satisfies

$$\mu(B(x,r)) \simeq r^{\alpha} \quad \text{for all } x \in M \text{ and } r > 0,$$

where

$$\alpha = \alpha_1 + \alpha_2.$$

Define the kernel $J(x,dy)$ on $\mathcal{X} \times \mathcal{B}(\mathcal{X})$ by

$$J(x,dy) := \frac{\mu_1(dy_1)}{d(x_1,y_1)^{\alpha_1+\beta}}\delta_{x_2}(dy_2) + \frac{\mu_2(dy_2)}{d(x_2,y_2)^{\alpha_2+\beta}}\delta_{x_1}(dy_1)$$

Then, it was proved in [10, Section 15] that conditions $(TJ)_W$ and $(PI)_W$ are satisfied for

$$W(x,r) := r^{\beta} \quad \text{for all } x \in \mathcal{X} \text{ and } r > 0.$$

Moreover, it was proved in [22, Example 4.1 and Lemma 6.2] that $(AB)_W$ is also satisfied.

Example 3.5. Let $\mathcal{X} = \mathbb{R}^2$ and $W(x,r) = r^{\beta}$ for some $\beta < 2$. For any $x = (x^{(1)}, x^{(2)}) \in \mathbb{R} \times \mathbb{R}$ and $y = (y^{(1)}, y^{(2)}) \in \mathbb{R} \times \mathbb{R}$, consider the following jump measure

$$J(x,dy) := \frac{dy^{(1)}}{|x^{(1)} - y^{(1)}|^{1+\beta}}\delta_{x^{(2)}}(dy^{(2)}) + \frac{dy_2}{|x^{(2)} - y^{(2)}|^{1+\beta}}\delta_{x^{(1)}}(dy^{(1)})$$

that generates a cylindrical stable process on $\mathbb{R}^2$. One can use the same method as in Example 3.4 to prove that J satisfies $(TJ)_W$ and $(PI)_W$. Moreover, it was proved in [25, Proposition 7.1] that $(AB)_W$ is also satisfied.

In the above Examples 3.4 and 3.5, the function W is independent of the space variable. Next, we give an example of a jump kernel J satisfying $(TJ)_W$, $(PI)_W$ and $(AB)_W$ with the scaling function $W(x,r)$ essentially depending on the space variable x (following the ideas of [25, Section 7.2][1]).

[1]Although the Dirichlet form theory was used in [25, Section 7.2], the idea of construction of J still works without using Dirichlet form.

Example 3.6. Let $(\mathcal{X}, d)$ be an arbitrary metric space, and μ be a measure on $\mathcal{X}$ with full support satisfying (VD) and $\mu(\mathcal{X}) = \infty$. Fix a point $o \in \mathcal{X}$ and let $0 < \varepsilon < \beta < 2$. For any $x \in \mathcal{X}$ and $r \in (0, \infty)$, set

$$W(x, r) := \left(\frac{d(o, x) + r}{r} \right)^{\varepsilon} r^{\beta}.$$

By a direct computation one can verify that this function W satisfies (2.2) with $\beta_1 = \beta - \varepsilon, \beta_2 = \beta$ and $C_W = 2^{\varepsilon}$. The following jump kernel

$$J^0(x, y) := \frac{1}{V(x, y)W(x, d(x, y))} + \frac{1}{V(y, x)W(y, d(x, y))}$$

satisfies $(\mathrm{TJ})_{\mathrm{W}}$ and $(\mathrm{PI})_{\mathrm{W}}$ by Lemma 3.1 and $(\mathrm{AB})_{\mathrm{W}}$ by Lemma 3.2. Now we construct another jump kernel J that satisfies the same conditions but which is much larger than J^0.

For that, we use two sequences $\{E_n\}$, $\{F_n\}$ of Borel subsets of $\mathcal{X}$ satisfying the following properties:

$$\begin{cases} \text{(i)} & \text{All the sets } \{E_n,\ F_n\}_{n\geq 1} \text{ are mutually disjoint;} \\ \text{(ii)} & \text{For any } n \geq 1,\ 1 \leq \mu(E_n) \leq 2 \text{ and } 1 \leq \mu(F_n) \leq 2; \\ \text{(iii)} & d(E_n, F_n) \to \infty \text{ as } n \to \infty. \end{cases} \tag{3.5}$$

An example of such sequences will be given below. Using the sequences $\{E_n\}$, $\{F_n\}$ as in (3.5), let us define the following jump kernel

$$J(x, y) := J^0(x, y) + \sum_{n\geq 1} \left(\frac{\mathbf{1}_{E_n\times F_n}(x, y)}{W(x, d(x, y))} + \frac{\mathbf{1}_{E_n\times F_n}(y, x)}{W(y, d(x, y))} \right) \tag{3.6}$$

and prove that it satisfies all the conditions $(\mathrm{TJ})_{\mathrm{W}}$, $(\mathrm{PI})_{\mathrm{W}}$ and $(\mathrm{AB})_{\mathrm{W}}$.

Since $J^0 \leq J$ and J^0 satisfies $(\mathrm{PI})_{\mathrm{W}}$, it follows that J also satisfies $(\mathrm{PI})_{\mathrm{W}}$. Let us verify that J satisfies $(\mathrm{TJ})_{\mathrm{W}}$. By (3.2), we have

$$J(x, y) \simeq J^0(x, y) + \sum_{n\geq 1} \frac{\mathbf{1}_{E_n\times F_n}(x, y) + \mathbf{1}_{F_n\times E_n}(x, y)}{W(x, d(x, y))}. \tag{3.7}$$

For any $x \in \mathcal{X}$, by the mutually disjointness of $\{E_n, F_n\}_{n\in\mathbb{N}}$, there exists at most one n_x or at most one m_x such that $x \in E_{n_x}$ or $x \in F_{m_x}$. Note that x can not lie in both E_{n_x} and F_{m_x} simultaneously. So, we may as well assume that $x \in E_{n_x}$. Then we have

$$\begin{aligned} \int_{B(x,r)^{\complement}} \sum_{n\geq 1} \frac{\mathbf{1}_{E_n\times F_n}(x, y) + \mathbf{1}_{F_n\times E_n}(x, y)}{W(x, d(x, y))}\, d\mu(y) &\leq \int_{B(x,r)^{\complement}} \frac{\mathbf{1}_{F_{n_x}}(y)}{W(x, d(x, y))}\, d\mu(y) \\ &\leq \frac{1}{W(x, r)} \cdot \mu(B(x, r)^{\complement} \cap F_{n_x}) \leq \frac{2}{W(x, r)}. \end{aligned}$$

This implies that J satisfies $(\mathrm{TJ})_{\mathrm{W}}$ since so does J^0.

Let us prove that J satisfies $(\mathrm{AB})_{\mathrm{W}}$. Fix three concentric balls $B_0 = (x_0, R)$, $B = B(x_0, R + r)$ and $\Omega = B(x_0, R')$ with $x_0 \in \mathcal{X}$ and $0 < R < R + r < R'$. Let

$$\phi(x) := 1 \wedge \frac{R + r - d(x_0, x)}{r} \vee 0 \quad \text{for all } \ x \in \mathcal{X}.$$

Clearly, $\phi \in \mathrm{cutoff}(B_0, B)$. Following the arguments in the proof of Lemma 3.2, in order to prove that J satisfies $(\mathrm{AB})_{\mathrm{W}}$, it suffices to verify that, for any $x \in \mathcal{X}$,

$$\int_{\mathcal{X}} |\phi(x) - \phi(y)|^2 J(x, y)\, d\mu(y) \lesssim \frac{1}{W(x, r)}.$$

By Lemma 3.2 and (3.7), we only need to show that, for any $x \in \mathcal{X}$,

$$\int_{\mathcal{X}} |\phi(x)-\phi(y)|^2 \sum_{n\geq 1} \frac{\mathbf{1}_{E_n\times F_n}(x,y)+\mathbf{1}_{F_n\times E_n}(x,y)}{W(x,d(x,y))}\, d\mu(y) \leq \frac{2}{W(x,r)}. \tag{3.8}$$

Indeed, for any $x, y \in \mathcal{X}$ and $r \in (0,\infty)$, observe that

$$|\phi(x)-\phi(y)| \leq \min\left\{1,\ r^{-1}d(x,y)\right\}$$

and

$$\frac{W(x,r)}{W(x,d(x,y))} = \left(\frac{r}{d(x,y)}\right)^{\beta}\left(\frac{1+\frac{d(o,x)}{r}}{1+\frac{d(o,x)}{d(x,y)}}\right)^{\varepsilon} \leq \left(\frac{r}{d(x,y)}\right)^{\beta} \max\left\{1,\ \left(\frac{d(x,y)}{r}\right)^{\varepsilon}\right\}.$$

Consequently, for any $x \in \mathcal{X}$, using the notation n_x as above (assuming without loss of generality that $x \in E_{n_x}$), we then derive

$$\begin{aligned}
&\int_{\mathcal{X}} |\phi(x)-\phi(y)|^2 \sum_{n\geq 1} \frac{\mathbf{1}_{E_n\times F_n}(x,y)+\mathbf{1}_{F_n\times E_n}(x,y)}{W(x,d(x,y))}\, d\mu(y)\\
&\leq \int_{\mathcal{X}} |\phi(x)-\phi(y)|^2 \frac{\mathbf{1}_{F_{n_x}}(y)}{W(x,d(x,y))}\, d\mu(y)\\
&\leq \frac{1}{W(x,r)} \int_{F_{n_x}} \min\left\{1,\ r^{-2}d(x,y)^2\right\}\left(\frac{r}{d(x,y)}\right)^{\beta} \max\left\{1,\ \left(\frac{d(x,y)}{r}\right)^{\varepsilon}\right\} d\mu(y)\\
&\leq \frac{1}{W(x,r)}\mu(F_{n_x})\\
&\leq \frac{2}{W(x,r)},
\end{aligned}$$

where the penultimate step holds because $\varepsilon < \beta < 2$ implies that the integrand is bounded by 1. This proves (3.8). Hence, we conclude that J satisfies $(\mathrm{AB})_{\mathrm{W}}$.

Now let us construct sequences $\{E_n\}$, $\{F_n\}$ satisfying (3.5). Recall that $o \in \mathcal{X}$ is a fixed point and C'_D, α_+ are the constants in (2.7). Let $\lambda > 0$ be large enough such that

$$2 \cdot 9^{\alpha_+} C'_D \leq V(o,\lambda).$$

Let $x_0 := o$. We are about to construct a sequence of balls $\{B_k := B(x_k, r_k)\}_{k\geq 1}$ satisfying the following properties: for any $k \in \mathbb{N}$,

$$\begin{cases}
\text{(i)}\ \ d(x_k,o) \geq \max\{\lambda,\ 2d(x_{k-1},o)\};\\
\text{(ii)}\ \ 1 \leq \mu(B_k) \leq 2;\\
\text{(iii)}\ \ r_k \leq d(x_k,o)/4;\\
\text{(iv)}\ \ \cup_{m=1}^{k} B_m \subset B(o,\ 5d(x_k,o)/4).
\end{cases} \tag{3.9}$$

Indeed, since $\mu(\mathcal{X}) = \infty$ and every ball has finite measure by (VD), we have $B(o,r)^{\complement} \neq \emptyset$ for any $r \in (0,\infty)$. Thus, we can inductively take a sequence of points $\{x_k\}_{k\in\mathbb{N}}$ that satisfy (i) of (3.9). From (i), it follows easily that $\{d(x_k,o)\}_{k\in\mathbb{N}}$ is increasing and that

$$d(x_k,o) \geq 2^{k-1}\lambda \quad \text{for all } k \in \mathbb{N}. \tag{3.10}$$

Once we have these points $\{x_k\}_{k\in\mathbb{N}}$, then for each $k \in \mathbb{N}$ we can choose a value $r_k \in (0,\infty)$ such that $1 \le \mu(B(x_k, r_k)) \le 2$, which induces (ii) of (3.9). Moreover, by the choice of λ, (VD) and by (i) and (ii) of (3.9), we have

$$9^{\alpha_+} C_D' \le \frac{V(o,\lambda)}{2} \le \frac{V(o,\, d(x_k,o)+r_k)}{V(x_k,r_k)} \le C_D'\left(\frac{2d(x_k,o)+r_k}{r_k}\right)^{\alpha_+},$$

which is equivalent to $r_k \le d(x_k,o)/4$, thereby leading to (iii) of (3.9). To see that (iv) of (3.9) holds, if $z \in B_m$ for some $m \in \{1,2,\dots,k\}$, then by the increasing property of $\{d(x_k,o)\}_{k\in\mathbb{N}}$ and (iii), we obtain

$$d(z,o) \le d(z,x_m) + d(x_m,o) < r_m + d(x_m,o) \le \frac{5}{4}d(x_m,o) \le \frac{5}{4}d(x_k,o)$$

and, hence, $\cup_{k=1}^m B_m$ is contained in the ball $B(0, \frac{5}{4}d(x_k,o))$. In this way, we have constructed a sequence of balls $\{B_k\}_{k\in\mathbb{N}}$ satisfying (i)-(ii)-(iii)-(iv) of (3.9).

For any $k \in \mathbb{Z}$, if $x \in B_{k+1}$ and $y \in \cup_{m=1}^k B_m$, then we have by (i), (iii) and (iv) in (3.9) that

$$\begin{aligned} d(x,y) &\ge d(x_{k+1},o) - d(x_{k+1},x) - d(o,y) \\ &\ge d(x_{k+1},o) - \frac{d(x_{k+1},o)}{4} - \frac{5d(x_k,o)}{4} \\ &= \frac{3d(x_{k+1},o)}{4} - \frac{5d(x_k,o)}{4} \ge \frac{d(x_k,o)}{4} \ge 2^{k-3}\lambda. \end{aligned} \tag{3.11}$$

This shows that the distance between B_{k+1} and $\cup_{m=1}^k B_m$ goes to infinity as $k \to \infty$. In particular, the sequence $\{B_k\}_{k\in\mathbb{N}}$ are mutually disjoint. Thus, for any $n \in \mathbb{N}$, upon letting

$$E_n := B_{2n} \quad \text{and} \quad F_n := B_{2n-1}, \tag{3.12}$$

we see that $\{E_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ are exactly two sequences of Borel sets satisfying (3.5).

Let us emphasize that, for the sets $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ in (3.12), the corresponding jump kernel J from (3.6) is essentially larger than J^0 on a large set. Indeed, for any $n \in \mathbb{N}$ and $(x,y) \in E_n \times F_n$, it follows from (3.10) and (3.11) that

$$d(x,y) \ge \mathrm{dist}(E_n,F_n) = \mathrm{dist}(B_{2n},B_{2n-1}) \ge \frac{d(x_{2n-1},o)}{4} \ge 2^{2n-4}\lambda,$$

which combined with (3.10) and $B_{2n-1} \subset B(o, 5d(x_{2n-1},o)/4)$ (see (iv) of (3.9)) yields that for any $z \in B(o, 2^{2n-4}\lambda)$,

$$d(z,x) \le d(z,o) + d(o,y) + d(y,x) < 2^{2n-4}\lambda + \frac{5}{4}d(x_{2n-1},o) + d(y,x) < 7d(x,y),$$

so that $B(o, 2^{2n-4}\lambda) \subset B(x,\, 7d(x,y))$ and, hence,

$$V(o, 2^{2n-4}\lambda) \le V(x, 7d(x,y)) \le C_D^3 V(x,y).$$

Thus, for any $(x,y) \in E_n \times F_n$, we obtain using (3.2) and $V(x,y) \simeq V(y,x)$ (see (VD)) that

$$\frac{\mathbf{1}_{E_n\times F_n}(x,y)}{W(x,d(x,y))} \cdot \frac{1}{J^0(x,y)} \simeq V(x,y) \gtrsim V(o, 2^{2n-4}\lambda).$$

This together with the assumption $\mu(\mathcal{X}) = \infty$ implies that, for the set $A = \bigcup_{n=N}^\infty (E_n \times F_n)$,

$$\lim_{N\to\infty}\left(\inf_{x,y\in A} \frac{J(x,y)}{J^0(x,y)}\right) \gtrsim \lim_{N\to\infty} V(o, 2^{2N-4}\lambda) = \infty.$$

Observe that the set A has an infinite measure:

$$(\mu \times \mu)(A) = \sum_{n=N}^{\infty} \mu(E_n)\mu(F_n) = \infty,$$

which follows from the mutually disjointness of $\{E_n \times F_n\}_{n\in\mathbb{N}}$ and Property (ii) of (3.5).

3.3 Product spaces

In this subsection, we study the conditions $(\mathrm{TJ})_W$, $(\mathrm{AB})_W$, $(\mathrm{PI})_W$ on product spaces. We show in Theorem 3.7 below that if on each metric measure space $(\mathcal{X}_i, d_i, \mu_i)$, all the conditions $(\mathrm{TJ})_W$, $(\mathrm{AB})_W$, $(\mathrm{PI})_W$ are satisfied with a common scaling function $W(r)$ (independent of space variable x), then the same conditions are satisfied also on the product space $(\mathcal{X}, d, \mu)$ with the same scaling function $W(r)$.

Let $m \in \mathbb{N}$. For any $i \in \{1, 2, \dots, m\}$, suppose that $(\mathcal{X}_i, d_i, \mu_i)$ is a doubling metric measure space. We consider the product space

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m.$$

An element $x \in \mathcal{X}$ can be written as $x = (x_1, \dots, x_m)$, where $x_i \in \mathcal{X}_i$ for any $i \in \{1, 2, \dots, m\}$. If $x_i, y_i \in \mathcal{X}_i$ and $r \in (0, \infty)$, we still adopt the notation

$$V_i(x_i, r) := \mu_i(B_i(x_i, r)) \quad \text{and} \quad V_i(x_i, y_i) := \mu_i(B_i(x_i,\ d_i(x_i, y_i))),$$

where each $B_i(x, r) = \{y \in \mathcal{X}_i :\ d_i(x, y) < r\}$. Define on $\mathcal{X}$ the following metric:

$$d(x, y) := \max_{1\le i\le m} d_i(x_i, y_i).$$

Clearly, for any $x \in \mathcal{X}$ and $r \in (0, \infty)$, the ball $B(x, r)$ in $\mathcal{X}$ takes the form of

$$B(x, r) = \prod_{i=1}^{m} B_i(x_i, r) = B_1(x_1, r) \times B_2(x_2, r) \times \cdots \times B_m(x_m, r).$$

Consider on the product space $\mathcal{X}$ the product measure μ:

$$d\mu := d\mu_1 \times \cdots \times d\mu_m.$$

Then, for any $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) = \prod_{i=1}^{m} \mu_i(B_i(x_i, r)) = \prod_{i=1}^{m} V_i(x_i, r).$$

Clearly, volume doubling condition (VD) is satisfied on this product space $(\mathcal{X}, d, \mu)$. Moreover, if each $(\mathcal{X}_i, d_i, \mu_i)$ satisfies (RVD) then $(\mathcal{X}, d, \mu)$ also satisfies (RVD) but with different constants.

For any $i \in \{1, 2, \dots, m\}$, let $J_i(x_i, dy_i)$ be a jump measure on $\mathcal{X}_i$. As in Definition 2.1, each $J_i(x_i, dy_i)$ determines a bilinear form $(\mathcal{E}_i, \mathcal{F}_i)$. Let δ_{x_i} be the Dirac measure in $\mathcal{X}_i$ at the point x_i. Define the *product jump measure* $J(x, dy)$ on $\mathcal{X}$ by

$$J(x, dy) := \sum_{i=1}^{m} \delta_{x_1}(dy_1) \cdots \delta_{x_{i-1}}(dy_{i-1})\, J_i(x,\ dy_i)\, \delta_{x_{i+1}}(dy_{i+1}) \cdots \delta_{x_m}(dy_m). \tag{3.13}$$

It is easy to verify that $J(x,dy)$ in (3.13) satisfies Definition 2.1. If f is a Borel measurable function on $\mathcal{X}\times\mathcal{X}$, then it follows from (3.13) that

$$\begin{aligned}
&\int_{\mathcal{X}}\int_{\mathcal{X}} f(x;y)\,J(x,dy)\,d\mu(x)\\
&=\sum_{i=1}^{m}\int_{\mathcal{X}}\int_{\mathcal{X}} f(x;y)\delta_{x_1}(dy_1)\cdots\delta_{x_{i-1}}(dy_{i-1})\,J_i(x,\,dy_i)\,\delta_{x_{i+1}}(dy_{i+1})\cdots\delta_{x_m}(dy_m)\,d\mu(x)\\
&=\sum_{i=1}^{m}\int_{\mathcal{X}}\int_{\mathcal{X}_i} f(x_1,\dots,x_m;x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m)J_i(x,\,dy_i)\,d\mu(x).
\end{aligned}$$

Consequently, the *product bilinear form* $(\mathcal{E},\mathcal{F})$ associated to $J(x,\,dy)$ in (3.13) is as follows: for any Borel measurable functions u,v on $\mathcal{X}$,

$$\begin{aligned}
\mathcal{E}(u,v) :&= \int_{\mathcal{X}}\int_{\mathcal{X}}(u(x)-u(y))(v(x)-v(y))\,J(x,dy)\,d\mu(x)\\
&=\sum_{i=1}^{m}\int_{\mathcal{X}}\int_{\mathcal{X}_i}(u(x_1,\dots,x_m)-u(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m))\\
&\qquad\times(v(x_1,\dots,x_m)-v(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m))\,J_i(x,\,dy_i)\,d\mu(x).
\end{aligned}\tag{3.14}$$

Assuming Theorems 2.9, 2.13 and 2.15 for the moment, we obtain the following result for the product bilinear form and its heat kernel.

