HEAT KERNELS, VOLUME GROWTH AND ANTI-ISOPERIMETRIC INEQUALITIES

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Abstract: we give on-diagonal large time lower bounds for heat kernels on non-compact Riemannian manifolds.

Résumé: nous donnons des bornes inférieures sur la diagonale et en grand temps pour le noyau de la chaleur sur les variétés non-compactes.

Version française abrégée: Soit M une variété non-compacte, et $p_t(x,y)$, $x,y \in M$, $t>0$ son noyau de la chaleur. Il est bien connu [A], [PE], [LY] qu'une des conséquences de l'inégalité de Harnack parabolique globale est une borne inférieure du noyau de la chaleur du type

$$p_t(x,y) \geq \frac{\text{const}}{V(x,\sqrt{t})} \exp\left(-\frac{d^2}{Ct}\right) \tag{0.1}$$

où $d = \text{dist}(x,y)$, $V(x,r)$ désigne le volume de la boule $B(x,r)$ de rayon r centrée en x, et $C, \text{const} > 0$ dépendent de la constante de Harnack. Une estimation supérieure du même style est alors vraie.

En particulier, (0.1) entraîne une borne inférieure sur la diagonale en termes de la fonction de croissance du volume

$$p_t(x,x) \geq \frac{\text{const}}{V(x,\sqrt{t})} \tag{0.2}$$

qui est optimale.

L'inégalité de Harnack fournit une information précise sur le noyau de la chaleur, mais elle ne peut être vraie que sous des conditions assez contraignantes sur la variété: on sait qu'elle équivaut à la conjonction d'une famille d'inégalités de Poincaré et de la propriété de doublement du volume (voir [S] , [G1]).

Si l'on s'intéresse seulement à une borne supérieure du noyau de la chaleur, il suffit d'avoir l'une des inégalités fonctionnelles équivalentes: inégalité de Sobolev, de Nash, de Faber-Krahn, inégalité de Sobolev logarithmique (voir [N] , [V] , [CKS] , [D1] , [G3] , [C1] , [BCLS]) qui sont nettement plus faibles que les inégalités de Poincaré.

En ce qui concerne les bornes inférieures, les conditions géométriques qui les assurent étaient inconnues jusqu'à présent. Nous n'avons pas fait de progrès sur les minorations en dehors de la diagonale. En revanche, nous avons obtenu des minorations sur la diagonale sous des hypothèses raisonnables.

Le premier pas dans cette direction est dû à F. Lust-Piquard [L] , qui a montré que des hypothèses sur la fonction de croissance du volume suffisent à obtenir des minorations du noyau de la chaleur sur la diagonale. Nous affaiblissons ces hypothèses et nous donnons des conclusions optimales.

Nous obtenons des bornes inférieures d'une part pour $p_t(x,x)$, où x est fixé dans M, d'autre part pour $\sup_{x\in M} p_t(x,x)$.

Nous faisons des hypothèses de deux sortes: dans un premier groupe de résultats, nous partons d'une borne supérieure sur la fonction $V(x,r)$, pour r grand, et d'une information sur la géométrie à l'intérieur d'une boule $B(x,r_0)$ de rayon fixé. Nous obtenons alors des bornes inférieures d'une part pour $p_t(x,x)$, où x est fixé dans M, d'autre part pour

$\sup_{x\in M} p_t(x,x)$. Dans le second cas, des hypothèses moins fortes sur la géométrie de $B(x,r_0)$ suffisent.

La deuxième famille d'hypothèses consiste en des inégalités de type anti-isopérimétrique. On sait qu'une inégalité de Faber-Krahn (qui minore la première valeur propre de Dirichlet d'un domaine par une fonction de son volume) implique une borne supérieure du noyau de la chaleur. Nous introduisons des anti-inégalités de Faber-Krahn, qui signifient qu'il existe une famille d'ensembles dont le volume décrit $\mathbb{R}_+^*$ et dont la première valeur propre de Dirichlet est majorée par une fonction du volume. Nous montrons qu'une anti-inégalité de Faber-Krahn entraîne une minoration de $\sup_{x\in M} p_t(x,x)$, et qu'elle est sous certaines conditions entraînée par une inégalité anti-isopérimétrique. Une partie de ces résultats vaut dans le cadre des semi-groupes sous-markoviens symétriques; ils s'étendent aussi aux chaînes de Markov sur les graphes.

La présente note est développée dans [CG] .

I. Introduction

Let us consider a complete non-compact Riemannian manifold M and the heat kernel $p_t(x,y)$ on M where $x,y\in M, t>0$. We discuss here lower bounds of $p_t(x,x)$ in terms of geometric properties of the manifold.

