Correction to “Sub-Gaussian estimates of heat kernels on infinite graphs” by A. Grigor’yan and A. Telcs

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This note contains a corrected version of Section 10 of the paper [4]. The purpose of that section in [4] was to prove the implication \((G) \Rightarrow (H)\) using \((G) \Rightarrow (HG) \Rightarrow (H)\). However, the proof of the first implication \((G) \Rightarrow (HG)\) contained an error. Despite that, the result \((G) \Rightarrow (H)\) remains true, which is proved below using a modified definition of \((HG)\).

10 The Harnack inequality and the Green kernel

Recall that the weighted graph \((\Gamma, \mu)\) satisfies the elliptic Harnack inequality \((H)\) if there exist constants \(H, K > 1\) such that, for all \(z \in \Gamma, R \geq 1,\) and for any nonnegative function \(u\) in \(B(z, KR)\) which is harmonic in \(B(z, KR)\), the following inequality is satisfied

\[
\max_{B(z,R)} u \leq H \min_{B(z,R)} u. \quad (H)
\]

Note that this inequality always holds for \(R < 1\) because in this case \(B(z,R) = \{z\}\).

In this section we establish that \((H)\) is implied by the condition \((G)\), where the latter means that \(C^{-1}d(x,y)^{-\gamma} \leq g(x,y) \leq Cd(x,y)^{-\gamma}, \ \forall x \neq y. \quad (G)\)

Consider the following Harnack inequality for the Green function \((HG)\): for some constants \(H', M > 2,\) for all \(z \in \Gamma, R \geq 1,\) and for any finite set \(U \supset B(z, MR),\)

\[
\max_{x \in B(z,R)} g_U(x,z) \leq H' \min_{y \in B(z,2R)} g_U(y,z). \quad (HG)
\]

It is easy to see that \((HG)\) can be equivalently stated as follows:

\[
\max_{B(z,2R) \setminus B(z,R)} g_U(\cdot,z) \leq H' \min_{B(z,2R) \setminus B(z,R)} g_U(\cdot,z).
\]

**Proposition 10.1** Assume that \((p_0)\) hold and the graph \((\Gamma, \mu)\) is transient. Then

\((G) \Rightarrow (HG) \Rightarrow (H)\).

The essential part of the proof is contained in the following lemma.

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1. It seems to be unknown whether in general condition \((H)\) with some value of \(K\) implies that for a smaller value of \(K\) (but possibly with a larger value of \(H\)). However, this is true in the presence of the doubling volume property.

2. A slightly different version of \((HG)\) – denote it by \((HG')\) – was considered in [5] and [1], where in the right hand side of \((HG)\) one takes the minimum over \(y \in B(z,R)\) rather than over \(y \in B(z,2R)\). It was shown in [1] that \((H) \Rightarrow (HG')\). It is easy to see that \((H) + (HG') \Rightarrow (HG)\) so that in fact \((H) \Rightarrow (HG)\). Proposition 10.1 contains the converse to that.
Lemma 10.2 Let $U_0 \subset U_1 \subset U_2 \subset U_3$ be a sequence of finite sets in $\Gamma$ such that $\overline{U}_i \subset U_{i+1}$, $i = 0, 1, 2$. Denote $A = U_2 \setminus U_1$, $B = U_0$ and $U = U_3$. Then, for any function $u$ which is nonnegative in $\overline{U}$ and harmonic in $U$, we have

$$\max_B u \leq H \min_B u,$$

where

$$H := \max_{x, y \in B} \max_{z \in A} \frac{g_U(x, z)}{g_U(y, z)}$$

(see Fig. 1).

Figure 1: The sets $B = U_0$, $A = U_2 \setminus U_1$ and $U = U_3$

Remark 10.1 Note that no a priori assumption has been made about the graph $(\Gamma, \mu)$ except for connectedness and unboundedness.

Proof. The following potential-theoretic argument is borrowed from [2]. Given a nonnegative function $u$ in $\overline{U}$, which is harmonic in $U$, denote by $S_u$ the following class of superharmonic functions in $U$:

$$S_u = \{ v : v \geq 0 \text{ in } \overline{U}, \ v \geq u \text{ in } \overline{U}_1, \ \text{ and } \Delta v \leq 0 \text{ in } U \},$$

and define the function $w$ on $\overline{U}$ by

$$w(x) = \min \{ v(x) : v \in S_u \}.$$  \hspace{1cm} (10.3)

Clearly, $w \in S_u$. Since the function $u$ itself is also in $S_u$, we have $w \leq u$ in $\overline{U}$. On the other hand, by definition of $S_u$, $w \geq u$ in $\overline{U}_1$, whence we see that $u = w$ in $\overline{U}_1$ (see Fig. 2). In particular, it suffices to prove (10.1) for $w$ instead of $u$.