Theorem 3.7. *Let $m\in\mathbb{N}$. For any $i\in\{1,2,\dots,m\}$, suppose that $(\mathcal{X}_i,d_i,\mu_i)$ is a metric measure space satisfying (VD) and (RVD). Moreover, on every $(\mathcal{X}_i,d_i,\mu_i)$, there is a bilinear form $(\mathcal{E}_i,\mathcal{F}_i)$, which is determined by a jump measure $J_i(x_i,dy_i)\,d\mu_i(x_i)$. If for some space/time scaling function $r\mapsto W(r)$ that is independent of the space variable x, the conditions*

$$\text{(TJ)}_W + \text{(PI)}_W + \text{(AB)}_W \tag{3.15}$$

are all satisfied for each $(\mathcal{E}_i,\mathcal{F}_i)$ and $J_i(x_i,dy_i)\,d\mu_i(x_i)$, then the product kernel $J(x,\,dy)\,d\mu(x)$ in (3.13) also satisfies (3.15). As a consequence, the associated product bilinear form $(\mathcal{E},\mathcal{F})$ in (3.14) is a regular Dirichlet form on $L^2(\mathcal{X})$ and the corresponding heat kernel $\{p_t\}_{t>0}$ is stochastic complete and satisfies (LLE)_W and (TP)_W.

Proof. Once we have obtained that the product jump measure $J(x,\,dy)$ satisfies $\text{(TJ)}_W+\text{(PI)}_W+\text{(AB)}_W$, then applying Theorem 2.9 we obtain the regularity of $(\mathcal{E},\mathcal{F})$. Further, applying Theorems 2.13 and 2.15 we obtain (LLE)_W, (TP)_W and the stochastic completeness of $\{p_t\}_{t>0}$.

Hence, let us show that the product jump measure $J(x,\,dy)$ satisfies $\text{(TJ)}_W+\text{(PI)}_W+\text{(AB)}_W$. For simplicity, we consider only the case $m=2$ as the proof for a general m is similar. For $m=2$, we have

$$J(x,\,dy) = J_1(x_1,\,dy_1)\,\delta_{x_2}(dy_2) + \delta_{x_1}(dy_1)\,J_2(x_2,\,dy_2) \tag{3.16}$$

and

$$\begin{aligned}
\mathcal{E}(u,v) = &\int_{\mathcal{X}}\int_{\mathcal{X}_1}(u(x_1,x_2)-u(y_1,x_2))(v(x_1,x_2)-v(y_1,x_2))\,J_1(x_1,\,dy_1)\,d\mu(x)\\
&+\int_{\mathcal{X}}\int_{\mathcal{X}_2}(u(x_1,x_2)-u(x_1,y_2))(v(x_1,x_2)-v(x_1,y_2))\,J_2(x_2,\,dy_2)\,d\mu(x).
\end{aligned}\tag{3.17}$$

Since W is independent of the space variable x, below we will simply write $W(r)$.

Step 1. Let us prove (TJ)$_{\mathrm{W}}$. For any $x = (x_1, x_2) \in \mathcal{X}$ and $R \in (0, \infty)$, we write

$$\begin{aligned}
J(x, B(x,R)^{\complement}) &= \int_{B(x,R)^{\complement}} J(x, dy) \\
&= \int_{(\mathcal{X}_1\times\mathcal{X}_2)\setminus(B_1(x_1,R)\times B_2(x_2,R))} J_1(x_1,\, dy_1)\, \delta_{x_2}(dy_2) \\
&\quad + \int_{(\mathcal{X}_1\times\mathcal{X}_2)\setminus(B_1(x_1,R)\times B_2(x_2,R))} \delta_{x_1}(dy_1)\, J_2(x_2,\, dy_2) \\
&= \int_{B_2(x_2,R)} \int_{\mathcal{X}_1\setminus B_1(x_1,R)} J_1(x_1,\, dy_1)\, \delta_{x_2}(dy_2) + \int_{B_1(x_1,R)} \int_{\mathcal{X}_2\setminus B_2(x_2,R)} \delta_{x_1}(dy_1)\, J_2(x_2,\, dy_2) \\
&= \int_{\mathcal{X}_1\setminus B_1(x_1,R)} J_1(x_1,\, dy_1) + \int_{\mathcal{X}_2\setminus B_2(x_2,R)} J_2(x_2,\, dy_2) \\
&\lesssim \frac{1}{W(R)},
\end{aligned}$$

as desired.

Step 2. Let us prove (PI)$_{\mathrm{W}}$. We will use the following formula: for any measurable set $E \subseteq \mathcal{X}$ and $u \in L^1(E) \cap L^2(E)$,

$$\int_E |u - u_E|^2\, d\mu = \frac{1}{2\mu(E)} \iint_{E\times E} |u(x) - u(y)|^2\, d\mu(x)\, d\mu(y), \tag{3.18}$$

where $u_E = \frac{1}{\mu(E)} \int_E u\, d\mu$. Indeed, (3.18) follows from the following identities:

$$|u - u_E|^2 = u^2 + (u_E)^2 - 2uu_E \quad \text{and} \quad |u(x) - u(y)|^2 = u^2(x) + u^2(y) - 2u(x)u(y).$$

Fix a point $a = (a_1, a_2) \in \mathcal{X}$, where $a_1 \in \mathcal{X}_1$ and $a_2 \in \mathcal{X}_2$. Take a ball $B(a, r) \subseteq \mathcal{X}$. In this step, we use the notation

$$B = B(a, r) = B_1 \times B_2,$$

where $B_1 := B_1(a_1, r) \subseteq \mathcal{X}_1$ and $B_2 := B_2(a_2, r) \subseteq \mathcal{X}_2$.

Suppose that $u \in \mathcal{F} \cap L^\infty(\mathcal{X})$. By (3.17), we then have

$$\begin{aligned}
\mathcal{E}(u, u) = &\int_{\mathcal{X}_2} \int_{\mathcal{X}_1} \int_{\mathcal{X}_1} |u(x_1, x_2) - u(y_1, x_2)|^2\, J_1(x_1,\, x_2,\, dy_1)\, d\mu_1(x_1)\, d\mu_2(x_2) \\
&+ \int_{\mathcal{X}_1} \int_{\mathcal{X}_2} \int_{\mathcal{X}_2} |u(x_1, x_2) - u(x_1, y_2)|^2\, J_2(x_1,\, x_2,\, dy_2)\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&< \infty.
\end{aligned}$$

From this, it follows that these two triple-integrals are finite. In particular, for μ_2-a.a. $x_2 \in \mathcal{X}_2$,

$$\int_{\mathcal{X}_1} \int_{\mathcal{X}_1} |u(x_1, x_2) - u(y_1, x_2)|^2\, J_1(x_1,\, x_2,\, dy_1)\, d\mu_1(x_1) < \infty$$

and for μ_1-a.a. $x_1 \in \mathcal{X}_1$,

$$\int_{\mathcal{X}_2} \int_{\mathcal{X}_2} |u(x_1, x_2) - u(x_1, y_2)|^2\, J_2(x_1,\, x_2,\, dy_2)\, d\mu_2(x_2) < \infty,$$

which implies that

$$u_{x_2}(\cdot) := u(\cdot, x_2) \in \mathcal{F}_1 \cap L^\infty(\mathcal{X}_1) \quad \text{and} \quad u^{x_1}(\cdot) := u(x_1, \cdot) \in \mathcal{F}_2 \cap L^\infty(\mathcal{X}_2).$$

Further, since each $J_i(x_i, dy_i)\, d\mu_i(x_i)$ and the associated bilinear form $(\mathcal{E}_i, \mathcal{F}_i)$ satisfies $(\mathrm{PI})_\mathrm{W}$, by (3.18), we deduce that for μ_2-a. a. $x_2 \in \mathcal{X}_2$,

$$\begin{aligned} &\frac{1}{2\mu(B_1)} \iint_{B_1\times B_1} |u(x_1, x_2) - u(y_1, x_2)|^2\, d\mu_1(x_1)\, d\mu_1(y_1) \\ &\quad = \int_{B_1} |u_{x_2} - (u_{x_2})_{B_1}|^2\, d\mu_1 \\ &\quad \le CW(R) \iint_{(\kappa B_1)\times(\kappa B_1)} |u_{x_2}(x_1) - u_{x_2}(y_1)|^2\, J_1(x_1,\, dy_1)\, d\mu_1(x_1) \end{aligned} \tag{3.19}$$

and for μ_1-a. a. $y_1 \in \mathcal{X}_1$,

$$\begin{aligned} &\frac{1}{2\mu(B_2)} \iint_{B_2\times B_2} |u(y_1, x_2) - u(y_1, y_2)|^2\, d\mu_2(x_2)\, d\mu_2(y_2) \\ &\quad \le CW(R) \iint_{(\kappa B_2)\times(\kappa B_2)} |u^{y_1}(x_2) - u^{y_1}(y_2)|^2\, J_2(x_2, dy_2)\, d\mu_2(x_2), \end{aligned} \tag{3.20}$$

where the constants $\kappa \in [1, \infty)$ and $C \in (0, \infty)$ are as in Definition 2.5. In both sides of (3.19), by integrating over B_2 with respect to the variable x_2 and the measure $d\mu_2$, we then obtain

$$\begin{aligned} &\frac{1}{2\mu(B_1)} \int_{B_2} \iint_{B_1\times B_1} |u(x_1, x_2) - u(y_1, x_2)|^2\, d\mu_1(x_1)\, d\mu_1(y_1) d\mu_2(x_2) \\ &\quad \le CW(R) \int_{B_2} \iint_{(\kappa B_1)\times(\kappa B_1)} |u(x_1, x_2) - u(y_1, x_2)|^2\, J_1(x_1,\, dy_1)\, d\mu_1(x_1)\, d\mu_2(x_2) \\ &\quad \le CW(R) \iint_{(\kappa B)\times(\kappa B)} |u(x_1, x_2) - u(y_1, y_2)|^2\, J_1(x_1,\, dy_1)\, \delta_{x_2}(dy_2)\, d\mu_1(x_1)\, d\mu_2(x_2). \end{aligned} \tag{3.21}$$

Similarly, in both sides of (3.20), by integrating over B_1 with respect to the variable y_1 and the measure $d\mu_1$, we then obtain

$$\begin{aligned} &\frac{1}{2\mu(B_2)} \int_{B_1} \iint_{B_2\times B_2} |u(y_1, x_2) - u(y_1, y_2)|^2\, d\mu_2(x_2)\, d\mu_2(y_2)\, d\mu_1(y_1) \\ &\quad \le CW(R) \int_{B_1} \iint_{(\kappa B_2)\times(\kappa B_2)} |u(y_1, x_2) - u(y_1, y_2)|^2\, J_2(x_2, dy_2)\, d\mu_2(x_2)\, d\mu_1(y_1) \\ &\quad = CW(R) \int_{B_1} \iint_{(\kappa B_2)\times(\kappa B_2)} |u(x_1, x_2) - u(x_1, y_2)|^2\, J_2(x_2, dy_2)\, d\mu_2(x_2)\, d\mu_1(x_1) \\ &\quad \le CW(R) \iint_{(\kappa B)\times(\kappa B)} |u(x_1, x_2) - u(y_1, y_2)|^2\, J_2(x_2, dy_2)\, \delta_{x_1}(dy_1)\, d\mu_2(x_2)\, d\mu_1(x_1). \end{aligned} \tag{3.22}$$

Meanwhile, using (3.18), (3.21), (3.22) and the following inequality: for any $x = (x_1, x_2)$, $y = (y_1, y_2) \in B = B_1 \times B_2$,

$$|u(x) - u(y)| = |u(x_1, x_2) - u(y_1, y_2)| \le |u(x_1, x_2) - u(y_1, x_2)| + |u(y_1, x_2) - u(y_1, y_2)|,$$

we obtain

$$\int_B |u(x) - u_B|^2\, d\mu(x)$$

$$
\begin{aligned}
&= \frac{1}{2\mu(B_1)\mu(B_2)} \iint_{B_1\times B_2} \iint_{B_1\times B_2} |u(x_1,x_2)-u(y_1,y_2)|^2 \, d\mu_1(x_1)\, d\mu_2(x_2)\, d\mu_1(y_1)\, d\mu_2(y_2) \\
&\le \frac{1}{2\mu(B_1)\mu(B_2)} \iint_{B_2\times B_2} \iint_{B_1\times B_1} |u(x_1,x_2)-u(y_1,x_2)|^2 \, d\mu_1(x_1)\, d\mu_1(y_1)\, d\mu_2(x_2)\, d\mu_2(y_2) \\
&\quad + \frac{1}{2\mu(B_1)\mu(B_2)} \iint_{B_1\times B_1} \iint_{B_2\times B_2} |u(y_1,x_2)-u(y_1,y_2)|^2 \, d\mu_2(x_2)\, d\mu_2(y_2)\, d\mu_1(y_1)\, d\mu_1(x_1) \\
&= \frac{1}{2\mu(B_1)} \int_{B_2} \iint_{B_1\times B_1} |u(x_1,x_2)-u(y_1,x_2)|^2 \, d\mu_1(x_1)\, d\mu_1(y_1)\, d\mu_2(x_2) \\
&\quad + \frac{1}{2\mu(B_2)} \int_{B_1} \iint_{B_2\times B_2} |u(y_1,x_2)-u(y_1,y_2)|^2 \, d\mu_2(x_2)\, d\mu_2(y_2)\, d\mu_1(y_1) \\
&\le CW(R) \iint_{(\kappa B)\times(\kappa B)} |u(x_1,x_2)-u(y_1,y_2)|^2 \, J_1(x_1,\, dy_1)\, \delta_{x_2}(dy_2)\, d\mu_1(x_1)\, d\mu_2(x_2) \\
&\quad + CW(R) \iint_{(\kappa B)\times(\kappa B)} |u(x_1,x_2)-u(y_1,y_2)|^2 \, J_2(x_2,dy_2)\, \delta_{x_1}(dy_1)\, d\mu_2(x_2)\, d\mu_1(x_1) \\
&= CW(R) \iint_{(\kappa B)\times(\kappa B)} |u(x_1,x_2)-u(y_1,y_2)|^2 \, J(x,dy)\, d\mu_2(x_2)\, d\mu_1(x_1). \quad \text{(by (3.16))}
\end{aligned}
$$

This proves that $J(x,\, dy)$ satisfies $(\mathrm{PI})_{\mathrm{W}}$.

Step 3. Let us prove $(\mathrm{AB})_{\mathrm{W}}$. Let $0 < R < R + r < R' < \infty$ and $a = (a_1, a_2) \in \mathcal{X}$, where $a_1 \in \mathcal{X}_1$ and $a_2 \in \mathcal{X}_2$. Take three concentric balls in the product space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, say

$$
\begin{cases}
B_0 = B(a,R) = B_1(a_1,R) \times B_2(a_2,R); \\
B = B(a,R+r) = B_1(a_1,R+r) \times B_2(a_2,R+r); \\
\Omega = B(a,R') = B_1(a_1,R') \times B_2(a_2,R').
\end{cases}
$$

To simplify the notation, below we set $B_i' := B_i(a_i,R')$ and $B_i := B_i(a_i,R+r)$, where $i = 1,2$.

Let $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$, where we recall that $\mathcal{F}' = \{\mathcal{F} + c : \ c \text{ is a constant}\}$. Just like the arguments in the beginning of **Step 2**, we now have

$$
u_{x_2}(\cdot) := u(\cdot, x_2) \in \mathcal{F}_1' \cap L^\infty(\mathcal{X}_1) \quad \text{and} \quad u^{x_1}(\cdot) := u(x_1,\cdot) \in \mathcal{F}_2' \cap L^\infty(\mathcal{X}_2),
$$

where each $\mathcal{F}_i'$ is defined in a similar manner.

Fix $i \in \{1,2\}$. Since on each $(\mathcal{X}_i, d_i, \mu_i)$ the conditions (VD), (RVD), $(\mathrm{TJ})_{\mathrm{W}}$, $(\mathrm{PI})_{\mathrm{W}}$ and $(\mathrm{AB})_{\mathrm{W}}$ are satisfied, we derive that Theorem 2.9(iii) holds for each $(\mathcal{X}_i, d_i, \mu_i)$. Then, using Lemmas 5.5 and 5.3 from Section 5 below (see also Remark 5.4 therein), we obtain that there exists a universal cutoff function

$$
\phi_i \in \mathrm{cutoff}(B_i(a_i,R),\ B_i(a_i,R+r))
$$

such that for all $u \in \mathcal{F}_i' \cap L^\infty(\mathcal{X}_i)$,

$$
\begin{aligned}
&\iint_{B_1'\times B_1'} |u_{x_2}(x_1)|^2 |\phi_1(x_1)-\phi_1(y_1)|^2 \, J_1(x_1,dy_1)\, d\mu_1(x_1) \\
&\quad \le \zeta \iint_{B_1\times B_1} |\phi_1(x_1)|^2 |u_{x_2}(x_1)-u_{x_2}(y_1)|^2 J_1(x_1,dy_1)\, d\mu_1(x_1) \\
&\qquad + \frac{C}{W(r)} \int_{B_1'} |u_{x_2}(x_1)|^2 \, d\mu_1(x_1)
\end{aligned}
\tag{3.23}
$$

and

$$\begin{aligned}\iint_{B_2'\times B_2'} &|u^{x_1}(x_2)|^2|\phi_2(x_2)-\phi_2(y_2)|^2\, J_2(x_2,dy_2)\,d\mu_2(x_2)\\ &\le \zeta\iint_{B_2\times B_2}|\phi_2(x_2)|^2|u^{x_1}(x_2)-u^{x_1}(y_2)|^2 J_2(x_2,dy_2)\,d\mu_2(x_2)\\ &\quad+\frac{C}{W(r)}\int_{B_2'}|u^{x_1}(x_2)|^2\,d\mu_2(x_2).\end{aligned}\tag{3.24}$$

Now, for any $x=(x_1,x_2)\in\mathcal{X}$, define

$$\phi(x)=\phi(x_1,x_2)=\phi_1(x_1)\phi_2(x_2).$$

Note that $\phi\equiv 1$ on $B_0=B_1(a_1,R)\times B_2(a_2,R)$. Also, $\operatorname{supp}\phi\subseteq B_1\times B_2=B$. The continuity of ϕ on $\mathcal{X}$ is obvious. Hence, $\phi\in\operatorname{cutoff}(B_0,B)$. For any $x=(x_1,x_2)\in\mathcal{X}$ and $y=(y_1,y_2)\in\mathcal{X}$, it holds

$$|\phi(x_1,x_2)-\phi(y_1,x_2)|=|\phi_1(x_1)-\phi_1(y_1)||\phi_2(x_2)|$$

and

$$|\phi(x_1,x_2)-\phi(x_1,y_2)|=|\phi_1(x_1)||\phi_2(x_2)-\phi_2(y_2)|.$$

By these and (3.16), we write

$$\begin{aligned}\iint_{\Omega\times\Omega}&|u(x)|^2|\phi(x)-\phi(y)|^2\,J(x,dy)\,d\mu(x)\\ &=\int_{B_2'}\left(\iint_{B_1'\times B_1'}|u_{x_2}(x_1)|^2|\phi_1(x_1)-\phi_1(y_1)|^2 J_1(x_1,\,dy_1)\,d\mu_1(x_1)\right)|\phi_2(x_2)|^2\,d\mu_2(x_2)\\ &\quad+\int_{B_1'}\left(\iint_{B_2'\times B_2'}|u^{x_1}(x_2)|^2|\phi_2(x_2)-\phi_2(y_2)|^2\,J_2(x_2,dy_2)\,d\mu_2(x_2)\right)|\phi_1(x_1)|^2 d\mu_1(x_1).\end{aligned}$$

If we integrate in both sides of (3.23) with respect to $|\phi_2(x_2)|^2\,d\mu_2(x_2)$, and also integrate in both sides of (3.24) with respect to $|\phi_1(x_1)|^2\,d\mu_1(x_1)$, then we continue the above estimate via

$$\begin{aligned}\iint_{\Omega\times\Omega}&|u(x)|^2|\phi(x)-\phi(y)|^2\,J(x,dy)\,d\mu(x)\\ &\le\zeta\int_{B_2'}\iint_{B_1\times B_1}|\phi_1(x_1)\phi_2(x_2)|^2|u_{x_2}(x_1)-u_{x_2}(y_1)|^2 J_1(x_1,dy_1)\,d\mu_1(x_1)\,d\mu(x_2)\\ &\quad+\zeta\int_{B_1'}\iint_{B_2\times B_2}|\phi_1(x_1)\phi_2(x_2)|^2|u^{x_1}(x_2)-u^{x_1}(y_2)|^2 J_2(x_2,dy_2)\,d\mu_2(x_2)\,d\mu_1(x_1)\\ &\quad+\frac{2C}{W(r)}\int_{B_2'}\int_{B_1'}|u(x_1,x_2)|^2\,d\mu_1(x_1)\,d\mu_2(x_2).\end{aligned}$$

From $\operatorname{supp}\phi\subseteq B$ and (3.16), it follows that the sum of the first two terms is equal to

$$\zeta\iint_{B\times B}|\phi(x)|^2|u(x)-u(y)|^2 J(x,dy)\,d\mu(x).$$

We therefore obtain

$$\begin{aligned}\iint_{\Omega\times\Omega}&|u(x)|^2|\phi(x)-\phi(y)|^2\,J(x,dy)\,d\mu(x)\\ &\le\zeta\iint_{B\times B}|\phi(x)|^2|u(x)-u(y)|^2 J(x,dy)\,d\mu(x)+\frac{2C}{W(r)}\int_\Omega|u(x)|^2\,d\mu(x).\end{aligned}$$

This proves the validity of $(\mathrm{AB})_W$ for the product jump measure $J(x,dy)$. □

4 Regularity of Dirichlet forms

This section is devoted to the proof of the main Theorem 2.9. The key ingredients are a self improvement property of $(\mathrm{AB})_W$ and a partition of unity based on cutoff functions.

4.1 A comparison of $(\mathrm{AB})_W$ and $(\mathrm{AB}')_W$

It is obvious that the condition $(\mathrm{AB}')_W$ is stronger than $(\mathrm{AB})_W$. Next, we show the following equivalent versions of $(\mathrm{AB})_W$ and $(\mathrm{AB}')_W$.