It is well known [A] , [PE] , [LY] that a good lower bound of the heat kernel follows from the Harnack inequality. Namely, a uniform parabolic Harnack inequality implies for all $x,y\in M$ and all $t>0$

$$p_t(x,y)\geq \frac{\text{const}}{V(x,\sqrt{t})}\exp\left(-\frac{d^2}{Ct}\right) \tag{1.1}$$

where $d=\text{dist}(x,y)$, $V(x,r)$ denotes the volume of the ball $B(x,r)$ of radius r centred at the point x, and the constants $C,\text{const}>0$ depend on the Harnack constant. In particular, (1.1) implies an optimal on-diagonal bound

$$p_t(x,x)\geq \frac{\text{const}}{V(x,\sqrt{t})}. \tag{1.2}$$

On the other hand, the Harnack inequality imposes rather rigid restrictions on the manifold - it requires a certain Poincaré inequality and the doubling volume property (see [S] , [G1]).

If one is only interested an upper bound of the heat kernel, then it suffices to know one out of the family of equivalent functional inequalities such as Sobolev inequality, Nash inequality, logarithmic Sobolev inequality, and Faber-Krahn inequality (see [N] , [V] , [CKS] , [D1] , [G3] , [C1] , [BCLS]) which are substantially weaker than the Poincaré inequality.

It was not known until recently which geometric properties are responsible for lower bounds. The question of the off-diagonal bounds is still open, and we do not touch it here. On the contrary, we have obtained on-diagonal lower bounds under nearly optimal assumptions.

The first step in that direction is due to F.Lust-Piquard [L] who has proved that certain assumptions about the volume growth alone imply heat kernel on-diagonal bounds. Our results go far beyond those of [L] both in making weaker hypotheses and in proving stronger conclusions. We show on counter-examples that our lower bounds are optimal.

We obtain two kinds of estimates:

- a lower bound for $p_t(x,x)$ for a fixed x which will be referred to as a pointwise lower bound;
- a lower bound for $\sup_{x\in M} p_t(x,x)$ which will be referred to as a sup-lower bound.

As was shown by [D2] , $p_t(x,x)$ and $\sup_x p_t(x,x)$ may have different asymptotics as $t \to \infty$.

The assumptions we make are also of two kinds.

1. We prove both pointwise and sup-lower bounds under the hypothesis that we are given
   - an upper bound of the volume function $V(x,r)$ for all large r's;
   - certain information about the geometry inside a ball $B(x,r_0)$ of a small fixed radius $r_0 > 0$.
2. We introduce a so called anti-Faber-Krahn inequality which means that there is a family of regions whose volume ranges over $\mathbb{R}_+^*$ and for which the first Dirichlet eigenvalue is bounded from above by a certain function of the volume. For example, in $\mathbb{R}^n$ all balls with a fixed centre form such a family. Our results state that an anti-Faber-Krahn inequality implies a sup-lower bound of the heat kernel.

These results can be proved also in the context of the abstract semigroup theory.

II. Volume growth

Let us fix a point $z \in M$ and some $r_0 > 0$ and suppose that for all $r > r_0$

$$V(z,r) \le v(r) \tag{2.1}$$

$v(r)$ being a continuous positive increasing function on $(r_0, +\infty)$. Let us assume that

$$\frac{r^2}{\log v(r)} \tag{2.2}$$

is strictly increasing on $(r_0, +\infty)$ and tends to ∞ as $r \to \infty$ (and also $v(r_0) > 2$). Then the function (2.2) has an inverse function $\mathcal{R}(t)$ defined on an interval (t_0, ∞).

Theorem 1 *Let us assume in addition to the above hypotheses that for some $v_0 > 0$*

$$V(z,r_0) \ge v_0. \tag{2.3}$$

Then we have for all $t > t_0$ and some $a = a(r_0, v_0) > 0$

$$\sup_{x\in M} p_t(x,x) \ge \frac{0.5}{V(z,\mathcal{R}(at))}. \tag{2.4}$$

Basically, this theorem says that a sup-lower bound of the heat kernel follows from an upper bound of the volume function at a single point z. The additional assumption (2.3) holds always - we can take $v_0 = V(z,r_0)$ - but we would like to underline that the constant a in (2.4) depends on it.

Under a stronger assumption, we can get the pointwise lower bound for $p_t(z,z)$. Let us denote by $\lambda_1(\Omega)$ the first Dirichlet eigenvalue of the Laplace operator in a region Ω.