Let us show that $w \in c_0(U)$, that is, $w$ vanish on $\overline{U} \setminus U$. Indeed, let $v(x)$ solve the Dirichlet problem

$$\begin{cases}
\Delta v = -1 & \text{in } U, \\
v = 0 & \text{on } \overline{U} \setminus U.
\end{cases}$$

Since $v$ is superharmonic, by the strong minimum principle $v$ is strictly positive in $U$. Hence, for a large enough constant $C$, we have $Cv \geq u$ in $\overline{U}_1$ whence $Cv \in S_u$ and $w \leq Cv$. Since $v = 0$ on $\overline{U} \setminus U$, this implies $w = 0$ on $\overline{U} \setminus U$. 

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Figure 2: The function $u$, a function $v \in S_u$ and the function $w = \min_{S_u} v$. The latter is harmonic in $U_1$ and in $U \setminus U_1$.

Set $f := -\Delta w$ and observe that by construction $f \geq 0$ in $U$. Since $w \in c_0(U)$, we have, for any $x \in U$,

$$w(x) = \sum_{z \in U} g_U(x, z) f(z) \mu(z).$$

(10.4)

Next we will prove that $f = 0$ outside $A$ so that the summation in (10.4) can be restricted to $z \in A$. Given that much, we obtain, for all $x, y \in B$,

$$\frac{w(x)}{w(y)} = \frac{\sum_{z \in A} g_U(x, z) f(z) \mu(z)}{\sum_{z \in A} g_U(y, z) f(z) \mu(z)} \leq H,$$

whence (10.1) follows.

We are left to verify that $w$ is harmonic in $U_1$ and outside $U_1$. Indeed, if $x \in U_1$ then

$$\Delta w(x) = \Delta u(x) = 0,$$

because $w = u$ in $U_1$. Let $\Delta w(x) \neq 0$ for some $x \in U \setminus U_1$. Since $w$ is superharmonic, we have $\Delta w(x) < 0$ and

$$w(x) > P w(x) = \sum_{y \sim x} P(x, y) w(y).$$

Consider the function $w'$ which is equal to $w$ everywhere in $\overline{U}$ except for the point $x$, and $w'$ at $x$ is defined to satisfy

$$w'(x) = \sum_{y \sim x} P(x, y) w'(y).$$

Clearly, $w'(x) < w(x)$, and $w'$ is superharmonic in $U$. Since $w' = w = u$ in $U_1$, we have $w' \in S_u$. Hence, by the definition (10.3) of $w$, $w \leq w'$ in $\overline{U}$ which contradicts $w(x) > w'(x)$.

Proof of Proposition 10.1. Let us prove $(G) \Rightarrow (HG)$. It will be sufficient to prove that if $U \supset B(z, MR)$ (where $M > 2$ is to be specified below) then

$$g_U(y, z) \geq \frac{1}{2} g(y, z) \quad \text{for all } y \in B(z, 2R).$$

(10.5)

Since also $g_U \leq g$, hypothesis $(G)$ and (10.5) will imply

$$\max_{x \in B(z, R)} g_U(x, z) \leq \max_{x \in B(z, R)} g(x, z) \leq C \min_{y \in B(z, 2R)} g(y, z) \leq 2C \min_{y \in B(z, 2R)} g_U(x, z).$$
The proof of (10.5) follows the approach of [3]. Consider the function \( u = g(\cdot, z) - g_U(\cdot, z) \) which is nonnegative and harmonic in \( U \). Since outside \( U \) the function \( u \) coincides with \( g(\cdot, z) \), we obtain by the maximum principle and \((G)\) that
\[
\max_U u = \max_{U^c} u = \max_{U^c} g(\cdot, z) \leq C (MR)^{-\gamma}.
\]
Therefore, for \( y \in B(x, 2R) \),
\[
g(y, z) \geq C^{-1} (2R)^{-\gamma} \geq 2C (MR)^{-\gamma} \geq 2 \max u
\]
provided \( M \) is large enough, whence it follows that
\[
g_U(y, z) \geq g(y, z) - \max u \geq \frac{1}{2} g(y, z).
\]
Let us prove \( (HG) \Rightarrow (H) \). Fix a point \( x_0 \in \Gamma \) and write for shortness \( B_r := B(x_0, r) \). Let \( u \) be a nonnegative harmonic function in \( U := B_{6MR} \), where \( R > 1 \). By Lemma 10.2, we have
\[
\max_{B_R} u \leq H \min_{B_R} u, \tag{10.6}
\]
where
\[
H := \max_{x,y \in B_R} \max_{z \in A} \frac{g_U(x, z)}{g_U(y, z)}, \tag{10.7}
\]
and \( A = B_{5R} \setminus B_{4R} \) (see Fig. 1). Let us show that \( H \leq H' \) where \( H' \) is the constant from \((HG)\). Indeed, if \( x, y \in B_R \) and \( z \in A \) then it is easy to see that \( x \in B(z, 3R)^c \) and \( y \in B(z, 6R) \). Since \( 5R + 3MR < 6MR \), we see that \( B(z, 3MR) \subset U \). By \((HG)\) we obtain, for all \( x, y \in B_R \),
\[
g_U(x, z) \leq H' g_U(y, z).
\]
Substituting into (10.7), we obtain that \((H)\) holds with \( K = 6M \) and \( H = H' \).

References