Lemma 4.1. *Let $\tau \in (0,\infty)$. Under $(\mathrm{TJ})_W$, the formulae (2.5) and (2.10) can be improved into the following:*

$$\iint_{\substack{\Omega\times\Omega \\ d(x,y)<\tau r}} |u(x)|^2|\phi(x)-\phi(y)|^2\, J(x,dy)\, d\mu(x)$$
$$\leq \zeta \iint_{\substack{B\times B \\ d(x,y)<\tau r}} |\phi(x)|^2|u(x)-u(y)|^2 J(x,dy)d\mu(x) + C_\tau \sup_{z\in\Omega}\frac{1}{W(z,r)}\int_\Omega |u(x)|^2\, d\mu(x) \tag{4.1}$$

and

$$\iint_{\substack{\Omega\times\Omega \\ d(x,y)<\tau r}} |u(x)|^2|\phi(x)-\phi(y)|^2\, J(x,dy)\, d\mu(x)$$
$$\leq \zeta \iint_{\substack{(B\setminus B_0)\times(B\setminus B_0) \\ d(x,y)<\tau r}} |\phi(x)|^2|u(x)-u(y)|^2 J(x,dy)d\mu(x) + C_\tau \sup_{z\in\Omega}\frac{1}{W(z,r)}\int_\Omega |u(x)|^2\, d\mu(x), \tag{4.2}$$

respectively, where C_τ is a positive constant depending on τ but independent of the main parameters involved.

Proof. We will prove here only (4.1) as the proof of (4.2) goes in a similar way.

Note that any $\phi \in \mathrm{cutoff}(B_0, B)$ satisfies $0 \leq \phi \leq 1$. From $(\mathrm{TJ})_W$ and (2.2), we derive that

$$\begin{aligned}\iint_{\substack{\Omega\times\Omega \\ d(x,y)\geq\tau r}} |u(x)|^2|\phi(x)-\phi(y)|^2\, J(x,dy)\, d\mu(x) &\leq \int_\Omega |u(x)|^2\left(\int_{d(x,y)\geq\tau r} J(x,dy)\right) d\mu(x)\\ &\leq C\int_\Omega \frac{|u(x)|^2}{W(x,\tau r)}\, d\mu(x)\\ &\leq CC_W \max\{\tau^{-\beta_1},\, \tau^{-\beta_2}\} \sup_{z\in\Omega}\frac{1}{W(z,r)}\int_\Omega |u(x)|^2\, d\mu(x),\end{aligned}$$

where C_W and β_1, β_2 are the constants in (2.2). Similarly, we have

$$\begin{aligned}\iint_{\substack{B\times B \\ d(x,y)\geq\tau r}} |\phi(x)|^2|u(x)-u(y)|^2\, J(x,dy)\, d\mu(x) &\leq 2\iint_{\substack{B\times B \\ d(x,y)\geq\tau r}} (|u(x)|^2+|u(y)|^2)\, J(x,dy)\, d\mu(x)\\ &= 4\iint_{\substack{B\times B \\ d(x,y)\geq\tau r}} |u(x)|^2\, J(x,dy)\, d\mu(x)\\ &\leq 4C\int_B \frac{|u(x)|^2}{W(x,\tau r)}\, d\mu(x)\\ &\leq 4CC_W \max\{\tau^{-\beta_1},\, \tau^{-\beta_2}\} \sup_{z\in B}\frac{1}{W(z,r)}\int_B |u(x)|^2\, d\mu(x).\end{aligned}$$

Thus, we obtain (4.1), which finishes the proof. ◻

Now we show that both (AB)$_W$ and (AB$'$)$_W$ imply that there are sufficiently many cutoff functions in the domain $\mathcal{F}$ of the bilinear form $\mathcal{E}$.

Lemma 4.2. *Under the assumption of* (TJ)$_W$, *if either* (AB)$_W$ *or* (AB$'$)$_W$ *holds, then the function* ϕ *in* (2.5) *or* (2.10) *can be taken to satisfy* $\phi \in \mathcal{F}$.

Proof. We prove this only for the condition (AB$'$)$_W$ as the same argument works for (AB)$_W$.

Suppose that the condition (AB$'$)$_W$ holds and $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$. For $x_0 \in \mathcal{X}$ and $0 < R < R + r < R' < \infty$, set the three concentric balls B_0, B, Ω as in (2.4). Our main aim is to find a function $\phi \in \text{cutoff}(B_0, B) \cap \mathcal{F}$ satisfying (2.10), but with different constants ζ and C.

To this end, for any given $\epsilon \in (0, \infty)$, via applying (AB$'$)$_W$ to the function

$$u_\epsilon := |u| + \epsilon,$$

we find a function $\phi^\epsilon \in \text{cutoff}(B_0, B)$ such that

$$\begin{aligned}&\iint_{\Omega\times\Omega} |u_\epsilon(x)|^2 |\phi^\epsilon(x) - \phi^\epsilon(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\quad \le \zeta \iint_{(B\setminus B_0)\times(B\setminus B_0)} |\phi^\epsilon(x)|^2 |u_\epsilon(x) - u_\epsilon(y)|^2 \, J(x, dy) \, d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z, r)} \int_\Omega |u_\epsilon|^2 \, d\mu.\end{aligned}$$

Further, from

$$|u(x)|^2 + \varepsilon^2 \le |u_\varepsilon(x)|^2 \le 2(|u(x)|^2 + \varepsilon^2)$$

and

$$|u_\epsilon(x) - u_\epsilon(y)| = ||u(x)| - |u(y)|| \le |u(x) - u(y)|,$$

it follows that

$$\begin{aligned}&\iint_{\Omega\times\Omega} |u_\epsilon(x)|^2 |\phi^\epsilon(x) - \phi^\epsilon(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\quad \le \zeta \iint_{(B\setminus B_0)\times(B\setminus B_0)} |\phi^\epsilon(x)|^2 |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x) + \sup_{z\in\Omega} \frac{2C}{W(z, r)} \left(\int_\Omega |u|^2 \, d\mu + \varepsilon^2 \mu(\Omega) \right).\end{aligned} \tag{4.3}$$

Choose a number $\epsilon_0 \in (0, \infty)$ that satisfies

$$\varepsilon_0^2 = \frac{1}{\mu(\Omega)} \int_\Omega |u(x)|^2 \, d\mu(x).$$

For this special ϵ_0, we derive from (4.3) that

$$\begin{aligned}&\iint_{\Omega\times\Omega} |u(x)|^2 |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\quad \le \zeta \iint_{(B\setminus B_0)\times(B\setminus B_0)} |\phi^{\epsilon_0}(x)|^2 |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x) + \sup_{z\in\Omega} \frac{4C}{W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x)\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&\varepsilon_0^2 \iint_{\Omega\times\Omega} |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\quad \le \zeta \iint_{(B\setminus B_0)\times(B\setminus B_0)} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x) + \sup_{z\in\Omega} \frac{4C}{W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x).\end{aligned} \tag{4.5}$$

Observe that (4.4) exactly shows that the cutoff function ϕ^{ϵ_0} satisfies (2.10).

It remains to show that $\phi^{\epsilon_0} \in \mathcal{F}$. Clearly, $\phi^{\epsilon_0} \in L^2(\mathcal{X})$. Note that $\phi^{\epsilon_0} \in \text{cutoff}(B_0, B)$, which implies that $\operatorname{supp} \phi^{\epsilon_0} \subseteq B$ and, hence, $\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y) \neq 0$ only if $x \in B$ or $y \in B$. By this and symmetry, we then obtain

$$\begin{aligned}\mathcal{E}(\phi^{\epsilon_0}, \phi^{\epsilon_0}) &= \int_{\mathcal{X}} \int_{\mathcal{X}} |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\leq 2 \int_B \int_{\mathcal{X}} |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x). \end{aligned} \tag{4.6}$$

If $x \in B$ and $d(y, x) < R' - (R + r)$, then

$$d(y, x_0) \leq d(y, x) + d(x, x_0) < R'$$

and, hence, $y \in \Omega = B(x_0, R')$. Combining (4.5) and $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$ gives

$$\int_B \int_\Omega |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) \leq \varepsilon_0^{-2} \left(\zeta \mathcal{E}(u, u) + \sup_{z \in \Omega} \frac{4C}{W(z, r)} \|u\|^2_{L^2(\mathcal{X})} \right) < \infty, \tag{4.7}$$

where the last finiteness property follows from (2.2) and the fact that

$$\sup_{z \in \Omega} \frac{1}{W(z, r)} \leq \frac{C_W}{W(x_0, R')} \left(\frac{R'}{r} \right)^{\beta_2} < \infty.$$

Meanwhile, by $0 \leq \phi^{\epsilon_0} \leq 1$ and (TJ)$_{\mathrm{W}}$, we obtain

$$\begin{aligned}\int_B \int_{d(y,x) \geq R' - (R+r)} |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) &\leq \int_B \int_{d(y,x) \geq R' - (R+r)} J(x, dy) \, d\mu(x) \\ &\leq \int_B \frac{C}{W(x, R' - (R + r))} \, d\mu(x) \\ &\leq \sup_{x \in B} \frac{C\mu(B)}{W(x, R' - (R + r))} < \infty. \end{aligned} \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6) leads to

$$\mathcal{E}(\phi^{\epsilon_0}, \phi^{\epsilon_0}) \leq 2 \left(\int_B \int_\Omega + \int_B \int_{d(y,x) \geq R' - (R+r)} \right) |\phi^{\epsilon_0}(x) - \phi^{\epsilon_0}(y)|^2 \, J(x, dy) \, d\mu(x) < \infty.$$

Letting $\phi = \phi^{\epsilon_0}$, we conclude that $\phi \in \text{cutoff}(B_0, B) \cap \mathcal{F}$ satisfies (2.10). This ends the proof. □

4.2 Self improvement property of (AB)$_{\mathrm{W}}$

In the next lemma we prove that the coefficient $\zeta > 0$ on the right hand side of (2.5) can be made arbitrarily small. This self-improvement property of condition (AB)$_{\mathrm{W}}$ was first observed by Andres and Barlow in [2] for local Dirichlet forms. For jump-type Dirichlet form this property was proved in [22] but in a more restricted setting.

Lemma 4.3. *Under the assumptions of* (TJ)$_{\mathrm{W}}$ *and* (AB)$_{\mathrm{W}}$, *the following holds: for any* $\lambda > 0$, *there exists* $C(\lambda) > 0$ *such that, for any*

$$u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$$

and for any three concentric balls

$$\begin{cases} B_0 = B(x_0, R); \\ B = B(x_0, R + r); \\ \Omega = B(x_0, R'), \end{cases}$$

with $x_0 \in \mathcal{X}$ *and* $0 < R < R + r < R' < \infty$, *there exists* $\phi^{(\lambda)} \in \mathrm{cutoff}(B_0, B) \cap \mathcal{F}$ *such that*

$$\iint_{\Omega\times\Omega} |u(x)|^2 |\phi^{(\lambda)}(x) - \phi^{(\lambda)}(y)|^2 J(x,\, dy)\, d\mu(x) \\ \leq \lambda \iint_{B\times B} |\phi^{(\lambda)}(x)|^2 |u(x) - u(y)|^2 J(x,\, dy)\, d\mu(x) + \sup_{z\in\Omega} \frac{C(\lambda)}{W(z, r)} \int_\Omega |u(x)|^2\, d\mu(x). \tag{4.9}$$

Proof. The argument here is similar to the proof of [22, Lemma 7.1], but does not rely on the regularity of the Dirichlet form.

Let B_0, B, Ω, x_0, r, R, R' and u be as in the statement of this lemma. If $u = 0$ holds μ-a.e. on Ω, then $\phi \equiv 0$ satisfies (4.9). So, in the rest of the proof, we may assume that $\|u\|_{L^2(\Omega)} > 0$. Let

$$\varepsilon := \left(\frac{1}{\mu(\Omega)} \int_\Omega u^2\, d\mu \right)^{\frac{1}{2}} \quad \text{and} \quad u_\varepsilon := |u| + \varepsilon.$$

Fix a number $q \in (1, \infty)$. For any integer $k \geq 0$, define the sequences

$$\begin{cases} r_k := (1 - q^{-k}) r; \\ s_k := r_k - r_{k-1} = (q-1) q^{-k} r; \\ B_k := B(x_0, R + r_k); \\ U_k := B_{k+1} \setminus B_k. \end{cases}$$

Note that $r_k \uparrow r$ and, hence, $B_k \uparrow B$ as $k \to +\infty$. Moreover, $\cup_{k=1}^\infty U_k = B \setminus B_1$. Applying $(\mathrm{AB})_{\mathrm{W}}$ to the function u_ε and to each triple (B_k, B_{k+1}, Ω), we obtain that there exists $\phi_k \in \mathrm{cutoff}(B_k, B_{k+1})$ such that

$$\iint_{\Omega\times\Omega} u_\varepsilon^2(x)(\phi_k(x) - \phi_k(y))^2 J(x,\, dy)\, d\mu(x) \\ \leq \zeta \iint_{B_{k+1}\times B_{k+1}} \phi_k(x)^2 (u_\varepsilon(x) - u_\varepsilon(y))^2 J(x,\, dy)\, d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z, s_{k+1})} \int_\Omega u_\varepsilon^2\, d\mu,$$

where ζ, C are universal constants in the definition of condition $(\mathrm{AB})_{\mathrm{W}}$. Since

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq |u(x) - u(y)| \quad \text{for all } x, y \in \mathcal{X},$$

and

$$\int_\Omega u_\varepsilon^2\, d\mu \leq 2 \int_\Omega u^2\, d\mu + 2\varepsilon^2 \mu(\Omega) = 4 \int_\Omega u^2\, d\mu,$$

we have

$$\iint_{\Omega\times\Omega} u_\varepsilon^2(x)(\phi_k(x) - \phi_k(y))^2 J(x,\, dy)\, d\mu(x) \\ \leq \zeta \iint_{B_{k+1}\times B_{k+1}} \phi_k(x)^2 (u(x) - u(y))^2 J(x,\, dy)\, d\mu(x) + \sup_{z\in\Omega} \frac{4C}{W(z, s_{k+1})} \int_\Omega u^2\, d\mu. \tag{4.10}$$

Set $b_0 := 1$. Let β_2 be the exponent determined in (2.2). For any $k \geq 1$, define the sequences

$$\begin{cases} b_k := q^{-\beta_2 k}; \\ a_k := b_{k-1} - b_k = (q^{\beta_2} - 1)q^{-\beta_2 k}, \end{cases}$$

and the function

$$\phi := \phi^{(q)} := \sum_{k=1}^{\infty} a_k \phi_k.$$

Since each ϕ_k is continuous and

$$\sum_{k=1}^{\infty} a_k = b_0 = 1,$$

we have that $\sum_{k=1}^{N} a_k \phi_k \to \phi$ uniformly as $N \to \infty$, and then $\phi \in C(\mathcal{X})$. In particular,

$$\begin{cases} 0 \leq \phi \leq 1 & \text{on } \mathcal{X}; \\ \phi = 1 & \text{on } B_0; \\ \phi = 0 & \text{on } B^{\complement}. \end{cases}$$

That is, $\phi \in \text{cutoff}(B_0, B)$. It remains to prove the following:

(i) $\phi \in \mathcal{F}$;

(ii) there is some $q \in (1, \infty)$ such that ϕ satisfies (4.9).

To verify (i), for any $k \geq 1$, since $u_\varepsilon \geq \varepsilon$ and $0 \leq \phi_k \leq 1$, we derive from (4.10) that

$$\iint_{\Omega \times \Omega} (\phi_k(x) - \phi_k(y))^2 J(x, dy)\, d\mu(x) \leq \zeta \varepsilon^{-2} \mathcal{E}(u, u) + \sup_{z \in \Omega} \frac{4C\varepsilon^{-2}}{W(z, s_{k+1})} \int_\Omega u^2\, d\mu.$$

From (2.2), it follows that

$$\frac{W(z, r)}{W(z, s_{k+1})} \leq C_W \left(\frac{r}{s_{k+1}} \right)^{\beta_2} = C_W \left(\frac{q^{k+1}}{q-1} \right)^{\beta_2}, \tag{4.11}$$

where C_W and β_2 are the same constants as in (2.2). This, together with the fact that $\operatorname{supp} \phi_k \subseteq B$ and the symmetry: $J(x, dy)\, d\mu(x) = J(y, dx)\, d\mu(y)$, yields that

$$\begin{aligned} \mathcal{E}(\phi_k, \phi_k) &= \iint_{\mathcal{X} \times \mathcal{X}} (\phi_k(x) - \phi_k(y))^2 J(x, dy)\, d\mu(x) \\ &= \iint_{\Omega \times \Omega} (\phi_k(x) - \phi_k(y))^2 J(x, dy)\, d\mu(x) \\ &\quad + \iint_{B \times \Omega^{\complement}} \phi_k^2(x) J(x, dy)\, d\mu(x) + \iint_{\Omega^{\complement} \times B} \phi_k^2(y) J(x, dy)\, d\mu(x) \\ &\leq \zeta \varepsilon^{-2} \mathcal{E}(u, u) + \frac{4CC_W \varepsilon^{-2} q^{(k+1)\beta_2}}{(q-1)^{\beta_2}} \sup_{z \in \Omega} \frac{1}{W(z, r)} \int_\Omega u^2\, d\mu + 2 \iint_{B \times \Omega^{\complement}} \phi_k^2(x) J(x, dy)\, d\mu(x). \end{aligned}$$

Note that $0 \leq \phi_k \leq 1$ and $d(B, \Omega^{\complement}) \geq R' - (R + r) > 0$. The latter, combined with (2.2) and the condition $(\mathrm{TJ})_W$, gives

$$\iint_{B \times \Omega^{\complement}} \phi_k^2(x) J(x, dy)\, d\mu(x) \leq C \int_B \frac{1}{W(x, R' - (R + r))}\, d\mu(x) \leq \frac{CC_W \mu(B)}{W(x_0, R')} \left(\frac{R'}{R' - (R + r)} \right)^{\beta_2} < \infty,$$

where C is the constant given in $(\mathrm{TJ})_W$. Consequently, there exists $C := C(u, q, x_0, R', R, r, \varepsilon) > 0$ (it depends on all variables in question except for k), such that

$$\mathcal{E}(\phi_k, \phi_k) \le C(u, q, x_0, R', R, r, \varepsilon).$$

Moreover, since $\phi = \sum_{k=1}^{\infty} a_k \phi_k$ and $\sum_{k=1}^{\infty} a_k = 1$, we obtain that

$$\sqrt{\mathcal{E}(\phi, \phi)} \le \sum_{k=1}^{\infty} a_k \sqrt{\mathcal{E}(\phi_k, \phi_k)} \le \sqrt{C(u, q, x_0, R', R, r, \varepsilon)} < \infty,$$

which finishes the proof of (i).

For the proof of (ii), applying (4.10), (4.11) and $(\mathrm{TJ})_W$, one can follow the second part of the proof of [22, Lemma 7.1] (see the arguments in [22, p. 138-142], and see also the arguments in [21, pp. 456-459]) to obtain that

$$\begin{aligned}
&\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy) \, d\mu(x) \\
&\quad \le 6\zeta \frac{q^{2\beta_2}(q^{\beta_2} - 1)}{q^{\beta_2} + 1} \iint_{B\times B} \phi^2(x)(u(x) - u(y))^2 \, J(x, dy) \, d\mu(x) \\
&\quad\quad + CC_W \frac{q^{2\beta_2}}{(q-1)^{\beta_2}} \sup_{z\in\Omega} \frac{1}{W(z, r)} \int_\Omega u^2 \, d\mu.
\end{aligned}$$

Here we omit the details. Moreover, for any $\lambda > 0$, we can choose $q_0 \in (1, \infty)$ sufficiently close to 1 such that

$$6\zeta \frac{q_0^{2\beta_2}(q_0^{\beta_2} - 1)}{q_0^{\beta_2} + 1} = \lambda.$$

In this case, the function $\phi = \phi^{(q_0)}$ satisfies (4.9). This completes the proof. □

Remark 4.4. Under $(\mathrm{TJ})_W$, the inequality (2.5) is equivalent to the following inequality:

$$\begin{aligned}
&\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy) \, d\mu(x) \\
&\quad \le \zeta \iint_{\Omega\times\Omega} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x)
\end{aligned} \tag{4.12}$$

(although (2.5) is a priori stronger than (4.12)). That is, the integration area $B \times B$ in the first integral on the right hand side of (2.5) can be replaced by $\Omega \times \Omega$ and there is no $\phi(x)^2$ in the first integral on the right hand side of (4.12).

Now, we verify that (4.12) implies (2.5). With B_0, B and Ω as in (2.4), we observe that for $B' = B(x_0, R + \frac{r}{2})$ there exists some $\phi \in \mathrm{cutoff}(B_0, B')$ that satisfies (4.12) with Ω therein replaced by $B = B(x_0, R + r)$. This can be applied to estimate the first double-integral in the right hand side of the forthcoming formula:

$$\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy) \, d\mu(x) = \left(\iint_{B\times B} + \iint_{(\Omega\setminus B)\times B} + \iint_{B\times(\Omega\setminus B)} \right) \cdots,$$

in which the latter two double-integrals can be treated by using $(\mathrm{TJ})_W$ and $\mathrm{supp}\, \phi \subseteq B'$. Thus, we obtain that (4.12) is equivalent to

$$\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy) \, d\mu(x)$$

$$\leq \zeta \iint_{B\times B} |u(x)-u(y)|^2 J(x,dy)\,d\mu(x) + \sup_{z\in\Omega} \frac{C}{W(z,r)} \int_\Omega |u(x)|^2\,d\mu(x).$$

Then, one can repeat the proof of Lemma 4.3 by using the above inequality instead of (2.5) to obtain (4.9). Finally, observe that (4.9) implies (2.5).