Theorem 2 *Let us assume in addition to the above hypotheses that for some $r_0, \kappa, \nu > 0$ and for any region $\Omega \subset B(z, r_0)$, a Faber-Krahn type inequality is true*

$$\lambda_1(\Omega) \geq \kappa(|\Omega|)^{-\nu}. \tag{2.5}$$

Then we have for any $t > t_0$ and some $b = b(r_0, \nu, \kappa) > 0$

$$p_t(z, z) \geq \frac{0.5}{V(z, \mathcal{R}(bt))}. \tag{2.6}$$

Let us note that the inequality (2.5) is always true with some positive κ (which in general depends on z) and $\nu = \frac{2}{n}$ where $n = \dim M$. This follows from the fact that it holds in the Euclidean space and extends to manifolds by a compactness argument. Hence, if we fix a point z then the lower bound (2.6) of the heat kernel is determined just by the upper bound of the volume function.

On the other hand, there is a counter-example which shows that it is impossible to get *any* pointwise lower bound for $p_t(z, z)$ only in terms of the upper bound function $v(r)$. Therefore, the constant b must really depend on the local geometry near the point z.

Examples. 1. Let $v(r) = r^N$, then the function $\mathcal{R}(t)$ looks as follows

$$\mathcal{R}(t) \sim \sqrt{t \log t}, \tag{2.7}$$

and we have by (2.6)

$$p_t(z, z) \geq \frac{\text{const}}{(t \log t)^{\frac{n}{2}}}.$$

In view of (1.2) , one may expect to have $\mathcal{R}(t) \sim \sqrt{t}$. It turns out that it cannot be achieved in general - our examples show that the function (2.7) is optimal.

2. Let $v(r) = e^{r^\alpha}$, $\alpha < 2$. Then $\mathcal{R}(t) \sim t^{\frac{1}{2-\alpha}}$ and

$$p_t(z, z) \geq \text{const} \exp(-\text{const} \cdot t^{\frac{\alpha}{2-\alpha}}). \tag{2.8}$$

The exponent $\frac{\alpha}{2-\alpha}$ in (2.8) is optimal for $\alpha \leq 1$.

Let us mention for comparison the result of [L] which gives a less sharp value $\frac{\alpha(1+\alpha)}{2}$ for the pointwise lower bound and the correct exponent $\frac{\alpha}{2-\alpha}$ for the sup-lower bound.

Finally, we discuss what assumptions ensure the double sided heat kernel estimate

$$\frac{C_1}{V(x, \sqrt{t})} \leq p_t(x, x) \leq \frac{C_2}{V(x, \sqrt{t})}. \tag{2.9}$$

For example, this inequality holds on manifolds of non-negative Ricci curvature [LY] . It also follows from the Harnack inequality whenever the latter is true (see [G2]).

Theorem 3 *Let us suppose that for some point $x \in M$, for some $\alpha, \beta > 0$, for all $R > 0$ and for all compact domains $\Omega \subset B(x, R)$*

$$\lambda_1(\Omega) \geq \frac{\alpha}{R^2} \left(\frac{V(x, R)}{|\Omega|} \right)^\beta. \tag{2.10}$$

Then the inequality (2.9) holds for all $t > 0$ with constants $C_{1,2} > 0$ depending on α, β.

The condition (2.10) is in some sense necessary for (2.9) . Indeed, as was proved in [G3] (Proposition 5.2) if we assume for all $x \in M$ and all $t > 0$ an upper bound $p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}$ as well as the doubling volume property, then (2.10) follows for all points x.

III. Anti-isoperimetric inequalities

Suppose that we are given a family of regions Ω_v (where v is a continuous parameter) whose volume is not greater than v and whose first eigenvalues $\lambda_1(\Omega_v)$ have a suitable upper bound in terms of v. Then we will derive a lower bound of $\sup_x p_t(x,x)$. The core of all estimates below is the following proposition which is true also in a very general setting of abstract semigroups.

Proposition 4 *We have always*

$$\sup_x p_t(x,x) \geq \sup_\Omega \frac{e^{-\lambda_1(\Omega)t}}{|\Omega|}.$$

Corollary 5 *Let us suppose that for a point $z \in M$, there exist an increasing one-to-one function φ from $(0,+\infty)$ onto itself such that $\lambda_1(B(z,r)) \leq \frac{1}{\varphi(r)}$ for any $r > 0$. Then for all $t > 0$*

$$\sup_x p_t(x,x) \geq \frac{e^{-1}}{V(z,\varphi^{-1}(t))}. \tag{3.1}$$

One can show that the doubling volume property

$$V(z,2r) \leq CV(z,r) \tag{3.2}$$

implies the estimate $\lambda_1(B(z,r)) \leq \frac{\text{const}}{r^2}$. Therefore, we have