Remark 4.5. Lemma 4.3 will be used to approximate elements in $\mathcal{F}$ via bounded compactly supported functions of $\mathcal{F}$ (see Lemma 4.10 below).

4.3 Partition of unity under $(\mathrm{AB})_{\mathrm{W}}$ or $(\mathrm{AB}')_{\mathrm{W}}$

This subsection is devoted to construction of a partition of unity on $\mathcal{X}$ by using the cutoff functions from the condition $(\mathrm{AB})_{\mathrm{W}}$ or $(\mathrm{AB}')_{\mathrm{W}}$. Due to similarity, we state and prove the result only for the condition $(\mathrm{AB})_{\mathrm{W}}$.

Proposition 4.6. *Suppose that* (VD), $(\mathrm{TJ})_{\mathrm{W}}$ *and* $(\mathrm{AB})_{\mathrm{W}}$ *hold. Then, for any* $\varepsilon \in (0,\infty)$*, there exists a family of maximal* ε*-separated points* $\{x_i\}_{i\in I}$ *in* $\mathcal{X}$*, where* I *is a countable index set, such that*

(i) $\{B_i = B(x_i,\varepsilon)\}_{i\in I}$ *is a covering of* $\mathcal{X}$*;*

(ii) $\{B(x_i,\varepsilon/2)\}_{i\in I}$ *is pairwise disjoint;*

(iii) *for any given number* $\kappa \in [1,\infty)$*, there exists a positive constant* $N = N(\kappa)$*, independent of* $\{x_i\}_{i\in I}$ *and* ε*, such that*

$$\sum_{i\in I} \mathbf{1}_{B(x_i,\,15\kappa\varepsilon)} \leq N. \tag{4.13}$$

Consequently, for any $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$*, there is a family of functions* $\{\phi_i\}_{i\in I}$ *in* $\mathcal{F}$ *such that the following hold:*

(a) *for each* $i\in I$*,* $\phi_i \in C_c(\mathcal{X})$*,* $0\leq \phi_i \leq 1$ *and* $\operatorname{supp}\phi_i \subseteq 2B_i$*;*

(b) *for any* $x\in\mathcal{X}$*, it holds* $\sum_{i\in I}\phi_i(x) \equiv 1$*;*

(c) *for each* $i\in I$*, setting* $\Lambda_i := \{j\in I:\ 3B_j\cap 3B_i \neq \emptyset\}$ *and* $u_{B_i} := \fint_{B_i} u\,d\mu = \frac{1}{\mu(B_i)}\int_{B_i} u\,d\mu$*, then*

$$\iint_{(3B_i)\times(3B_i)} |u(x)-u_{B_i}|^2 |\phi_i(x)-\phi_i(y)|^2 J(x,\,dy)\,d\mu(x)$$
$$\leq C\left(\sum_{j\in\Lambda_i}\iint_{(3B_j)\times(3B_j)} |u(x)-u(y)|^2 J(x,\,dy)\,d\mu(x) + \frac{1}{W(x_i,\varepsilon)}\int_{13B_i} |u-u_{B_i}|^2\,d\mu\right), \tag{4.14}$$

where the constant $C\in(0,\infty)$ *is independent of* u*,* ε *and* $\{x_i\}_{i\in I}$*. Moreover, the cardinality of* Λ_i *is bounded by* N*.*

Proof. Let $\{x_i\}_{i\in I}$ be a family of maximal ε-separated points in $\mathcal{X}$, that is

$$\inf_{i\neq j} d(x_i,x_j) \geq \varepsilon \quad \text{and} \quad \inf_{i\in I} d(x,x_i) < \varepsilon.$$

Then, it is easy to validate (i) and (ii). The countable property of I is guaranteed by (ii) and (VD).

To show (iii), we fix a number $\kappa \in [1,\infty)$. For any $x \in \mathcal{X}$, suppose that $x \in 15\kappa B_i$ for some $i \in I$. So, the proof of (iii) falls into validating that the cardinality of $\{j \in I: (15\kappa B_j) \cap (15\kappa B_i) \neq \emptyset\}$ is bounded by a number N. To this end, if $(15\kappa B_j) \cap (15\kappa B_i) \neq \emptyset$, then $d(x_j, x_i) < 30\kappa\varepsilon$, so that

$$\frac{1}{2} B_j = B(x_j, \varepsilon/2) \subseteq B(x_i, 31\kappa\varepsilon) = 31\kappa B_i$$

and, hence, by (2.7),

$$\frac{\mu(31\kappa B_i)}{\mu(\frac{1}{2}B_j)} \leq C'_D \left(\frac{d(x_i, x_j) + 31\kappa\varepsilon}{\varepsilon/2} \right)^{\alpha_+} \leq C'_D (122\kappa)^{\alpha_+},$$

where C'_D and α_+ are as defined in (2.7). This, along with the mutually disjointness of $\{\frac{1}{2}B_j\}_{j\in I}$ from (ii), yields that

$$\begin{aligned} \sharp(\{j \in I: (15\kappa B_j) \cap (15\kappa B_i) \neq \emptyset\}) &\leq C'_D (122\kappa)^{\alpha_+} \sum_{\{j\in I: (15\kappa B_j)\cap(15\kappa B_i)\neq\emptyset\}} \frac{\mu(\frac{1}{2}B_j)}{\mu(31\kappa B_i)} \\ &\leq C'_D (122\kappa)^{\alpha_+}. \end{aligned} \tag{4.15}$$

Thus, setting $N = N(\kappa) = C'_D(122\kappa)^{\alpha_+}$ gives (iii).

From (4.15), it follows directly that $\sharp\Lambda_i \leq N$. The proof of (a)-(b)-(c) is split into the following three steps.

Step 1: construction of $\{\phi_i\}_{i\in I}$. Suppose that $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$. For any $i \in I$, if we apply $(\mathrm{AB})_W$ to the function

$$\tilde{u}_i := |u - u_{B_i}| + \sum_{j\in\Lambda_i} |u_{B_j} - u_{B_i}|,$$

with $x_0 = x_i$, $R = \varepsilon$, $r = \varepsilon/4$ and $R' = 3\varepsilon$ therein, then there exists $\psi_i \in \mathrm{cutoff}(B_i, \frac{5}{4}B_i) \cap \mathcal{F}$ such that

$$\begin{aligned} &\iint_{(3B_i)\times(3B_i)} |\tilde{u}_i(x)|^2 |\psi_i(x) - \psi_i(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\quad \leq \zeta \iint_{(3B_i)\times(3B_i)} |\psi_i(x)|^2 |\tilde{u}_i(x) - \tilde{u}_i(y)|^2 \, J(x, dy)\, d\mu(x) + \frac{C}{W(x_i, \varepsilon)} \int_{3B_i} |\tilde{u}_i(x)|^2 \, d\mu(x), \end{aligned}$$

where the constants ζ and C are nonnegative and independent of u, $\{x_i\}_{i\in I}$ and ε. Note that here ψ_i can be required to belong to $\mathcal{F}$ because of Lemma 4.2. Moreover, here we also used the fact that for any $\tau \geq 1$ the assumption (2.2) implies that

$$\sup_{z\in\tau B_i} \frac{1}{W(z, \varepsilon)} \leq C_W \frac{\tau^{\beta_2}}{W(x_i, \varepsilon)}.$$

Next, from the fact that

$$|\tilde{u}_i(x) - \tilde{u}_i(y)| = ||u(x) - u_{B_i}| - |u(y) - u_{B_i}|| \leq |u(x) - u(y)|,$$

we then derive

$$\iint_{(3B_i)\times(3B_i)} (|u(x) - u_{B_i}| + \sum_{j\in\Lambda_i} |u_{B_j} - u_{B_i}|)^2 |\psi_i(x) - \psi_i(y)|^2 \, J(x, dy)\, d\mu(x)$$

$$\leq \zeta \iint_{(3B_i)\times(3B_i)} |\psi_i(x)|^2 |u(x)-u(y)|^2 \, J(x,\,dy)\,d\mu(x)$$
$$+ \frac{2C}{W(x_i,\varepsilon)} \int_{3B_i} |u(x)-u_{B_i}|^2 d\mu(x) + \frac{2C\mu(3B_i)}{W(x_i,\varepsilon)} \left(\sum_{j\in\Lambda_i} |u_{B_j}-u_{B_i}|\right)^2. \tag{4.16}$$

Before going further, we deal with the third term in the right hand side of (4.16). For any $i \in I$ and $j \in \Lambda_i$, we have $d(x_j, x_i) < 6\varepsilon$, so that

$$B_j = B(x_j,\varepsilon) \subseteq B(x_i, 7\varepsilon) = 7B_i.$$

From this, the Hölder inequality and (VD), we deduce

$$\begin{aligned}
|u_{B_j}-u_{B_i}|^2 &= \left(\fint_{B_j} (u-u_{B_i})\,d\mu\right)^2 \\
&\leq \fint_{B_j} |u-u_{B_i}|^2\,d\mu \\
&\leq \frac{1}{\mu(B_j)} \int_{7B_i} |u-u_{B_i}|^2\,d\mu \\
&\leq C_D^3 \fint_{7B_i} |u-u_{B_i}|^2\,d\mu,
\end{aligned} \tag{4.17}$$

which, along with (4.15) and $\sharp\Lambda_i \leq N$, further yields

$$\left(\sum_{j\in\Lambda_i} |u_{B_j}-u_{B_i}|\right)^2 \leq C_D^3 \left(\sum_{j\in\Lambda_i} \left(\fint_{7B_i} |u-u_{B_i}|^2\,d\mu\right)^{\frac{1}{2}}\right)^2 \leq C_D^3 N^2 \fint_{7B_i} |u-u_{B_i}|^2\,d\mu.$$

Thus, the formula (4.16) amounts to saying that

$$\iint_{(3B_i)\times(3B_i)} \Big(|u(x)-u_{B_i}| + \sum_{j\in\Lambda_i} |u_{B_j}-u_{B_i}|\Big)^2 |\psi_i(x)-\psi_i(y)|^2 \, J(x,\,dy)\,d\mu(x)$$
$$\lesssim \iint_{(3B_i)\times(3B_i)} |\psi_i(x)|^2 |u(x)-u(y)|^2 \, J(x,\,dy)\,d\mu(x) + \frac{1}{W(x_i,\varepsilon)} \int_{7B_i} |u(x)-u_{B_i}|^2\,d\mu(x). \tag{4.18}$$

Now, for any $x \in \mathcal{X}$, define

$$\Psi(x) := \sum_{i\in I} \psi_i(x).$$

By (4.13) and $0 \leq \psi_i \leq \mathbf{1}_{2B_i}$, we know immediately that $1 \leq \Psi(x) \leq N$. Moreover, let

$$\phi_i(x) := \frac{\psi_i(x)}{\Psi(x)}.$$

It is obvious that $\{\phi_i\}_{i\in I}$ satisfy (a) and (b). So, we are left to validate $\phi_i \in \mathcal{F}$ and (4.14) in (c).

Step 2: verification that $\phi_i \in \mathcal{F}$. Clearly, each $\phi_i \in L^2(\mathcal{X})$. Fix $i \in I$ and let

$$\Lambda_i' := I \setminus \Lambda_i = \left\{j \in I :\ B(x_j, 3\varepsilon) \cap B(x_i, 3\varepsilon) = \emptyset\right\}.$$

For any $x, y \in \mathcal{X}$, write

$$\begin{aligned}
|\phi_i(x) - \phi_i(y)| &= \left|\frac{\psi_i(x)\Psi(y) - \psi_i(y)\Psi(x)}{\Psi(x)\Psi(y)}\right| \\
&\le |\psi_i(x)\Psi(y) - \psi_i(y)\Psi(x)| \\
&\le |\psi_i(x) - \psi_i(y)|\,\Psi(y) + |\Psi(x) - \Psi(y)|\,\psi_i(y) \\
&\le N\,|\psi_i(x) - \psi_i(y)| + \sum_{j\in I} \left|\psi_j(x) - \psi_j(y)\right| \psi_i(x). \qquad (4.19)
\end{aligned}$$

If $j \in \Lambda_i'$, then by $\operatorname{supp}\phi_i = \operatorname{supp}\psi_i \subseteq B(x_i, \frac{5}{4}\varepsilon)$, we obtain $\psi_j(x)\psi_i(x) = 0$ and $d(x_j, x_i) \ge 3\varepsilon$, where the latter implies that $\psi_j(y)\psi_i(x) \ne 0$ only if $d(y, x) > \epsilon$. From this and (4.19), it follows that

$$|\phi_i(x) - \phi_i(y)| \le N\,|\psi_i(x) - \psi_i(y)| + \sum_{j\in\Lambda_i} \left|\psi_j(x) - \psi_j(y)\right| \mathbf{1}_{2B_i}(x) + \sum_{\{j\in\Lambda_i' :\, d(x,y)>\varepsilon\}} \psi_j(y)\psi_i(x). \qquad (4.20)$$

Combining (4.20) with the Minkowski inequality yields

$$\begin{aligned}
\sqrt{\mathcal{E}(\phi_i, \phi_i)} &= \left(\int_{\mathcal{X}}\int_{\mathcal{X}} |\phi_i(x) - \phi_i(y)|^2\, J(x, dy)\, d\mu(x)\right)^{\frac{1}{2}} \\
&\le N\left(\int_{\mathcal{X}}\int_{\mathcal{X}} |\psi_i(x) - \psi_i(y)|^2\, J(x, dy)\, d\mu(x)\right)^{\frac{1}{2}} \\
&\quad + \sum_{j\in\Lambda_i}\left(\int_{\mathcal{X}}\int_{\mathcal{X}} |\psi_j(x) - \psi_j(y)|^2 \mathbf{1}_{2B_i}(x)\, J(x, dy)\, d\mu(x)\right)^{\frac{1}{2}} \\
&\quad + \left(\iint_{d(x,y)>\varepsilon} \left|\sum_{j\in\Lambda_i'} \psi_j(y)\psi_i(x)\right|^2 J(x, dy)\, d\mu(x)\right)^{\frac{1}{2}}. \qquad (4.21)
\end{aligned}$$

Note that the first two terms in the right hand side of (4.21) is bounded by

$$N\sqrt{\mathcal{E}(\psi_i, \psi_i)} + \sum_{j\in\Lambda_i} \sqrt{\mathcal{E}(\psi_j, \psi_j)},$$

which is a finite number (may depend on i) by terms of $\sharp\Lambda_i \le N$ and each $\psi_j \in \mathcal{F}$. For the third term in the right hand side of (4.21), we have by (iii), $(\mathrm{TJ})_{\mathrm{W}}$ and (2.2) that

$$\begin{aligned}
\iint_{d(x,y)>\varepsilon} \left|\sum_{j\in\Lambda_i'} \psi_j(y)\psi_i(x)\right|^2 J(x, dy)\, d\mu(x) &\le \int_{2B_i}\int_{d(x,y)>\varepsilon} \left(\sum_{j\in\Lambda_i'} \mathbf{1}_{2B_j}(y)\right)^2 J(x, dy)\, d\mu(x) \\
&\le N^2 \int_{2B_i}\int_{d(x,y)>\varepsilon} J(x, dy)\, d\mu(x) \\
&\lesssim N^2 \int_{2B_i} \frac{1}{W(x, \varepsilon)}\, d\mu(x) \\
&\lesssim N^2 \frac{\mu(2B_i)}{W(x_i, \varepsilon)} < \infty.
\end{aligned}$$

So, it follows from (4.21) that $\mathcal{E}(\phi_i, \phi_i) < \infty$ and, hence, $\phi_i \in \mathcal{F}$.

Step 3: proof of (4.14). Since we have assumed $(\mathrm{TJ})_W$, the argument in the proof of Lemma 4.1 shows that it suffices to verify that

$$\iint_{\substack{(3B_i)\times(3B_i)\\ d(x,y)<\varepsilon/4}} |u(x)-u_{B_i}|^2|\phi_i(x)-\phi_i(y)|^2\,J(x,\,dy)\,d\mu(x)$$

can be controlled by the right hand side of (4.14). Based on (4.20), we write

$$\begin{aligned}
&\iint_{\substack{(3B_i)\times(3B_i)\\ d(x,y)<\varepsilon/4}} |u(x)-u_{B_i}|^2|\phi_i(x)-\phi_i(y)|^2\,J(x,\,dy)\,d\mu(x)\\
&\le N\iint_{\substack{(3B_i)\times(3B_i)\\ d(x,y)<\varepsilon/4}} |u(x)-u_{B_i}|^2|\psi_i(x)-\psi_i(y)|^2\,J(x,\,dy)\,d\mu(x)\\
&\quad+\sum_{j\in\Lambda_i}\iint_{\substack{(3B_i)\times(3B_i)\\ d(x,y)<\varepsilon/4}} |u(x)-u_{B_i}|^2|\psi_j(x)-\psi_j(y)|^2\,J(x,\,dy)\,d\mu(x)\\
&=: \mathrm{I}+\mathrm{II}.
\end{aligned}$$

Invoking (4.18) and the fact $0\le\psi_i\le\mathbf{1}_{2B_i}(x)$, we get

$$\mathrm{I}\lesssim N\left(\iint_{(3B_i)\times(3B_i)} |u(x)-u(y)|^2\,J(x,\,dy)\,d\mu(x)+\frac{1}{W(x_i,\varepsilon)}\int_{7B_i}|u(x)-u_{B_i}|^2\,d\mu(x)\right),$$

as desired.

To estimate II, for any $j\in\Lambda_i$, we have $i\in\Lambda_j$, $3B_i\subseteq 9B_j$ and

$$|u(x)-u_{B_i}|\le|u(x)-u_{B_j}|+|u_{B_j}-u_{B_i}|\le|u(x)-u_{B_j}|+\sum_{k\in\Lambda_j}|u_{B_k}-u_{B_j}|=\tilde u_j.$$

Note that, if $d(x,y)<\varepsilon/4$ and $\psi_j(x)-\psi_j(y)\ne 0$, then by $\operatorname{supp}\psi_j\subseteq\frac{5}{4}B_j$, we derive that both $x,y\in 2B_j$. Thus, applying (4.18) we obtain that, for any $j\in\Lambda_i$,

$$\begin{aligned}
&\iint_{\substack{(3B_i)\times(3B_i)\\ d(x,y)<\varepsilon/4}} |u(x)-u_{B_i}|^2|\psi_j(x)-\psi_j(y)|^2\,J(x,\,dy)\,d\mu(x)\\
&\le\iint_{\substack{(2B_j)\times(2B_j)\\ d(x,y)<\varepsilon/4}} |\tilde u_j(x)|^2|\psi_j(x)-\psi_j(y)|^2\,J(x,\,dy)\,d\mu(x)\\
&\lesssim\iint_{(3B_j)\times(3B_j)} |\psi_j(x)|^2|u(x)-u(y)|^2\,J(x,\,dy)\,d\mu(x)+\frac{1}{W(x_j,\varepsilon)}\int_{7B_j}|u(x)-u_{B_j}|^2\,d\mu(x). \qquad (4.22)
\end{aligned}$$

If $j\in\Lambda_i$, then $d(x_j,x_i)<6\varepsilon$ and, hence, $7B_j\subseteq 13B_i$. Combining this with (4.17) and (VD) yields

$$\begin{aligned}
\int_{7B_j}|u(x)-u_{B_j}|^2\,d\mu(x)&\le 2\int_{7B_j}\left(|u(x)-u_{B_i}|^2+|u_{B_i}-u_{B_j}|^2\right)d\mu(x)\\
&\lesssim\int_{13B_i}|u(x)-u_{B_i}|^2\,d\mu(x).
\end{aligned}$$

Meanwhile, by the aforementioned fact $7B_j\subseteq 13B_i$ and (2.2), we see that

$$W(x_i,\varepsilon)\simeq W(x_j,\varepsilon).$$

Using this, let us sum up in $j \in \Lambda_i$ the both sides of (4.22). Applying also (4.15) and the fact $0 \le \psi_j \le \mathbf{1}_{2B_j}$, we obtain

$$\mathrm{II} \lesssim \sum_{j\in\Lambda_i} \iint_{(3B_j)\times(3B_j)} |u(x)-u(y)|^2\, J(x,\,dy)\, d\mu(x) + \frac{1}{W(x_i,\varepsilon)} \int_{13B_i} |u(x)-u_{B_i}|^2\, d\mu(x).$$

A combination of the above upper estimates of I and II implies (4.14). □

Remark 4.7. The proof of Proposition 4.6 gives also a corresponding partition of unity under the condition $(\mathrm{AB}')_{\mathrm{W}}$. Indeed, if $(\mathrm{AB}')_{\mathrm{W}}$ holds, then items (i)-(ii)-(iii) and (a)-(b) of Proposition 4.6 remain true, but with (c) therein replaced by the following (c′):

(c′) for each $i \in I$,

$$\begin{aligned}\iint_{3B_i\times 3B_i} &|u(x)-u_{B_i}|^2|\phi_i(x)-\phi_i(y)|^2\, J(x,\,dy)\, d\mu(x)\\ &\le C\left(\sum_{j\in\Lambda_i}\iint_{S_j\times S_j} |u(x)-u(y)|^2\, J(x,\,dy)\, d\mu(x) + \frac{1}{W(x_i,\varepsilon)}\int_{13B_i}|u(x)-u_{B_i}|^2\, d\mu(x)\right),\end{aligned}$$

where $S_j := B(x_j, 5\varepsilon/4) \setminus B(x_j,\varepsilon)$.

The proof follows from the same arguments as that of Proposition 4.6, but now we use $(\mathrm{AB}')_{\mathrm{W}}$ instead of $(\mathrm{AB})_{\mathrm{W}}$.

4.4 Proof of main Theorem 2.9

Assuming (VD) + $(\mathrm{TJ})_{\mathrm{W}}$ + $(\mathrm{PI})_{\mathrm{W}}$ + $(\mathrm{AB})_{\mathrm{W}}$, we need to prove that the bilinear form $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form, which amounts to the following three statements: $\mathcal{F}$ is dense in $L^2(\mathcal{X})$, $\mathcal{F}\cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$ as well as in $\mathcal{F}$. These statements are proved in Propositions 4.8, 4.9, 4.11 below, thus constituting the proof of Theorem 2.9.

Proposition 4.8. *If* $\mathrm{cutoff}(B_1,B_2)\cap\mathcal{F}\neq\emptyset$ *for all open balls* B_1,B_2 *with* $B_1\Subset B_2$*, then* $\mathcal{F}\cap C_c(\mathcal{X})$ *is dense in* $C_c(\mathcal{X})$*. In particular, if the condition* $(\mathrm{AB})_{\mathrm{W}}$ *holds, then* $\mathcal{F}\cap C_c(\mathcal{X})$ *is dense in* $C_c(\mathcal{X})$*.*

Proof. Since $(\mathrm{AB})_{\mathrm{W}}$ holds, we have by Lemma 4.2 that $\mathrm{cutoff}(B_1,B_2)\cap\mathcal{F}$ is non-empty for all balls B_1,B_2 with $B_1\Subset B_2$. Thus, it suffices to prove the density of $\mathcal{F}\cap C_c(\mathcal{X})$ in $C_c(\mathcal{X})$ under the assumption of $\mathrm{cutoff}(B_1,B_2)\cap\mathcal{F}\neq\emptyset$ for all balls B_1,B_2 with $B_1\Subset B_2$.

Fix $x_o\in\mathcal{X}$. Let $\mathscr{A}$ be the closure of $\mathcal{F}\cap C_c(\mathcal{X})$ in $C_0(\mathcal{X})$ under the norm $\|\cdot\|_{L^\infty(\mathcal{X})}$, where

$$C_0(\mathcal{X}) = \left\{ f\in C(\mathcal{X}) : \lim_{d(x,x_o)\to\infty} f(x) = 0\right\}.$$

According to the Stone-Weierstrass theorem (see [18, p. 147, Corollary 8.3]), we have

$$\mathscr{A} = C_0(\mathcal{X}),$$

provided that $\mathscr{A}$ satisfies the following properties:

(a) $\mathscr{A}$ is a subalgebra of $C_0(\mathcal{X})$, that is, $fg\in\mathscr{A}$ if $f,g\in\mathscr{A}$;

(b) for any $x_1,x_2\in\mathcal{X}$ with $x_1\neq x_2$, there exists $\phi\in\mathscr{A}$ such that $\phi(x_1)\neq\phi(x_2)$;

(c) for any $x\in\mathcal{X}$, there exists $\phi\in\mathscr{A}$ such that $\phi(x)\neq 0$.

Once we have proved (a)-(b)-(c), we then derive from $C_c(\mathcal{X}) \subseteq C_0(\mathcal{X})$ that $\mathcal{F} \cap C_c(\mathcal{X})$ is dense in $C_c(\mathcal{X})$.

To see (a), it is obvious that for any $f, g \in \mathcal{F} \cap C_c(\mathcal{X})$ we have $fg \in C_c(\mathcal{X})$ and, moreover, the Minkowski inequality implies

$$\begin{aligned}\mathcal{E}(fg, fg) &= \int_{\mathcal{X}} \int_{\mathcal{X}} |f(x)g(x) - f(y)g(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\leq 2 \int_{\mathcal{X}} \int_{\mathcal{X}} |f(x)[g(x) - g(y)]|^2 \, J(x, dy)\, d\mu(x) \\ &\quad + 2 \int_{\mathcal{X}} \int_{\mathcal{X}} |[f(x) - f(y)]g(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\leq 2\|f\|^2_{L^\infty(\mathcal{X})}\mathcal{E}(g, g) + 2\|g\|^2_{L^\infty(\mathcal{X})}\mathcal{E}(f, f),\end{aligned}$$

that is, $fg \in \mathcal{F}$. In general, if $f, g \in \mathscr{A}$, then there exist sequences $\{f_j\}_{j\in\mathbb{N}}$ and $\{g_j\}_{j\in\mathbb{N}}$ in $\mathcal{F} \cap C_c(\mathcal{X})$ that respectively converges to f and g under the L^∞-norm. Note that every $f_j g_j \in \mathcal{F} \cap C_c(\mathcal{X})$ and $\{f_j g_j\}_{j\in\mathbb{N}}$ converges to fg under $\|\cdot\|_{L^\infty(\mathcal{X})}$. This proves that $fg \in \mathscr{A}$, so that (a) holds.

To show (b), for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$, we consider two disjoint balls $B_0 := B(x_1, r)$ and $B := B(x_1, 2r)$ with $r < d(x_1, x_2)/3$ and obtain by the assumption $\operatorname{cutoff}(B_0, B) \cap \mathcal{F} \neq \emptyset$, that there exists a cutoff function

$$\phi \in \operatorname{cutoff}(B_0, B) \cap \mathcal{F}.$$

Obviously, $\phi \in \mathscr{A}$. Moreover, ϕ separates the points x_1 and x_2, since

$$\phi(x_1) = 1 \neq 0 = \phi(x_2).$$

This last formula also indicates that ϕ does not vanish on any point $x_1 \in \mathcal{X}$, which implies (c). Thus, we complete the proof of Proposition 4.8.

□

Proposition 4.9. *If* $\operatorname{cutoff}(B_1, B_2) \cap \mathcal{F} \neq \emptyset$ *for all open balls* B_1, B_2 *with* $B_1 \Subset B_2$, *then* $\mathcal{F} \cap C_c(\mathcal{X})$ *is dense in* $L^2(\mathcal{X})$. *In particular, if* (AB)$_W$ *holds, then* $\mathcal{F} \cap C_c(\mathcal{X})$ *is dense in* $L^2(\mathcal{X})$.

Proof. As in the proof of Proposition 4.8, it suffices to show the density of $\mathcal{F} \cap C_c(\mathcal{X})$ in $L^2(\mathcal{X})$ under the assumption that $\operatorname{cutoff}(B_1, B_2) \cap \mathcal{F} \neq \emptyset$ for all open balls B_1, B_2 with $B_1 \Subset B_2$.

Suppose that $f \in L^2(\mathcal{X})$. It is known that $C_c(\mathcal{X})$ is dense in $L^2(\mathcal{X})$. Thus, for any $\varepsilon \in (0, \infty)$, there exists a function $g \in C_c(\mathcal{X})$ such that

$$\|f - g\|_{L^2(\mathcal{X})} < \epsilon.$$

Since $g \in C_c(\mathcal{X})$, we may as well assume that $\operatorname{supp} g \subseteq B$ for some ball B of $\mathcal{X}$. By Proposition 4.8, there exists a function $h \in \mathcal{F} \cap C_c(\mathcal{X})$ such that

$$\|h - g\|_{L^\infty(\mathcal{X})} < (2\mu(2B))^{-\frac{1}{2}}\varepsilon.$$

This last estimate implies

$$\sup_{x\in B} |h(x) - g(x)| < (2\mu(2B))^{-\frac{1}{2}}\varepsilon \quad \text{and} \quad \sup_{x\notin B} |h(x)| < (2\mu(2B))^{-\frac{1}{2}}\varepsilon.$$

Further, by the assumption, there exists a function

$$\phi \in \operatorname{cutoff}(B, 2B) \cap \mathcal{F}.$$

Consider the function $\psi = h\phi$. Clearly, $\psi \in C_c(\mathcal{X})$. From the statement (a) in the proof of Proposition 4.8, it follows that $\psi \in \mathcal{F}$. Moreover, using $\phi \in \text{cutoff}(B, 2B)$ and $\operatorname{supp} g \subseteq B$, we derive

$$\begin{aligned}\int_{\mathcal{X}} |\psi - g|^2 \, d\mu &= \int_B |h - g|^2 \, d\mu + \int_{(2B)\setminus B} |h\phi|^2 \, d\mu \\ &\le \mu(B) \sup_{x\in B} |h(x) - g(x)|^2 + \mu((2B)\setminus B) \sup_{x\notin B} |h(x)|^2 \le \varepsilon^2,\end{aligned}$$

thereby leading to

$$\|\psi - f\|_{L^2(\mathcal{X})} \le \|\psi - g\|_{L^2(\mathcal{X})} + \|g - f\|_{L^2(\mathcal{X})} < 2\varepsilon.$$

This proves the density of $\mathcal{F} \cap C_c(\mathcal{X})$ in $L^2(\mathcal{X})$. □

Before the proof of density of $C_c(\mathcal{X}) \cap \mathcal{F}$ in $\mathcal{F}$, we will apply $(\text{AB})_W$ and the self-improvement property of $(\text{AB})_W$ of Lemma 4.3 to establish the following lemma.

Lemma 4.10. *Suppose that the conditions* $(\text{AB})_W$ *and* $(\text{TJ})_W$ *hold. Then, for any* $u \in \mathcal{F}$*, there exists a sequence of bounded and compactly supported functions* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ *such that*

$$\lim_{n\to\infty} \left(\|u - u_n\|^2_{L^2(\mathcal{X})} + \mathcal{E}(u - u_n, u - u_n) \right) = 0.$$

Proof. For any $u \in \mathcal{F}$ and $n \in \mathbb{N}$, define $u_n := u \wedge n$, which are bounded functions and satisfy

$$|u_n(x) - u_n(y)| \le |u(x) - u(y)| \quad \text{for all } x, y \in \mathcal{X}.$$

Thus, $u_n \in \mathcal{F}$ and, moreover, the dominated convergence theorem for integrals shows that

$$\lim_{n\to\infty} \mathcal{E}(u - u_n, u - u_n) = 0.$$

Thus, we may as well assume that $u \in \mathcal{F} \cap L^\infty(\mathcal{X})$ and we need to approximate u by a sequence of bounded and compactly supported functions $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$.

To this end, we fix a reference point $x_o \in \mathcal{X}$. For any $k \in \mathbb{N}$, set $B_k := B(x_o, 2^k)$. In what follows, we will often use the following fact that

$$\sup_{z\in B_{k+3}} \frac{1}{W(z, 2^k)} \simeq \frac{1}{W(x_o, 2^k)},$$

which follows from (2.2). By Lemma 4.3, given any $\lambda > 0$, there exists a constant $C(\lambda) > 0$ such that for any integer $k \ge 3$, there exists $\psi_k^{(\lambda)} \in \text{cutoff}(B_k, B_{k+1}) \cap \mathcal{F}$ such that

$$\begin{aligned}&\iint_{B_{k+3}\times B_{k+3}} |u(x)|^2 |\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|^2 \, J(x, dy) \, d\mu(x) \\ &\quad \le \lambda \iint_{B_{k+3}\times B_{k+3}} |\psi_k^{(\lambda)}(x)|^2 |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x) + \frac{C(\lambda)}{W(x_o, 2^k)} \int_{B_{k+3}} |u(x)|^2 \, d\mu(x). \qquad (4.23)\end{aligned}$$

Note that both λ and $C(\lambda)$ are independent of x_o, k and u. Define

$$u_k^\lambda := u\psi_k^{(\lambda)}.$$

Clearly, each u_k^λ is in $L^\infty(\mathcal{X})$ and has bounded support. So, it suffices to show that

$$\lim_{\lambda\to 0} \lim_{k\to\infty} \left(\|u - u_k^\lambda\|_{L^2(\mathcal{X})} + \mathcal{E}(u - u_k^\lambda, u - u_k^\lambda) \right) = 0.$$

Obviously, for any $\lambda > 0$, applying $\psi_k^{(\lambda)} \equiv 1$ on $B(x_o, 2^k)$ yields that

$$\lim_{k\to\infty} \|u - u_k^\lambda\|_{L^2(\mathcal{X})} = 0.$$

Moreover, we write

$$\mathcal{E}(u - u_k^\lambda, u - u_k^\lambda) = \int_{\mathcal{X}} \int_{\mathcal{X}} \left| u(x)(1 - \psi_k^{(\lambda)}(x)) - u(y)(1 - \psi_k^{(\lambda)}(y)) \right|^2 J(x, dy)\, d\mu(x).$$

For any $x, y \in \mathcal{X}$, observe that

$$|u(x)(1 - \psi_k^{(\lambda)}(x)) - u(y)(1 - \psi_k^{(\lambda)}(y))| \le |u(x) - u(y)||1 - \psi_k^{(\lambda)}(y)| + |u(x)||\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|.$$

By $u \in \mathcal{F}$ and the fact that $\psi_k^{(\lambda)} \equiv 1$ on $B(x_o, 2^k)$, we apply the dominated convergence theorem to deduce that

$$\begin{aligned} &\lim_{k\to\infty} \int_{\mathcal{X}} \int_{\mathcal{X}} |u(x) - u(y)|^2 |1 - \psi_k^{(\lambda)}(x)|^2 J(x, dy)\, d\mu(x) \\ &= \int_{\mathcal{X}} \int_{\mathcal{X}} \left(\lim_{k\to\infty} |u(x) - u(y)|^2 |1 - \psi_k^{(\lambda)}(x)|^2 \right) J(x, dy)\, d\mu(x) = 0. \end{aligned}$$

So, the proof of

$$\lim_{\lambda\to 0} \lim_{k\to\infty} \mathcal{E}(u - u_k^\lambda, u - u_k^\lambda) = 0$$

is reduced to proving that

$$\lim_{\lambda\to 0} \lim_{k\to\infty} \int_{\mathcal{X}} \int_{\mathcal{X}} |u(x)|^2 |\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|^2 J(x, dy)\, d\mu(x) = 0. \tag{4.24}$$

In (4.24), we may restrict the integral domain to those $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that

$$\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y) \neq 0.$$

Hence, the fact $\operatorname{supp} \psi_k^{(\lambda)} \subseteq B(x_o, 2^{k+1})$ implies that either $d(x, x_o) < 2^{k+1}$ or $d(y, x_o) < 2^{k+1}$.

In the case $d(x, y) < 2^{k-2}$ we have $x, y \in B(x_o, 2^{k+3})$, which, alongside with (4.23) yields

$$\begin{aligned} &\iint_{\substack{\mathcal{X}\times\mathcal{X} \\ d(x,y)<2^{k-2}}} |u(x)|^2 |\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|^2 J(x, dy)\, d\mu(x) \\ &= \iint_{\substack{B_{k+3}\times B_{k+3} \\ d(x,y)<2^{k-2}}} |u(x)|^2 |\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|^2 J(x, dy)\, d\mu(x) \\ &\le \lambda \iint_{B_{k+3}\times B_{k+3}} |\psi_k^{(\lambda)}(x)|^2 |u(x) - u(y)|^2 J(x, dy)\, d\mu(x) + \frac{C(\lambda)}{W(x_o, 2^k)} \int_{B_{k+3}} |u(x)|^2\, d\mu(x) \\ &\le \lambda \mathcal{E}(u, u) + \frac{C(\lambda)}{W(x_o, 2^k)} \|u\|^2_{L^2(\mathcal{X})}. \end{aligned} \tag{4.25}$$

Consider now the case $d(x, y) \ge 2^{k-2}$. If $d(x, x_o) < 2^{k+1}$, then using $u \in L^2(\mathcal{X})$, $0 \le \psi_k^{(\lambda)} \le 1$, $(\mathrm{TJ})_W$ and (2.2), we obtain that

$$\begin{aligned} &\iint_{\substack{d(x,y)\ge 2^{k-2} \\ d(x,x_o)<2^{k+1}}} |u(x)|^2 |\psi_k^{(\lambda)}(x) - \psi_k^{(\lambda)}(y)|^2 J(x, dy)\, d\mu(x) \\ &\le \int_{d(x,x_o)<2^{k+1}} \left(\int_{d(x,y)\ge 2^{k-2}} J(x, dy) \right) |u(x)|^2\, d\mu(x) \end{aligned}$$

$$\begin{aligned}&\leq \int_{d(x,x_o)<2^{k+1}} \frac{C}{W(x,2^{k-2})}|u(x)|^2\,d\mu(x)\\ &\leq \frac{C}{W(x_o,2^k)}\|u\|^2_{L^2(\mathcal{X})}\end{aligned} \tag{4.26}$$

holds for some constant $C>0$ independent of k and u.

Still in the case $d(x,y)\geq 2^{k-2}$, let now $d(x,x_o)\geq 2^{k+1}$. Then

$$|\psi_k^{(\lambda)}(x)-\psi_k^{(\lambda)}(y)|=\psi_k^{(\lambda)}(y),$$

which is nonzero only if $d(y,x_o)<2^{k+1}$. From this and

$$|u(x)|^2\leq 2|u(x)-u(y)|^2+2|u(y)|^2$$

it follows that

$$\begin{aligned}&\iint_{\substack{d(x,y)\geq 2^{k-2}\\ d(x,x_o)\geq 2^{k+1}}} |u(x)|^2|\psi_k^{(\lambda)}(x)-\psi_k^{(\lambda)}(y)|^2\,J(x,\,dy)\,d\mu(x)\\ &\quad\leq \iint_{\substack{d(x,x_o)<2^{k+1}\\ d(x,y)\geq 2^{k-2}}} |u(x)|^2\,J(x,\,dy)\,d\mu(x)\\ &\quad\leq 2\iint_{d(x,y)\geq 2^{k-2}} |u(x)-u(y)|^2\,J(x,\,dy)\,d\mu(x)+2\iint_{\substack{d(y,x_o)<2^{k+1}\\ d(x,y)\geq 2^{k-2}}} |u(y)|^2\,J(x,\,dy)\,d\mu(x).\end{aligned} \tag{4.27}$$

For the second term, applying the symmetry property (J2), $(\mathrm{TJ})_{\mathrm{W}}$ and (2.2), we proceed the arguments in (4.26) and obtain

$$\begin{aligned}\iint_{\substack{d(y,x_o)<2^{k+1}\\ d(x,y)\geq 2^{k-2}}} |u(y)|^2\,J(x,\,dy)\,d\mu(x) &= \iint_{\substack{d(x,x_o)<2^{k+1}\\ d(y,x)\geq 2^{k-2}}} |u(x)|^2\,J(x,\,dy)\,d\mu(x)\\ &\leq \frac{C}{W(x_o,2^k)}\|u\|^2_{L^2(\mathcal{X})}.\end{aligned} \tag{4.28}$$

Combining (4.25)-(4.26)-(4.27)-(4.28) yields

$$\begin{aligned}&\iint_{\mathcal{X}\times\mathcal{X}} |u(x)|^2|\psi_k^{(\lambda)}(x)-\psi_k^{(\lambda)}(y)|^2\,J(x,\,dy)\,d\mu(x)\\ &\quad\leq \lambda\mathcal{E}(u,u)+\frac{3C+C(\lambda)}{W(x_o,2^k)}\|u\|^2_{L^2(\mathcal{X})}+2\iint_{d(x,y)\geq 2^{k-2}} |u(x)-u(y)|^2\,J(x,\,dy)\,d\mu(x).\end{aligned}$$

In both sides of the last formula, letting $k\to\infty$ and using $\lim_{r\to\infty} W(x_o,r)=\infty$ yields

$$\lim_{k\to\infty}\iint_{\mathcal{X}\times\mathcal{X}} |u(x)|^2|\psi_k^{(\lambda)}(x)-\psi_k^{(\lambda)}(y)|^2\,J(x,\,dy)\,d\mu(x)\leq \lambda\mathcal{E}(u,u).$$

Due to the arbitrariness of λ, we find that (4.24) holds. This concludes the proof of Lemma 4.10. □

Proposition 4.11. *If* (VD), $(\mathrm{AB})_{\mathrm{W}}$, $(\mathrm{TJ})_{\mathrm{W}}$ *and* $(\mathrm{PI})_{\mathrm{W}}$ *hold, then* $\mathcal{F}\cap C_c(\mathcal{X})$ *is dense in* $\mathcal{F}$.

Proof. Given any $u\in\mathcal{F}$, we need to find an approximation sequence in $\mathcal{F}\cap C_c(\mathcal{X})$. In view of Lemma 4.10, we may as well assume further that $u\in L^\infty(\mathcal{X})$ has bounded support.

Let $\varepsilon \in (0,1)$. According to Proposition 4.6, there exists a family of functions $\{\phi_i\}_{i\in I}$ which form a partition of unity. In the succedent argument, we adopt all the notation used in (i)-(ii)-(iii) and (a)-(b) of Proposition 4.6. Define

$$u_\varepsilon := \sum_{i\in I} u_{B_i}\phi_i.$$

By the fact that u has bounded support, $\phi_i \in C_c(\mathcal{X})$, supp $\phi_i \subseteq 2B_i$ and $\{\frac{1}{2}B_i\}_{i\in I}$ are mutually disjoint, we find that the summation in defining u_ε is valid for a finite number of i, which further induces that

$$u_\varepsilon \in \mathcal{F} \cap C_c(\mathcal{X}).$$

We will prove that u_ε is the desired approximation sequence.

For any $x \in \mathcal{X}$, we apply $\sum_{i\in I}\phi_i \equiv 1$ to write

$$u(x) - u_\varepsilon(x) = \sum_{i\in I} (u(x) - u_{B_i})\,\phi_i(x) =: \sum_{i\in I} u_i(x)\phi_i(x). \tag{4.29}$$

Denote by κ the constant determined in the condition (PI)$_{\mathrm{W}}$. Let $N = N(\kappa)$ be the number that is determined in Proposition 4.6(iii), which is independent of ε. Then, there is a partition $\{J_j\}_{j=1}^N$ of the index set I, such that for any $j \in \{1, 2, \dots, N\}$, the family of balls $\{15\kappa B_l\}_{l\in J_j}$ are pairwise disjoint.