Theorem 6 *Suppose that for some point $z \in M$, the doubling volume property (3.2) holds for all $r > 0$. Then for all $t > 0$*

$$\sup_x p_t(x,x) \geq \frac{e^{-4C}}{V(z,\sqrt{t})}.$$

Let us note that this result is stronger than the corresponding estimate in [L] (theorem 13). The latter required in addition to the doubling volume property also an upper bound of the volume function as (2.1) with $v(r) = r^N$ and yielded the heat kernel estimate in terms of the function $v(r)$ rather than in terms of $V(z,r)$.

In the next theorem, we assume the existence of a family Ω_v where v ranges over $(0,\infty)$ such that for any $v > 0$ we have $|\Omega_v| \leq v$ and $\lambda_1(\Omega_v) \leq \Lambda(v)$ where the function $\Lambda(v)$ is a positive continuous decreasing function on $(0,+\infty)$.

Given the function $\Lambda(\cdot)$, we associate with it another function $\gamma(t)$ from the identity

$$t = \int_0^{\gamma(t)} \frac{dv}{v\Lambda(v)} \tag{3.3}$$

(assuming that the integral in (3.3) converges at 0). The function $\gamma(t)$ is an increasing differentiable function which tends to ∞ as $t \to \infty$. Let us impose the following regularity condition on the function γ which will be referred to as (D) condition:

(D) *there exists $\alpha > 0$ such that for all $t > 0$ and all $s \in (t,2t)$ we have $f(s) \geq \alpha f(t)$ where $f(t) = \frac{\gamma'(t)}{\gamma(t)}$.*

This condition does not restrict the rate of increasing $\gamma(t)$ as $t \to \infty$.

Theorem 7 *Under the above assumptions we have for all $t > 0$*

$$\sup_x p_t(x,x) \geq \frac{1}{\gamma(\frac{2}{\alpha}t)}. \tag{3.4}$$

Examples. 1. Let $\Lambda(v) = v^{-\nu}$ (in $\mathbb{R}^n$ we have $\nu = \frac{2}{n}$). Then we have by (3.3) $\gamma(t) = \mathrm{const} t^{\frac{1}{\nu}}$ and by (3.4) $\sup_x p_t(x,x) \geq \mathrm{const} t^{-\frac{1}{\nu}}$. This estimate is optimal as seen in $\mathbb{R}^n$.

2. Let $\Lambda(v) = \log^{-\beta} v$, $\beta > 0$ for large v (and is as in the first example for small v). Then we have, for large t, $\gamma(t) \sim e^{\mathrm{const}\cdot t^{\frac{1}{\beta+1}}}$ and respectively, $\sup_x p_t(x,x) \geq \mathrm{const} e^{-\mathrm{const}\cdot t^{\frac{1}{\beta+1}}}$ which appears to be optimal as well.

Remark. If $\lambda_1(\Omega) \geq \Lambda(|\Omega|)$ for any compact region $\Omega \subset M$, then it implies ([G3] , [C2]) an upper bound of the heat kernel $p_t(x,y) \leq \frac{\mathrm{const}}{\gamma(ct)}$. This shows that the results in both directions are optimal up to constant multiples.

Let us now consider an anti-isoperimetric inequality between the boundary area and the volume. Let us denote $S(x,R) = \mathrm{Area}\partial B(x,R)$. We say that a $\varphi-$anti-isoperimetric inequality holds at the point x, if, for all $r > 0$, $S(x,r) \leq \frac{V(x,r)}{\varphi(V(x,r))}$.

Proposition 8 *If for some point $x \in M$ a $\varphi-$anti-isoperimetric inequality is true and for all $r > 0$ the area $S(x,r)$ is increasing in r, then $\lambda_1(B(x,r)) \leq \frac{4}{\varphi^2(V(x,r))}$.*

Together with Theorem 7 it implies

Theorem 9 *Suppose that a $\varphi-$anti-isoperimetric inequality holds at some point $x_0 \in M$, and let $S(x_0,r)$ be increasing in r. Let us define a function $\gamma(t)$ by*

$$t = \int_0^{\gamma(t)} \frac{\varphi^2(v)}{v} dv,$$

and let $\gamma(t)$ satisfy the condition (D). Then we have for all $t > 0$

$$\sup_{x \in M} p_t(x,x) \geq \frac{1}{\gamma(\frac{8}{\alpha}t)}.$$

A similar theory can be developed for Markov chains on graphs.

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