We first prove that $\|u - u_\varepsilon\|_{L^2(\mathcal{X})} \to 0$ as $\varepsilon \to 0$. Indeed, by the construction of u_ε, we can choose a bounded set $K \subseteq \mathcal{X}$ such that supp $u \subseteq K$ and supp $u_\varepsilon \subseteq K$ for all $\varepsilon \in (0,1)$. Moreover, since u_ε is bounded by $\sup_{x\in\mathcal{X}}|u(x)|$ uniformly in $\varepsilon \in (0,1)$, we have

$$|u - u_\varepsilon| \le 2\sup_{x\in\mathcal{X}}|u(x)|\cdot \mathbf{1}_K < \infty.$$

Hence, to prove that $\lim_{\varepsilon\to 0}\|u - u_\varepsilon\|_{L^2(\mathcal{X})} = 0$, by the dominated convergence theorem, it suffices to prove that

$$u - u_\varepsilon \overset{\mu\text{-a.e.}}{\longrightarrow} 0 \quad \text{as } \varepsilon \to 0. \tag{4.30}$$

Indeed, for any $x \in \mathcal{X}$, by (4.29) we write

$$|u(x) - u_\varepsilon(x)| \le \sum_{j=1}^N \sum_{i\in J_j} |u(x) - u_{B_i}|\phi_i(x).$$

Note that N is independent of ε, supp $\phi_i \subseteq 2B_i$ and $\{2B_i\}_{i\in J_j}$ are mutually disjoint. So, there exists one and only one $i_x \in J_i$ such that $2B_{i_x} \ni x$, which implies that

$$B_{i_x} \subseteq B(x, 3\varepsilon) \subseteq 4B_{i_x}.$$

By (VD), we have $\mu(B(x,3\varepsilon)) \le C_D^2\mu(B_{i_x})$. Using these and the above inequality, we obtain for any $x \in \mathcal{X}$,

$$|u(x) - u_\varepsilon(x)| \le \sum_{j=1}^N \fint_{B_{i_x}} |u(x) - u(y)|\,d\mu(y) \le \frac{NC_D^2}{\mu(B(x,3\varepsilon))}\int_{B(x,3\varepsilon)} |u(x) - u(y)|\,d\mu(y).$$

Therefore, by Lebesgue's differential theorem (see [29, Eq. (2.8), p. 12]), we obtain (4.30). This proves that u can be approximated by the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^2(\mathcal{X})$.

We still need to validate that

$$\lim_{\varepsilon\to 0}\mathcal{E}(u-u_\varepsilon,u-u_\varepsilon)=0.$$

Indeed, since $\operatorname{supp}\phi_l\subseteq 2B_l$, it follows that $u_l(x)\phi_l(x)-u_l(y)\phi_l(y)\neq 0$ implies that either $x\in 2B_l$ or $y\in 2B_l$, whence

$$|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|\le |u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|(\mathbf{1}_{2B_l}(x)+\mathbf{1}_{2B_l}(y)).$$

This, together with the symmetry $J(x,dy)\,d\mu(x)=J(y,dx)\,d\mu(y)$ and the Minkowski inequality, further induces

$$\begin{aligned}
\mathcal{E}(u-u_\varepsilon,u-u_\varepsilon)^{\frac12}&=\left(\int_{\mathcal X}\int_{\mathcal X}\left|\sum_{j=1}^N\sum_{l\in J_j}(u_l(x)\phi_l(x)-f_l(y)\phi_l(y))\right|^2J(x,dy)\,d\mu(x)\right)^{\frac12}\\
&\le\sum_{j=1}^N\left(\int_{\mathcal X}\int_{\mathcal X}\left[\sum_{l\in J_j}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|\mathbf{1}_{2B_l}(x)\right]^2J(x,dy)\,d\mu(x)\right)^{\frac12}\\
&\quad+\sum_{j=1}^N\left(\int_{\mathcal X}\int_{\mathcal X}\left[\sum_{l\in J_j}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|\mathbf{1}_{2B_l}(y)\right]^2J(x,dy)\,d\mu(x)\right)^{\frac12}\\
&=2\sum_{j=1}^N\left(\int_{\mathcal X}\int_{\mathcal X}\left[\sum_{l\in J_j}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|\mathbf{1}_{2B_l}(x)\right]^2J(x,dy)\,d\mu(x)\right)^{\frac12}.
\end{aligned}$$

Further, invoking the fact that $\{2B_l\}_{l\in J_j}$ are mutually disjoint, we obtain

$$\left[\sum_{l\in J_j}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|\mathbf{1}_{2B_l}(x)\right]^2=\sum_{l\in J_j}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|^2\mathbf{1}_{2B_l}(x),$$

thereby deriving

$$\mathcal{E}(u-u_\varepsilon,u-u_\varepsilon)^{\frac12}\le 2\sum_{j=1}^N\left(\sum_{l\in J_j}\int_{\mathcal X}\int_{\mathcal X}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|^2\mathbf{1}_{2B_l}(x)\,J(x,dy)\,d\mu(x)\right)^{\frac12}. \tag{4.31}$$

To continue, for any $j\in\{1,2,\dots,N\}$, we write

$$\begin{aligned}
\mathrm{I}_j&:=\sum_{l\in J_j}\int_{\mathcal X}\int_{\mathcal X}|u_l(x)\phi_l(x)-u_l(y)\phi_l(y)|^2\mathbf{1}_{2B_l}(x)\,J(x,dy)\,d\mu(x)\\
&\le 2\sum_{l\in J_j}\int_{2B_l}\int_{\mathcal X}|u_l(x)|^2|\phi_l(x)-\phi_l(y)|^2\,J(x,dy)\,d\mu(x)\\
&\quad+2\sum_{l\in J_j}\int_{2B_l}\int_{\mathcal X}|u_l(x)-u_l(y)|^2|\phi_l(y)|^2\,J(x,dy)\,d\mu(x)=:2\mathrm{Y}_j+2\mathrm{Z}_j.
\end{aligned}\tag{4.32}$$

Noting that $u_l(x)-u_l(y)=u(x)-u(y)$ and $0\le\phi_l\le\mathbf{1}_{2B_l}$, we then apply the mutually disjointedness of $\{2B_l\}_{l\in J_j}$ to derive

$$\mathrm{Z}_j\le\sum_{l\in J_j}\int_{2B_l}\int_{2B_l}|u_l(x)-u_l(y)|^2\,J(x,dy)\,d\mu(x)$$

$$\leq \int_{\mathcal{X}} \int_{B(x,4\varepsilon)} |u_l(x) - u_l(y)|^2 \, J(x,dy)\, d\mu(x)$$
$$= \int_{\mathcal{X}} \int_{B(x,4\varepsilon)} |u(x) - u(y)|^2 \, J(x,dy)\, d\mu(x)$$

which tends to zero as $\varepsilon \to 0$.

In order to estimate Υ_j, for any $l \in J_j$, we first write

$$\Upsilon_{j,l} := \int_{2B_l} \int_{\mathcal{X}} |u_l(x)|^2 |\phi_l(x) - \phi_l(y)|^2 \, J(x,dy)\, d\mu(x)$$
$$= \left(\int_{2B_l} \int_{(3B_l)^{\complement}} + \int_{2B_l} \int_{3B_l} \right) \dots \, J(x,dy)\, d\mu(x) =: \Upsilon_{j,l}^{(1)} + \Upsilon_{j,l}^{(2)}.$$

Recall that in (4.29) we have defined $u_l = u - u_{B_l}$. If $x \in 2B_l$ and $y \notin 3B_l$, then $d(x,y) > \varepsilon$. So, applying $(\mathrm{TJ})_{\mathrm{W}}$ and (2.2) gives

$$\Upsilon_{j,l}^{(1)} \leq \int_{2B_l} |u_l(x)|^2 \left(\int_{d(x,y)\geq\varepsilon} J(x,dy) \right) d\mu(x)$$
$$\leq C \int_{2B_l} \frac{|u_l(x)|^2}{W(x,\varepsilon)}\, d\mu(x)$$
$$\leq \frac{C}{W(x_l,\varepsilon)} \int_{2B_l} |u(x) - u_{B_l}|^2 \, d\mu(x).$$

Meanwhile, combining (4.14) with the fact that $u_l = u - u_{B_l}$ yields

$$\Upsilon_{j,l}^{(2)} \leq \iint_{(3B_l)\times(3B_l)} |u_l(x)|^2 |\phi_l(x) - \phi_l(y)|^2 \, J(x,dy)\, d\mu(x)$$
$$\lesssim \sum_{j\in\Lambda_l} \iint_{(3B_j)\times(3B_j)} |u(x) - u(y)|^2 \, J(x,dy)\, d\mu(x) + \frac{1}{W(x_l,\varepsilon)} \int_{13B_l} |u(x) - u_{B_l}|^2 \, d\mu(x).$$

By the definitions of Λ_l in Proposition 4.6, we know that $3B_j \subseteq 9B_l$ whenever $j \in \Lambda_l$. From this and the mutually disjointness of $\{3B_j\}_{j\in\Lambda_l}$, implies

$$\sum_{j\in\Lambda_l} \iint_{(3B_j)\times(3B_j)} |u(x) - u(y)|^2 \, J(x,dy)\, d\mu(x)$$
$$\leq \sum_{j\in\Lambda_l} \int_{3B_j} \int_{d(x,y)<6\varepsilon} |u(x) - u(y)|^2 \, J(x,dy)\, d\mu(x)$$
$$\leq \int_{9B_l} \int_{d(x,y)<6\varepsilon} |u(x) - u(y)|^2 \, J(x,dy)\, d\mu(x). \tag{4.33}$$

Next, for any $l \in I$ and $x \in 13B_l$, we have by (VD) (see also (2.7)) and the Hölder inequality that

$$|u(x) - u_{B_l}| \leq |u(x) - u_{13B_l}| + |u_{B_l} - u_{13B_l}|$$
$$\leq |u(x) - u_{13B_l}| + \fint_{B_l} |u(z) - u_{13B_l}|\, d\mu(z)$$
$$\leq |u(x) - u_{13B_l}| + C \fint_{13B_l} |u(z) - u_{13B_l}|\, d\mu(z)$$

$$\le |u(x) - u_{13B_l}| + C\left(\fint_{13B_l} |u(x) - u_{13B_l}|^2 \, d\mu(x)\right)^{\frac{1}{2}},$$

which, together with (PI)$_{\mathrm{W}}$ and the Minkowski inequality, further yields

$$\left(\int_{13B_l} |u(x) - u_{B_l}|^2 \, d\mu(x)\right)^{\frac{1}{2}} \lesssim \left(\int_{13B_l} |u(x) - u_{13B_l}|^2 \, d\mu(x)\right)^{\frac{1}{2}}$$
$$\lesssim \left(W(x_l, 13\varepsilon) \iint_{(13\kappa B_l)\times(13\kappa B_l)} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x)\right)^{\frac{1}{2}}.$$

Consequently, by (2.2), we have

$$\frac{1}{W(x_l, \varepsilon)} \int_{13B_l} |u(x) - u_{B_l}|^2 \, d\mu(x) \lesssim \frac{W(x_l, 13\varepsilon)}{W(x_l, \varepsilon)} \iint_{(13\kappa B_l)\times(13\kappa B_l)} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x)$$
$$\lesssim \int_{13\kappa B_l} \int_{d(x,y)<26\kappa\varepsilon} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x). \tag{4.34}$$

Inserting (4.33) and (4.34) into the estimates of $\mathrm{Y}^{(1)}_{j,l}$ and $\mathrm{Y}^{(2)}_{j,l}$, we then conclude that

$$\mathrm{Y}_{j,l} \lesssim \int_{13\kappa B_l} \int_{d(x,y)<26\kappa\varepsilon} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x),$$

which, combined with the mutually disjointness of the family of balls $\{15\kappa B_l\}_{l\in J_j}$, leads to

$$\mathrm{Y}_j = \sum_{l\in J_j} \mathrm{Y}_{j,l} \lesssim \sum_{l\in J_j} \int_{13\kappa B_l} \int_{d(x,y)<26\kappa\varepsilon} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x)$$
$$\lesssim \int_{\mathcal{X}} \int_{d(x,y)<26\kappa\varepsilon} |u(x) - u(y)|^2 \, J(x, dy) \, d\mu(x).$$

Since $u \in \mathcal{F}$, it is obvious that this last double-integral tends to zero as $\varepsilon \to 0$.

From the estimates of Y_j and Z_j, we derive from (4.32) and (4.31) that

$$\mathcal{E}(u - u_\varepsilon, u - u_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Summarizing all, we complete the proof of Proposition 4.11. □

Remark 4.12. In the following comments, we assume that (VD), (TJ)$_{\mathrm{W}}$ and (PI)$_{\mathrm{W}}$ are satisfied.

(i) Since (AB′)$_{\mathrm{W}}$ is stronger than (AB)$_{\mathrm{W}}$, it follows directly that all the conclusions in this subsection are true if we replace the hypothesis (AB)$_{\mathrm{W}}$ by (AB′)$_{\mathrm{W}}$. Consequently, under (VD), (TJ)$_{\mathrm{W}}$ and (PI)$_{\mathrm{W}}$,

$$(\mathrm{AB}')_{\mathrm{W}} \Rightarrow \text{“}(\mathcal{E}, \mathcal{F}) \text{ is a regular Dirichlet form on } L^2(\mathcal{X})\text{”}. \tag{4.35}$$

(ii) One may show (4.35) directly without referring to the self improvement property of (AB)$_{\mathrm{W}}$. To see this, observe that the arguments in the proofs of Propositions 4.8 and 4.9 run smoothly under (AB′)$_{\mathrm{W}}$. Regarding Lemma 4.10, note that the self-improvement property of (AB)$_{\mathrm{W}}$ is only used in (4.25). Now, instead of (4.23), we apply the condition (AB′)$_{\mathrm{W}}$ to find a function $\psi_k \in \mathrm{cutoff}(B_k, B_{k+1}) \cap \mathcal{F}$ such that

$$\iint_{B_{k+3}\times B_{k+3}} |u(x)|^2 |\psi_k(x) - \psi_k(y)|^2 \, J(x, dy) \, d\mu(x)$$

$$\leq \zeta \iint_{S_k\times S_k} |\psi_k(x)|^2|u(x)-u(y)|^2\,J(x,dy)\,d\mu(x) + \frac{C}{W(x_o,2^k)}\int_{B_{k+3}} |u(x)|^2\,d\mu(x),$$

where $S_k := B_{k+2}\setminus B_k$, and the constants ζ, C are positive and independent of x_o, k and u. With this, we can replace (4.25) by the following estimate:

$$\begin{aligned}
&\iint_{\substack{X\times X\\ d(x,y)<2^{k-2}}} |u(x)|^2|\psi_k(x)-\psi_k(y)|^2\,J(x,dy)\,d\mu(x)\\
&= \iint_{\substack{B_{k+3}\times B_{k+3}\\ d(x,y)<2^{k-2}}} |u(x)|^2|\psi_k(x)-\psi_k(y)|^2\,J(x,dy)\,d\mu(x)\\
&\leq \zeta \iint_{S_k\times S_k} |u(x)-u(y)|^2\,J(x,dy)\,d\mu(x) + \frac{C}{W(x_o,2^k)}\int_X |u(x)|^2\,d\mu(x).
\end{aligned}$$

Note that the right hand side of the above formula tends to 0 as $k\to\infty$ as $u\in\mathcal{F}$ and $\lim_{r\to\infty} W(x_o,r)=\infty$. This crucial estimate implies the conclusion of Lemma 4.10. In this argument, we only have used $(AB')_W$ but do not use the self-improvement property of $(AB)_W$ in Lemma 4.3. As for Proposition 4.11, we now apply the partition of unity under $(AB')_W$ that is given in Remark 4.7.

(iii) In view of the arguments in (ii), we may say that $(AB')_W$ is a replacement of both $(AB)_W$ and its self improvement property in Lemma 4.3.

5 Heat kernel estimates

In this section, we apply Theorem 2.9 to prove Theorems 2.13, 2.15 and 2.22.

5.1 From $(S)_W$ to $(AB')_W$

Proposition 5.1. *Suppose that $(\mathcal{E},\mathcal{F})$ is a bilinear form on $L^2(X)$. If $\mathcal{F}\cap C(X)$ is dense in $C_c(X)$, then, for any compact set K and any open set Ω with $K\subseteq\Omega$, we have*

$$\mathcal{F}\cap \mathrm{cutoff}(K,\Omega)\neq\emptyset.$$

In particular, the above statement is true when $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form.

Proof. Since (X,d) is locally compact and K is compact, it is known that there exists $\phi_0\in C_c(X)$ such that

$$\begin{cases} 0\leq\phi_0\leq 1 & \text{on } X;\\ \phi_0=1 & \text{on } K;\\ \phi_0=0 & \text{on } \Omega^{\complement}.\end{cases}$$

By the density of $\mathcal{F}\cap C(X)$ in $C_c(X)$, there exists $\phi_1\in\mathcal{F}\cap C(X)$ such that

$$\sup_{x\in X}|\phi_0(x)-\phi_1(x)|<\frac{1}{3}.$$

In particular, $\frac{2}{3}\leq\phi_1\leq\frac{4}{3}$ on K and $-\frac{1}{3}\leq\phi_1\leq\frac{1}{3}$ on $\Omega^{\complement}$. Let

$$\phi_2 := \phi_1 - \left(\frac{1}{3}\wedge\phi_1\right)\vee\left(-\frac{1}{3}\right)\in C(X).$$

It follows from [19, Theorem 1.4.2(iv)] that $\phi_2 \in \mathcal{F}$. Moreover, $\phi_2 \geq \frac{1}{3}$ on K and $\phi_2 = 0$ on $\Omega^{\complement}$. Define

$$\phi := (3\phi_2) \wedge 1 \in \mathcal{F},$$

and we have $\phi \in \text{cutoff}(K, \Omega)$, that is, $\mathcal{F} \cap \text{cutoff}(K, \Omega) \neq \emptyset$. □

Suppose that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. For any non-empty open set $\Omega \subseteq \mathcal{X}$, let $\mathcal{F}(\Omega)$ be as in (2.11). Then $(\mathcal{E}, \mathcal{F}(\Omega))$ is also a regular Dirichlet form. Denote the corresponding heat semigroup and heat kernel (if it exists) respectively by $\{P_t^{\Omega}\}_{t>0}$ and $\{p_t^{\Omega}\}_{t>0}$.

Definition 5.2. We say that a *survival condition* $(S)_W$ is satisfied if there exist constants $\epsilon, \delta > 0$ such that, for any ball $B := B(x_0, r) \subseteq \mathcal{X}$ of radius $r \in (0, \text{diam}\, \mathcal{X})$ the following inequality holds:

$$\operatorname*{essinf}_{\frac{1}{4}B} P_t^B 1 \geq \epsilon,$$

provided that $t \leq \delta W(x_0, r)$.

As can be seen from the following lemma, for a regular Dirichlet form, the condition $(S)_W$ implies $(AB')_W$ provided that $(TJ)_W$ holds.

Lemma 5.3. *Suppose that* $(\mathcal{E}, \mathcal{F})$ *in* (2.1) *is a regular jump-type Dirichlet form. Then*

$$(TJ)_W + (S)_W \Rightarrow (AB')_W.$$

Proof. The proof follows essentially ideas from [21, Lemma 2.4] and [17, Proposition 3.6]. Let

$$\begin{cases} B_0 = B(x_0, R); \\ B = B(x_0, R + r); \\ B_1 := B(x_0, R + 2r/5); \\ U := B(x_0, R + 4r/5) \setminus B(x_0, R); \\ U_0 := B(x_0, R + 3r/5) \setminus B(x_0, R + r/5); \\ \Omega = B(x_0, R'), \end{cases}$$

where $x_0 \in \mathcal{X}$ and $0 < R < R + r < R' < \infty$. We divide the the proof into three steps.

Step 1: estimating $G_\lambda^U \mathbf{1}_U$ for

$$\lambda := \left(\inf_{z \in B} W(z, r)\right)^{-1}.$$

According to [19, p. 17, Eq. (1.3.3)], we let $\{G_\gamma^U\}_{\gamma \in (0,\infty)}$ be the resolvent associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$. Firstly, it follows from [19, Theorem 4.4.1] that

$$G_\lambda^U \mathbf{1}_U \in \mathcal{F}(U) \subseteq \mathcal{F}.$$

Secondly, by [19, p. 17, Eq. (1.3.1)]), for any $0 \leq f \in L^1(U)$, we have

$$(G_\lambda^U \mathbf{1}_U, f) = \int_0^\infty e^{-\lambda t} (P_t^U \mathbf{1}_U, f)\, dt \leq \|f\|_{L^1(U)} \int_0^\infty e^{-\lambda t} dt = \|f\|_{L^1(U)} \lambda^{-1} = \inf_{z \in B} W(z, r) \|f\|_{L^1(U)}.$$

Since $0 \leq f \in L^1(U)$ is arbitrary and $G_\lambda^U \mathbf{1}_U = 0$ μ-a.e. outside of U, we obtain that

$$G_\lambda^U \mathbf{1}_U \leq \inf_{z \in B} W(z, r) \qquad \mu\text{-a.e. on } \mathcal{X}. \tag{5.1}$$

For any $x \in U_0$, we have $B_x := B(x, r/5) \subseteq U$, and hence, by (S)$_W$, for any $0 \le f \in L^1(\frac{1}{4}B_x)$,

$$\begin{aligned}(G_\lambda^U \mathbf{1}_U, f) \ge (G_\lambda^{B_x}\mathbf{1}_{B_x}, f) &= \int_0^\infty e^{-\lambda t}(P_t^{B_x}\mathbf{1}_{B_x}, f)\,dt \\ &\ge \int_0^{\delta W(x,r/5)} e^{-\lambda t}(P_t^{B_x}\mathbf{1}_{B_x}, f)\,dt \\ &\ge \varepsilon \|f\|_{L^1(\frac{1}{4}B_x)} \int_0^{\delta W(x,r/5)} e^{-\lambda t}\,dt \\ &= \frac{\varepsilon}{\lambda}\left(1 - e^{-\lambda\delta W(x,r/5)}\right)\|f\|_{L^1(\frac{1}{4}B_x)} \\ &\ge \frac{\varepsilon}{\lambda}\left(1 - e^{-\lambda\delta \inf_{x\in U_0} W(x,r/5)}\right)\|f\|_{L^1(\frac{1}{4}B_x)}.\end{aligned}$$

Moreover, by the right inequality in (2.2), we have $W(x, 5/r) \ge 5^{-\beta_2}C_W^{-1}W(x,r)$, and hence,

$$\inf_{x\in U_0} W(x, 5/r) \ge 5^{-\beta_2}C_W^{-1} \inf_{x\in U_0} W(x,r) \ge 5^{-\beta_2}C_W^{-1} \inf_{x\in B} W(x,r).$$

Combining the above two inequalities, we obtain

$$\begin{aligned}(G_\lambda^U \mathbf{1}_U, f) &\ge \frac{\varepsilon}{\lambda}\left(1 - e^{-\lambda\delta\cdot 5^{-\beta_2}C_W^{-1}\inf_{x\in B} W(x,r)}\right)\|f\|_{L^1(\frac{1}{4}B_x)} \\ &= \varepsilon\left(1 - e^{-5^{-\beta_2}C_W^{-1}\delta}\right)\inf_{z\in B} W(z,r)\|f\|_{L^1(\frac{1}{4}B_x)}.\end{aligned}$$

Due to the arbitrariness of $0 \le f \in L^1(\frac{1}{4}B_x)$, we obtain from the above inequality that

$$G_\lambda^U \mathbf{1}_U \ge \varepsilon(1 - e^{-5^{-\beta_2}C_W^{-1}\delta})\inf_{z\in B} W(z,r) \qquad \mu\text{-a.e. in } \frac{1}{4}B_x.$$

Moreover, since U_0 can be covered by finitely many balls like $\frac{1}{4}B_x$, we obtain

$$G_\lambda^U \mathbf{1}_U \ge \varepsilon(1 - e^{-5^{-\beta_2}C_W^{-1}\delta})\inf_{z\in B} W(z,r) \qquad \mu\text{-a.e. in } U_0. \tag{5.2}$$

Let us show that $G_\lambda^U\mathbf{1}_U \in C(\mathcal{X})$. Indeed, it follows from [21, Theorem 2.10, p. 460 and Lemma 5.12, p. 504] that $P_t^U\mathbf{1}_U(x)$ is jointly continuous in $(t,x) \in (0,\infty)\times U$ since $\mathbf{1}_U \in L^1(\mathcal{X}) \cap L^2(\mathcal{X})$. Then, by the dominated convergence theorem and the fact that $P_t^U\mathbf{1}_U(x) \le 1$ for all $t > 0$ and $x \in \mathcal{X}$, we obtain that $G_\lambda^U\mathbf{1}_U(x) = \int_0^\infty e^{-\lambda t}P_t^U\mathbf{1}_U(x)\,dt$ is continuous in $x \in \mathcal{X}$.

Step 2: constructing a function

$$\phi \in \operatorname{cutoff}(B(x_0, R + 3r/5), B(x_0, R + 4r/5)) \cap \mathcal{F} \subseteq \operatorname{cutoff}(B_0, B) \cap \mathcal{F}.$$

Define

$$\kappa := \frac{1}{\varepsilon(1 - e^{-5^{-\beta_2}C_W^{-1}\delta})} \quad \text{and} \quad g := \frac{\kappa G_\lambda^U \mathbf{1}_U}{\inf_{z\in B} W(z,r)}.$$

Since $G_\lambda^U\mathbf{1}_U \in \mathcal{F} \cap C(\mathcal{X})$, so does g. Moreover, by (5.1) and (5.2), we have

$$\begin{cases} 0 \le g \le \kappa & \text{on } \mathcal{X}; \\ g \ge 1 & \text{on } U_0; \\ g = 0 & \text{on } U^{\complement}. \end{cases}$$

By the regularity of $(\mathcal{E}, \mathcal{F})$ and Proposition 5.1, there exists

$$\tilde{\phi} \in \mathrm{cutoff}(B(x_0, R + 2r/5), B(x_0, R + r/5)) \cap \mathcal{F}.$$

Now, we define

$$\phi := (\tilde{\phi} + g) \wedge 1. \tag{5.3}$$

It follows from $\tilde{\phi}, g \in \mathcal{F}$ and [19, Theorem 1.4.2(i)] that $\phi \in \mathcal{F}$. Moreover, since $\tilde{\phi} = 1$ on $B(x_0, R + r/5)$ and $g \geq 1$ on $U_0 = B(x_0, R + 3r/5) \setminus B(x_0, R + r/5)$, we have

$$\phi = 1 \quad \text{on } B(x_0, R + 3r/5).$$

Since $\tilde{\phi} = 0$ on $B(x_0, R + 2r/5)^{\complement}$ and $g = 0$ on $U^{\complement} = B(x_0, R + 4r/5)^{\complement} \cup B(x_0, R)$, we have

$$\phi = 0 \quad \text{on } B(x_0, 4r/5)^{\complement}.$$

This gives the desired result.

Step 3: verification that ϕ satisfies the inequality in (AB′)$_{\mathrm{W}}$ for all $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$.

Recall that $B_1 = B(x_0, R + 2r/5)$ and $B = B(x_0, R + r)$. Under $d(x, y) < r/5$, if either $x \in B_1$ or $y \in B_1$, then we always have both $x, y \in B(x_0, R + 3r/5)$ and, hence, $\phi(x) = \phi(y) = 1$. Moreover, still under $d(x, y) < r/5$, if either $x \notin B$ or $y \notin B$, then we always have both $x, y \notin B(x_0, R + 4r/5)$ and, hence, $\phi(x) = \phi(y) = 0$. From these observations, we derive that if $d(x, y) < r/5$ then

$$|\phi(x) - \phi(y)| \neq 0 \quad \text{only if } x, y \in B \setminus B_1.$$

Therefore,

$$\begin{aligned}\iint_{\Omega\times\Omega} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy)\, d\mu(x) &= \left(\iint_{\substack{\Omega\times\Omega \\ d(x,y)\geq r/5}} + \iint_{\substack{(B\setminus B_1)\times(B\setminus B_1) \\ d(x,y)<r/5}} \right) \cdots \\ &\leq \left(\iint_{\substack{\Omega\times\Omega \\ d(x,y)\geq r/5}} + \iint_{(B\setminus B_1)\times(B\setminus B_1)} \right) \cdots =: \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

By (TJ)$_{\mathrm{W}}$ and $0 \leq \phi \leq 1$, we have

$$\mathrm{I}_1 \leq \int_\Omega |u(x)|^2 \left(\int_{d(x,y)\geq r/5} J(x, dy) \right) d\mu(x) \leq \frac{C}{\inf_{z\in B} W(z, r)} \int_\Omega |u(x)|^2 \, d\mu(x).$$

To estimate I_2, since $\tilde{\phi}$ is supported in B_1, we have by the definition (5.3) of ϕ that

$$|\phi(x) - \phi(y)| = |(g(x) \wedge 1) - (g(y) \wedge 1)| \leq |g(x) - g(y)| \quad \text{for all } x, y \in B \setminus B_1.$$

This, together with the symmetry $J(x, dy)\, d\mu(x) = J(y, dx)\, d\mu(y)$, yields

$$\begin{aligned}\mathrm{I}_2 &= \iint_{(B\setminus B_1)\times(B\setminus B_1)} |u(x)|^2 |\phi(x) - \phi(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\leq \iint_{(B\setminus B_1)\times(B\setminus B_1)} |u(x)|^2 |g(x) - g(y)|^2 \, J(x, dy)\, d\mu(x) \\ &\leq \iint_{(B\setminus B_0)\times(B\setminus B_0)} |u(x)|^2 |g(x) - g(y)|^2 \, J(x, dy)\, d\mu(x) \\ &= \frac{1}{2} \iint_{(B\setminus B_0)\times(B\setminus B_0)} (|u(x)|^2 + |u(y)|^2) |g(x) - g(y)|^2 \, J(x, dy)\, d\mu(x).\end{aligned}$$

Moreover, for any $x, y \in \mathcal{X}$, the following pointwise inequality

$$\begin{aligned}
&\frac{1}{2}(|u(x)|^2 + |u(y)|^2)|g(x) - g(y)|^2 \\
&\quad \le 2\,(g(x) - g(y))\left(u^2(x)g(x) - u^2(y)g(y)\right) + 2\left(g^2(x) + g^2(y)\right)(u(x) - u(y))
\end{aligned}$$

holds (see the proof of Lemma 2.2 in [21, p. 447]). Hence,

$$\begin{aligned}
\mathrm{I}_2 &\le 2 \iint_{(B\setminus B_0)\times(B\setminus B_0)} (g(x) - g(y))\left(u^2(x)g(x) - u^2(y)g(y)\right) J(x, dy)\, d\mu(x) \\
&\quad + 4 \iint_{(B\setminus B_0)\times(B\setminus B_0)} g^2(x)|u(x) - u(y)|^2 J(x, dy)\, d\mu(x) \\
&=: 2\,\mathrm{I}_{21} + 4\,\mathrm{I}_{22}.
\end{aligned}$$

Since $g \le \kappa\phi$ by the definition (5.3), we have

$$\mathrm{I}_{22} \le \kappa^2 \iint_{(B\setminus B_0)\times(B\setminus B_0)} \phi^2(x)|u(x) - u(y)|^2 J(x, dy)\, d\mu(x),$$

which is just the first term in the right hand side of (2.10) in the $(\mathrm{AB}')_{\mathrm{W}}$ condition.

Consider now the estimate of I_{21}. Note that if $x, y \notin B \setminus B_0$, then by the fact

$$\operatorname{supp} g \subseteq U = B(x_0, R + 4r/5) \setminus B(x_0, R) \subseteq B \setminus B_0,$$

we see that $g(x) = g(y) = 0$, which further implies that

$$\begin{aligned}
\mathrm{I}_{21} &= \mathcal{E}(u^2 g, g) - \left(\iint_{(B\setminus B_0)^\complement\times(B\setminus B_0)} + \iint_{(B\setminus B_0)\times(B\setminus B_0)^\complement} + \iint_{(B\setminus B_0)^\complement\times(B\setminus B_0)^\complement}\right)\cdots \\
&= \mathcal{E}(u^2 g, g) - \iint_{(B\setminus B_0)^\complement\times(B\setminus B_0)} u^2(y)g^2(y)\, J(x, dy)\, d\mu(x) \\
&\quad - \iint_{(B\setminus B_0)\times(B\setminus B_0)^\complement} u^2(x)g^2(x)\, J(x, dy)\, d\mu(x) \\
&\le \mathcal{E}(u^2 g, g).
\end{aligned}$$

For any $u, v \in \mathcal{F}$, define

$$\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda(u, v). \tag{5.4}$$

Further, it follows from the definition of g, [19, Theorem 4.4.1(i)] and the fact that $g \le \kappa\phi$ that

$$\begin{aligned}
\mathcal{E}(u^2 g, g) \le \mathcal{E}(u^2 g, g) + \lambda(u^2 g, g) &= \sup_{z\in B} \frac{\kappa}{W(z, r)} \mathcal{E}_\lambda\left(u^2 g, G_\lambda^U \mathbf{1}_U\right) \\
&= \sup_{z\in B} \frac{\kappa}{W(z, r)} (u^2 g, \mathbf{1}_U) \\
&\le \sup_{z\in B} \frac{\kappa^2}{W(z, r)} (u^2 \phi, \mathbf{1}_U) \\
&\le \sup_{z\in B} \frac{\kappa^2}{W(z, r)} \int_U u^2\, d\mu.
\end{aligned}$$

Finally, combining the estimates of I_1, I_2, I_{21} and I_{22}, we prove that the function ϕ defined in (5.3) satisfies the inequality in condition $(\mathrm{AB}')_{\mathrm{W}}$. This completes the proof. □

Remark 5.4. Note that we have proved in Lemma 5.3 a conclusion that is stronger than $(\mathrm{AB}')_{\mathrm{W}}$. Indeed, we see in the above proof that the function ϕ in (2.10) and, hence, in (2.5), can be chosen to be independent of the function $u \in \mathcal{F}' \cap L^\infty(\mathcal{X})$.

5.2 From (LLE)$_W$ to (S)$_W$ and (PI)$_W$

The following lemma was proved in [22, Lemma 7.14].

Lemma 5.5. *Under* (VD), *we have* (LLE)$_W$ $\Rightarrow$ (S)$_W$.

The stochastic completeness comes as a consequence of (S)$_W$. The following lemma was proved in [22, Corollary 8.9(2) and Remark 2.15].

Lemma 5.6. *Suppose that* $(\mathcal{E},\mathcal{F})$ *in* (2.1) *is a regular jump-type Dirichlet form. If* (S)$_W$ *is satisfied, then* $(\mathcal{E},\mathcal{F})$ *is conservative.*

The remaining part of this subsection is devoted to the proof of (LLE)$_W$ $\Rightarrow$ (PI)$_W$ (see Lemma 5.10 below). Note that this implication was proved in [22, Subsections 7.3 and 7.4] under a more general setting, where some deep Dirichlet form theory was used. Here we give a direct and self-contained proof.

Indeed, for any ball $B\subseteq\mathcal{X}$, we will construct a regular pure jump-type Dirichlet form $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ on $L^2(\overline{B})$, where $\overline{B}$ is the closure of B, and then use $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ to prove the implication (LLE)$_W$ $\Rightarrow$ (PI)$_W$. Define

$$\begin{cases}\overline{\mathcal{E}}(u,v):=\displaystyle\iint_{\overline{B}\times\overline{B}}(u(x)-u(y))(v(x)-v(y))\,J(x,dy)\,d\mu(x);\\ \mathrm{Dom}(\overline{\mathcal{E}}):=\{u\in L^2(\overline{B}):\ u \text{ is Borel measurable on } \overline{B},\ \overline{\mathcal{E}}(u,u)<\infty\},\end{cases}\tag{5.5}$$

whenever the above double integral makes sense for Borel measurable functions u,v on $\overline{B}$. Let

$$\mathcal{F}|_{\overline{B}}:=\left\{u\in L^2(\overline{B}):\ \text{there exists } v\in\mathcal{F} \text{ such that }\ u=v|_B\ \mu\text{-a.e. on } \overline{B}\right\}.$$

In other words, each function u in $\mathcal{F}|_{\overline{B}}$ is the restriction of some function in $\mathcal{F}$ on $\overline{B}$. In this case, we use the same letter $u\in\mathcal{F}$ to denote its restriction on $\overline{B}$. It follows from the definition of $\overline{\mathcal{E}}$ that

$$\overline{\mathcal{E}}(u|_{\overline{B}},\,u|_{\overline{B}})=\overline{\mathcal{E}}(u,\,u)\le\mathcal{E}(u,u)<\infty\quad\text{ for all } u\in\mathcal{F}.$$

Hence,

$$\mathcal{F}|_{\overline{B}}\subseteq\mathrm{Dom}(\overline{\mathcal{E}}).$$

We remark that functions in $\mathrm{Dom}(\overline{\mathcal{E}})$ may not be defined outside of $\overline{B}$, but each function in $\mathcal{F}|_{\overline{B}}$ is corresponding to an element in $\mathcal{F}$ that is defined on the whole space $\mathcal{X}$.

Note that the kernel $J(x,dy)$ satisfies condition (J2) in Definition 2.1. Following the arguments after Definition 2.1, we obtain that $(\overline{\mathcal{E}},\mathrm{Dom}(\overline{\mathcal{E}}))$ is a Dirichlet form on $L^2(\overline{B})$ provided that $\mathrm{Dom}(\overline{\mathcal{E}})$ is dense in $L^2(\overline{B})$. Moreover, since $(\mathcal{E},\mathcal{F})$ is regular, we have that $\mathcal{F}|_{\overline{B}}\cap C_c(\overline{B})$ is dense in $L^2(\overline{B})$. This, together with the fact that

$$\left(\mathcal{F}|_{\overline{B}}\cap C_c(\overline{B})\right)\subseteq\mathcal{F}|_{\overline{B}}\subseteq\mathrm{Dom}(\overline{\mathcal{E}}),$$

implies that $\mathrm{Dom}(\overline{\mathcal{E}})$ is also dense in $L^2(\overline{B})$. Therefore, $(\overline{\mathcal{E}},\mathrm{Dom}(\overline{\mathcal{E}}))$ is indeed a Dirichlet form on $L^2(\overline{B})$. Moreover, setting

$$\overline{\mathcal{F}}:=\overline{\mathcal{F}|_{\overline{B}}\cap C_c(\overline{B})}^{\|\cdot\|_{\overline{\mathcal{E}}_1}},\tag{5.6}$$

we have that $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is also a Dirichlet form on $L^2(\overline{B})$. Here, for $\lambda>0$, $\overline{\mathcal{E}}_\lambda$ is defined as follows:

$$\overline{\mathcal{E}}_\lambda(u,v):=\overline{\mathcal{E}}(u,v)+\lambda(u,v)_{L^2(\overline{B})}\quad\text{ for all } u,v\in\mathrm{Dom}(\overline{\mathcal{E}})$$

and

$$\|u\|_{\overline{\mathcal{E}}_\lambda}:=\sqrt{\overline{\mathcal{E}}(u,u)+\lambda\|u\|^2_{L^2(\overline{B})}}\quad\text{ for all } u\in\mathrm{Dom}(\overline{\mathcal{E}}).$$

Proposition 5.7. *Let $(\mathcal{E},\mathcal{F})$ in (2.1) be a regular jump-type Dirichlet form. Let $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ be the Dirichlet form defined in (5.5) and (5.6). Then, the following hold:*

(i) *$(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\overline{B})$;*

(ii) *$\mathcal{F}|_{\overline{B}} \subseteq \overline{\mathcal{F}}$.*

Proof. Let us first show (i). Since $\mathcal{F}|_{\overline{B}} \cap C_c(\overline{B}) \subseteq \overline{\mathcal{F}}$, by the definition of $\overline{\mathcal{F}}$ in (5.6), we know that $\overline{\mathcal{F}} \cap C_c(\overline{B})$ is dense in $\overline{\mathcal{F}}$ with respect to $\|\cdot\|_{\overline{\mathcal{E}}_1}$-norm. For any $u \in C_c(\overline{B})$, we can extend it to a function in $C_c(\mathcal{X})$, which is also denoted by u. Since $(\mathcal{E},\mathcal{F})$ is regular, there exists $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{F} \cap C_c(\mathcal{X})$ such that

$$\sup_{x\in\mathcal{X}} |u_n(x) - u(x)| \to 0 \quad \text{as } n \to \infty.$$

For each $n \in \mathbb{N}$, it is clear that

$$u_n|_{\overline{B}} \in \left(\mathcal{F}|_{\overline{B}} \cap C_c(\overline{B})\right) \subseteq \left(\overline{\mathcal{F}} \cap C_c(\overline{B})\right)$$

and

$$\sup_{x\in\overline{B}} |u_n(x) - u(x)| \to 0 \quad \text{as } n \to \infty.$$

That is, $\overline{\mathcal{F}} \cap C_c(\overline{B})$ is dense in $C_c(\overline{B})$. Therefore, $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\overline{B})$.

Next, we show (ii). Fix $u \in \mathcal{F}|_{\overline{B}}$. Since $(\mathcal{E},\mathcal{F})$ is regular, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{F} \cap C_c(\mathcal{X})$ such that

$$\lim_{n\to\infty} \mathcal{E}_1(u_n - u, u_n - u) = 0.$$

For each $n \in \mathbb{N}$, observe that $u_n|_{\overline{B}} \in \mathcal{F}|_{\overline{B}} \cap C_c(\overline{B})$ and

$$\overline{\mathcal{E}}_1(u_n|_{\overline{B}} - u, u_n|_{\overline{B}} - u) \le \mathcal{E}_1(u_n - u, u_n - u).$$

From this and the definition of $\overline{\mathcal{F}}$ in (5.6), we deduce that $u \in \overline{\mathcal{F}}$. This proves that $\mathcal{F}|_{\overline{B}} \subseteq \overline{\mathcal{F}}$. □

Now, let us consider the part of $\overline{\mathcal{E}}$ on the open ball B, that is, the part Dirichlet form $(\overline{\mathcal{E}},\overline{\mathcal{F}}(B))$ of $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ on the open ball B, where

$$\overline{\mathcal{F}}(B) := \overline{\overline{\mathcal{F}} \cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}}. \tag{5.7}$$

Lemma 5.8. $\overline{\mathcal{F}}(B) = \overline{\mathcal{F}|_{\overline{B}} \cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}}$.

Proof. Since $\mathcal{F}|_{\overline{B}} \subseteq \overline{\mathcal{F}}$ by Proposition 5.7(ii), we have by (5.7) that

$$\overline{\mathcal{F}|_{\overline{B}} \cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}} \subseteq \overline{\mathcal{F}}(B).$$

It suffices to prove the converse part. To this end, we fix $u \in \overline{\mathcal{F}}(B)$ and will show that u can be approximated by functions in $\mathcal{F}|_{\overline{B}} \cap C_c(B)$ with respect to the $\|\cdot\|_{\overline{\mathcal{E}}_1}$-norm.

Step 1. By (5.7), we choose $\{v_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathcal{F}} \cap C_c(B)$ such that

$$\lim_{n\to\infty} \overline{\mathcal{E}}_1(v_n - u, v_n - u) = 0.$$

Step 2. Fix $n \in \mathbb{N}$. By the definition (5.6) of $\overline{\mathcal{F}}$, we choose $\{w_m\}_{m\in\mathbb{N}} \subseteq \mathcal{F}|_{\overline{B}} \cap C_c(\overline{B})$ such that

$$\lim_{m\to\infty} \overline{\mathcal{E}}_1(w_m - v_n, w_m - v_n) = 0.$$

Since v_n is bounded on $\overline{B}$, by [19, Theorem 1.4.2(v)], we can assume that

$$\sup_{m\in\mathbb{N}}\sup_{x\in\overline{B}}|w_m(x)|\leq\|v_n\|_{L^\infty(\overline{B})}.$$

Step 3. Since $v_n\in C_c(B)$ and $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is regular, by Proposition 5.1, we can choose

$$\phi\in\mathrm{cutoff}(\operatorname{supp}\,v_n,B)\cap\overline{\mathcal{F}}.$$

Then, by [19, Theorem 1.4.2(ii)], we have

$$\begin{aligned}\sup_{m\in\mathbb{N}}\overline{\mathcal{E}}_1(\phi w_m,\phi w_m)&\leq\sup_{m\in\mathbb{N}}\left(2\|\phi\|_{L^\infty(\overline{B})}\overline{\mathcal{E}}_1(w_m,w_m)+2\|w_m\|_{L^\infty(\overline{B})}\overline{\mathcal{E}}_1(\phi,\phi)\right)\\&\leq 2\|\phi\|_{L^\infty(\overline{B})}\sup_{m\in\mathbb{N}}\overline{\mathcal{E}}_1(w_m,w_m)+2\|v_n\|_{L^\infty(\overline{B})}\overline{\mathcal{E}}_1(\phi,\phi)<\infty.\end{aligned}$$

From this and the Banach-Alaoglu theorem, it follows that there exists a subsequence $\{w_{m_i}\}_{i\in\mathbb{N}}$ of $\{w_m\}_{m\in\mathbb{N}}$ such that $\{\phi w_{m_i}\}_{i\in\mathbb{N}}$ converges $\overline{\mathcal{E}}_1$-weakly to a certain element $w\in\overline{\mathcal{F}}$. Consequently, the Cesáro mean of a subsequence of $\{\phi w_{m_i}\}_{i\in\mathbb{N}}$ (still denoted it by $\{\phi w_{m_i}\}_{i\in\mathbb{N}}$) satisfies that

$$\tilde{w}_k:=\frac{1}{k}\sum_{i=1}^{k}\phi w_{m_i}\xrightarrow{\|\cdot\|_{\overline{\mathcal{E}}_1}}w\quad\text{ as }\;k\to\infty.$$

On the other hand, by **Step 2**, we have

$$\tilde{w}_k=\frac{1}{k}\sum_{i=1}^{k}\phi w_{m_i}\xrightarrow{L^2(\overline{B})}\phi v_n=v_n\quad\text{ as }\;k\to\infty.$$

Hence, we have $w=v_n$ and then

$$\lim_{k\to\infty}\overline{\mathcal{E}}_1(\tilde{w}_k-v_n,\,\tilde{w}_k-v_n)=0.$$

Step 4. For any $\epsilon>0$, by **Step 1**, choose $v_n\in\overline{\mathcal{F}}\cap C_c(B)$ such that

$$\overline{\mathcal{E}}_1(v_n-u,v_n-u)<\epsilon.$$

For this specific n, by **Step 3**, we may choose k large enough (depending on n and ε) such that

$$\overline{\mathcal{E}}_1(\tilde{w}_k-v_n,\,\tilde{w}_k-v_n)\leq\varepsilon.$$

Combining the last two formulae derives that

$$\overline{\mathcal{E}}_1(\tilde{w}_k-u,\tilde{w}_k-u)\leq 2\overline{\mathcal{E}}_1(v_n-u,v_n-u)+2\overline{\mathcal{E}}_1(\tilde{w}_k-v_n,\,\tilde{w}_k-v_n)<4\varepsilon.$$

This gives

$$\lim_{k\to\infty}\overline{\mathcal{E}}_1(\tilde{w}_k-u,\tilde{w}_k-u)=0.$$

Since $\{w_m\}_{m\in\mathbb{N}}\subseteq\mathcal{F}|_{\overline{B}}\cap C_c(\overline{B})$ and $\operatorname{supp}\,\phi\subseteq B$, by the definition of $\tilde{w}_k$, we have

$$\tilde{w}_k\in\mathcal{F}|_{\overline{B}}\cap C_c(B).$$

Hence, we obtain $u\in\overline{\mathcal{F}|_{\overline{B}}\cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}}$. Consequently, $\overline{\mathcal{F}}(B)\subseteq\overline{\mathcal{F}|_{\overline{B}}\cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}}$. $\square$

Let $\{\overline{P}_t\}_{t>0}$, $\{\overline{P}_t^B\}_{t>0}$ and $\{P_t^B\}_{t>0}$ be the heat semigroups associated with $(\overline{\mathcal{E}},\overline{\mathcal{F}})$, $(\overline{\mathcal{E}},\overline{\mathcal{F}}(B))$ and $(\mathcal{E},\mathcal{F}(B))$, respectively. One may compare the domains of these three Dirichlet forms. Indeed, by (5.6), Lemma 5.8 and the definition of $\mathcal{F}(B)$, we have

$$\overline{\mathcal{F}} = \overline{\mathcal{F}|_{\overline{B}} \cap C_c(\overline{B})}^{\|\cdot\|_{\overline{\mathcal{E}}_1}}, \quad \overline{\mathcal{F}}(B) = \overline{\mathcal{F}|_{\overline{B}} \cap C_c(B)}^{\|\cdot\|_{\overline{\mathcal{E}}_1}} \quad \text{and} \quad \mathcal{F}(B) = \overline{\mathcal{F} \cap C_c(B)}^{\|\cdot\|_{\mathcal{E}_1}}.$$

Suppose that $f \in L^2(\mathcal{X})$. Then it is obvious that $f|_{\overline{B}} \in L^2(\overline{B})$ and $f|_B \in L^2(B)$. Since $\{\overline{P}_t\}_{t>0}$ is a semigroup defined on $L^2(\overline{B})$, we take it for granted that $\overline{P}_t f = \overline{P}_t(f|_{\overline{B}})$. In a similar manner, we understood $\overline{P}_t^B f$ and $P_t^B f$ as $\overline{P}_t^B(f|_B)$ and $P_t^B(f|_B)$, respectively.

Proposition 5.9. *For any $t \in (0,\infty)$ and $0 \le f \in L^2(\mathcal{X})$, it holds that*

$$\overline{P}_t f \ge \overline{P}_t^B f \ge P_t^B f \quad \text{on } \overline{B}.$$

Proof. It is clear that $\overline{P}_t f = \overline{P}_t(f|_{\overline{B}}) \ge \overline{P}_t^B(f|_B) = \overline{P}_t^B f$. It suffices to prove the second inequality. Indeed, by $\mathcal{F} \cap C_c(B) \subseteq \mathcal{F}|_{\overline{B}} \cap C_c(B)$ and the fact that $\overline{\mathcal{E}}_1(u|_{\overline{B}}, u|_{\overline{B}}) \le \mathcal{E}_1(u,u)$, we then apply Lemma 5.8 to derive that

$$\mathcal{F}(B)|_{\overline{B}} \subseteq \overline{\mathcal{F}}(B). \tag{5.8}$$

Fix $\lambda > 0$, $0 \le f \in L^2(\mathcal{X})$ and $0 \le g \in \mathcal{F} \cap C_c(B)$. Let $\{\overline{G}_\lambda^B\}_{\lambda>0}$ and $\{G_\lambda^B\}_{\lambda>0}$ be the resolvents of $(\overline{\mathcal{E}},\overline{\mathcal{F}}(B))$ and $(\mathcal{E},\mathcal{F}(B))$, respectively. It follows from (5.8) and [19, Theorem 4.4.1(i)] that for

$$G_\lambda^B f|_{\overline{B}} \in \mathcal{F}(B)|_{\overline{B}} \subseteq \overline{\mathcal{F}}(B), \quad \overline{G}_\lambda^B f \subseteq \overline{\mathcal{F}}(B),$$

and

$$\overline{\mathcal{E}}_\lambda(\overline{G}_\lambda^B f, g|_{\overline{B}}) = (f, g|_{\overline{B}})_{L^2(\overline{B})} = (f, g)_{L^2(\mathcal{X})} = \mathcal{E}_\lambda(G_\lambda^B f, g). \tag{5.9}$$

We remark that $\overline{G}_\lambda^B f$ and $G_\lambda^B f$ respectively means $\overline{G}_\lambda^B(f|_B)$ and $G_\lambda^B(f|_B)$. The value of $\overline{G}_\lambda^B f$ outside of $\overline{B}$ is not defined because for $\overline{\mathcal{E}}$ everything happens inside the closed ball $\overline{B}$. But $G_\lambda^B f$ has a precise value at each point of $\mathcal{X}$ and, moreover, one has $G_\lambda^B f \in L^2(\mathcal{X})$ whenever $f \in L^2(\mathcal{X})$.

By (5.5) and the definition of $\mathcal{E}_\lambda$ (see (5.4)), we have

$$\overline{\mathcal{E}}_\lambda(\overline{G}_\lambda^B f, g|_{\overline{B}}) = \iint_{\overline{B}\times\overline{B}} (\overline{G}_\lambda^B f(x) - \overline{G}_\lambda^B f(y))(g(x) - g(y))\, J(x,dy)\, d\mu(x) + \lambda \int_{\overline{B}} g\overline{G}_\lambda^B f\, d\mu$$

and

$$\mathcal{E}_\lambda(G_\lambda^B f, g) = \iint_{\mathcal{X}\times\mathcal{X}} (\overline{G}_\lambda^B f(x) - \overline{G}_\lambda^B f(y))(g(x) - g(y))\, J(x,dy)\, d\mu(x) + \lambda \int_{\mathcal{X}} g\overline{G}_\lambda^B f\, d\mu.$$

This last two formulae, together with (5.9) the fact $\operatorname{supp} g \subseteq B$, further yields

$$\begin{aligned}
&\overline{\mathcal{E}}_\lambda(\overline{G}_\lambda^B f - G_\lambda^B f|_{\overline{B}}, g|_{\overline{B}}) \\
&\quad = \mathcal{E}_\lambda(G_\lambda^B f, g) - \overline{\mathcal{E}}_\lambda(G_\lambda^B f|_{\overline{B}}, g|_{\overline{B}}) \\
&\quad = \left(\iint_{\overline{B}\times\overline{B}^c} + \iint_{\overline{B}^c\times\overline{B}} + \iint_{\overline{B}^c\times\overline{B}^c}\right)(\overline{G}_\lambda^B f(x) - \overline{G}_\lambda^B f(y))(g(x) - g(y))\, J(x,dy)\, d\mu(x) \\
&\quad = 2\int_{\overline{B}}\int_{\overline{B}^c} G_\lambda^B f(x) g(x)\, J(x,dy)\, d\mu(x) \\
&\quad \ge 0.
\end{aligned}$$

Since $g|_{\overline{B}} \in \mathcal{F}|_{\overline{B}} \cap C_c(B)$ and $\mathcal{F}|_{\overline{B}} \cap C_c(B)$ is dense in $\overline{\mathcal{F}}(B)$, it follows that the above inequality holds true also for all $g \in \overline{\mathcal{F}}(B)$. In particular, for all $0 \le h \in L^2(\overline{B})$, we have

$$(\overline{G}_\lambda^B f - G_\lambda^B f|_{\overline{B}}, h) = \overline{\mathcal{E}}_\lambda(\overline{G}_\lambda^B f - G_\lambda^B f|_{\overline{B}}, \overline{G}_\lambda^B h) \ge 0.$$

Since $0 \le h \in L^2(\overline{B})$ is arbitrary, we obtain that

$$\overline{G}_\lambda^B f = \overline{G}_\lambda^B \left(f|_{\overline{B}}\right) \ge \left(G_\lambda^B f\right)|_{\overline{B}}.$$

Applying this inequality and the fact $G_\lambda^B f \in L^2(\mathcal{X})$ yields

$$\overline{G}_\lambda^B \left(\overline{G}_\lambda^B f\right) \ge \overline{G}_\lambda^B \left(\left(G_\lambda^B f\right)|_{\overline{B}}\right) = \overline{G}_\lambda^B \left(G_\lambda^B f\right) \ge \left(G_\lambda^B \left(G_\lambda^B f\right)\right)|_{\overline{B}} = \left((G_\lambda^B)^2 f\right)|_{\overline{B}}.$$

For general $n \in \mathbb{N}$, repeating this argument n-times gives

$$\left(\overline{G}_\lambda^B\right)^n f \ge \left((G_\lambda^B)^n f\right)|_{\overline{B}}.$$

Moreover, since $\lambda > 0$ is arbitrary, it follows from [19, p. 20, (1.3.5)] that

$$\overline{P}_t^B f = \lim_{\lambda\to\infty} e^{-t\lambda} \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \left(\lambda \overline{G}_\lambda^B\right)^n f \ge \lim_{\lambda\to\infty} e^{-t\lambda} \sum_{n=0}^{\infty} \frac{(t\lambda)^n}{n!} \left(\lambda G_\lambda^B\right)^n f = P_t^B f \quad \text{ on } \overline{B}.$$

This ends the proof. □

Lemma 5.10. *Let $(\mathcal{E},\mathcal{F})$ in (2.1) be a regular jump-type Dirichlet form. Then, under (VD), we have*

$$(\mathrm{LLE})_W \Rightarrow (\mathrm{PI})_W.$$

Proof. Suppose that $u \in \mathcal{F} \cap L^\infty(\mathcal{X})$. To show that $(\mathrm{PI})_W$ holds under (VD) and $(\mathrm{LLE})_W$, we fix a ball $B := B(x_0, R)$ with $x_0 \in \mathcal{X}$ and $R \in (0,\infty)$. Let $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ be the regular Dirichlet form on $L^2(\overline{B})$ defined in (5.5) and (5.6). Since $u|_{\overline{B}} \in \mathcal{F}|_{\overline{B}} \subseteq \overline{\mathcal{F}}$ by Proposition 5.7(ii), we have for any $t > 0$,

$$\begin{aligned}
&\iint_{\overline{B}\times\overline{B}} |u(x)-u(y)|^2\, J(x,dy)\, d\mu(x)\\
&\quad = \overline{\mathcal{E}}(u|_{\overline{B}}, u|_{\overline{B}}) \ge \frac{1}{t}\left(u|_{\overline{B}} - \overline{P}_t u|_{\overline{B}}, u|_{\overline{B}}\right) \qquad \text{(by [19, Lemma 1.3.4(i)])}\\
&\quad \ge \frac{1}{2t}\int_{\overline{B}} \left(\overline{P}_t 1(x) u^2(x) + \overline{P}_t(u^2)(x) - 2u(x)\overline{P}_t u(x)\right) d\mu(x) \qquad (\text{by } \overline{P}_t 1 \le 1)\\
&\quad = \frac{1}{2t}\int_{\overline{B}} \overline{P}_t\left((u(x)-u(\cdot))^2\right)(x)\, d\mu(x)\\
&\quad \ge \frac{1}{2t}\int_{B} P_t^B\left((u(x)-u(\cdot))^2\right)(x)\, d\mu(x) \qquad \text{(by Proposition 5.9)}\\
&\quad = \frac{1}{2t}\int_B\int_B p_t^B(x,y)(u(x)-u(y))^2\, d\mu(y)\, d\mu(x). \qquad (\text{by } (\mathrm{LLE})_W)
\end{aligned}$$

Let $\delta \in (0,1)$ be the constant from $(\mathrm{LLE})_W$. Setting $t := W(x_0, \delta R)$ (that is, $W^{-1}(x_0, t) = \delta R$) in the above inequality, we obtain

$$\iint_{\overline{B}\times\overline{B}} |u(x)-u(y)|^2\, J(x,dy)\, d\mu(x)$$

$$
\begin{aligned}
&\geq \frac{1}{2t}\int_{\delta^2 B}\int_{\delta^2 B} p_t^B(x,y)(u(x)-u(y))^2\,d\mu(y)\,d\mu(x)\\
&\geq \frac{1}{2W(x_0,\delta R)}\int_{\delta^2 B}\int_{\delta^2 B}\frac{c}{V(x_0,\delta R)}(u(x)-u(y))^2\,d\mu(y)\,d\mu(x) \quad (\text{by (LLE)}_W)\\
&\geq \frac{c}{W(x_0,\delta^2 R)V(x_0,\delta^2 R)}\int_{\delta^2 B}\int_{\delta^2 B}(u(x)-u(y))^2\,d\mu(y)\,d\mu(x). \quad (\text{by (VD) and (2.2)}) \qquad (5.10)
\end{aligned}
$$

Next, for any $n \in \mathbb{N}$, we let $R_n = R - \frac{1}{n}$. Then $R_n \uparrow R$ and $B_n := B(x_0, R_n) \uparrow B$ as $n \to \infty$. Applying (5.10) for each B_n, we obtain

$$
\begin{aligned}
&\iint_{B\times B}|u(x)-u(y)|^2\,J(x,dy)\,d\mu(x)\\
&\quad\geq \iint_{\overline{B}_n\times\overline{B}_n}|u(x)-u(y)|^2\,J(x,dy)\,d\mu(x)\\
&\quad\geq \frac{c}{W(x_0,\delta^2 R_n)V(x_0,\delta^2 R_n)}\int_{\delta^2 B_n}\int_{\delta^2 B_n}|u(x)-u(y)|^2\,d\mu(y)\,d\mu(x)\\
&\quad\geq \frac{c}{W(x_0,\delta^2 R)V(x_0,\delta^2 R)}\int_{\delta^2 B_n}\int_{\delta^2 B_n}|u(x)-u(y)|^2\,d\mu(y)\,d\mu(x).
\end{aligned}
$$

Passing to the limit yields

$$
\iint_{B\times B}|u(x)-u(y)|^2\,J(x,dy)\,d\mu(x) \geq \frac{c}{W(x_0,\delta^2 R)V(x_0,\delta^2 R)}\int_{\delta^2 B}\int_{\delta^2 B}|u(x)-u(y)|^2\,d\mu(y)\,d\mu(x).
$$

By the above inequality and (3.18) (for $E = \delta^2 B$), we obtain the inequality (2.3) in condition $(\text{PI})_W$ for $\kappa := \delta^{-2}$ and for the ball $\kappa^{-1}B$. Since B and u are arbitrary, we obtain $(\text{PI})_W$. □

5.3 Proof of Theorem 2.13

Proof of Theorem 2.13. The implication of (i) $\Rightarrow$ (ii) is obvious. Assuming (ii), we then apply Theorem 2.9 and derive that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Hence, to show that (ii) $\Rightarrow$ (iii), it remains to observe that, under (VD) and (RVD), the following implication holds

$$
(\text{AB})_W + (\text{TJ})_W + (\text{PI})_W \Rightarrow (\text{LLE})_W, \qquad (5.11)
$$

which was proved in [24, Theorem 2.10].

Let us show that (iii) $\Rightarrow$ (i). If $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form, then Lemmas 5.3 and 5.5 imply

$$
\begin{aligned}
(\text{VD}) + (\text{LLE})_W &\Rightarrow (\text{S})_W \qquad (5.12)\\
&\Rightarrow (\text{AB}')_W,
\end{aligned}
$$

while by Lemma 5.10, we have

$$
(\text{VD}) + (\text{LLE})_W \Rightarrow (\text{PI})_W.
$$

Finally, if any of the conditions (i), (ii), (iii) holds, then Lemma 5.6 (see also [20, Lemma 4.6]) yields the stochastic completeness of $\{p_t\}_{t>0}$. □

5.4 Proof of Theorem 2.15

Proof of Theorem 2.15. Again, the implication of (i) $\Rightarrow$ (ii) is obvious. To prove the implication (ii) $\Rightarrow$ (iii), by Theorem 2.13 it suffices to show that

$$(\mathrm{AB})_{\mathrm{W}} + (\mathrm{TJ})_{\mathrm{W}} + (\mathrm{PI})_{\mathrm{W}} \Rightarrow (\mathrm{TP})_{\mathrm{W}},$$

which follows from (5.11), (5.12) and [23, Theorem 10.5].

It remains to show that (iii) $\Rightarrow$ (i). Again by Theorem 2.13 it suffices to verify that if $(\mathcal{E}, \mathcal{F})$ is regular, then

$$(\mathrm{TP})_{\mathrm{W}} \Rightarrow (\mathrm{TJ})_{\mathrm{W}},$$

which was proved in [23, Lemma 10.1]. □

5.5 Proof of Theorem 2.22

Proof of Theorem 2.22. Note that the implications of (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious.

Let us show that (ii) $\Rightarrow$ (iii). Based on the discussions in Section 3.1, we observe that, under the hypothesis $(\mathrm{J})_\beta$, the jump measure satisfies $(\mathrm{TJ})_{\mathrm{W}}$ and $(\mathrm{PI})_{\mathrm{W}}$ with $W(x, r) = r^\beta$. Since also $(\mathrm{AB})_{\mathrm{W}}$ is satisfied, Theorem 2.9 yields that the bilinear form $(\mathcal{E}, \mathcal{F})$ given by (2.14)-(2.15) is a regular Dirichlet form. Moreover, by [24, (2.33)] (see also [21, Theorem 2.10] and [17, Theorem 1.13]), we have in this setting that

$$(\mathrm{AB})_\beta \Leftrightarrow (\mathrm{ULE})_\beta. \tag{5.13}$$

Thus, the heat kernel $\{p_t\}_{t>0}$ of $(\mathcal{E}, \mathcal{F})$ satisfies $(\mathrm{ULE})_\beta$.

Let us verify (iv) $\Rightarrow$ (i). Suppose that $(\mathcal{E}, \mathcal{F})$ defined in (2.14) and (2.15) is a regular jump-type Dirichlet form, whose jump kernel J satisfies $(\mathrm{J})_\beta$ and heat kernel $\{p_t\}_{t>0}$ satisfies $(\mathrm{ULE})_\beta$. By (5.13) (see also [17, Theorem 1.13]), we see that $(\mathrm{AB})_\beta$ holds, which together with Corollary 2.16 gives also $(\mathrm{AB}')_\beta$.

Finally, if any of the conditions (i), (ii), (iii) and (iv) holds, then we obtain by Theorem 2.13 that the heat kernel $\{p_t\}_{t>0}$ is jointly Hölder continuous (by [17, Lemma 5.6]) and stochastically complete.

This concludes the proof of Theorem 2.22. □

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