On homology theories of cubical digraphs

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Abstract

We prove the equivalence of the singular cubical homology and the path homology on the category of cubical digraphs. As a corollary we obtain new relations between the singular cubical homology of digraphs and simplicial homology.

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1 Introduction

The path homology theory and the singular cubical homology theory for the category of digraphs were introduced recently in the papers [1], [2], [3], and [4]. In this category, there is a natural transformation of the cubical homology theory to the path homology theory, that induces an isomorphism of homology groups in dimensions 0 and 1. Additionally, in [1] is given an example of a digraph for which the path homology are trivial in dimension 2 but singular cubical homology are non-trivial in this dimension.

In this paper we prove the equivalence of the singular cubical homology and the path homology theories on the category of cubical digraphs. As an intermediate result we prove that the image of every map of a digraph cube to a cubical digraph is contractible. As a corollary we obtain a relation of the singular cubical homology of digraphs to simplicial homology.

The paper is organized as follows. In Section 2, we recall the basic definitions from graph theory and describe some properties of singular cubical homology $H_c^*$ and the path homology $H_p^*$ on the category of digraphs [1], [2], [3], and [4].

In Section 3, we recall the definition of cubical digraph from [4] and prove contractibility of the image of a digraph cube in a cubical digraph for any digraph map. Then we state and prove the main result of the paper:

**Theorem 1.1.** On the category of cubical digraphs the singular cubical homology theory is equivalent to the path homology theory.

Then we obtain several corollaries that describe relation of the singular cubical homology theory of digraphs to simplicial homology.
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2 Singular cubical and path homology theories

In this Section we give necessary preliminary material about digraphs and homology theories on the category of digraphs. We shall consider only finite digraphs in the paper.

Definition 2.1. A digraph $G$ is a pair $(V_G, E_G)$ of a set $V = V_G$ of vertices and a subset $E_G \subset \{V_G \times V_G \setminus \text{diagonal}\}$ of ordered pairs $(v, w)$ of vertices which are called arrows and are denoted $v \rightarrow w$. The vertex $v = \text{orig}(v \rightarrow w)$ is called the origin of the arrow and the vertex $w = \text{end}(v \rightarrow w)$ is called the end of the arrow.

For two vertices $v, w \in V_G$, we write $v \equiv w$ if either $v = w$ or $v \rightarrow w$.

A subgraph $H$ of a digraph $G$ is a digraph whose set of vertices is a subset of that of $G$ and set edges of $H$ is the subset of edges of $G$. In this case we write $G \subset H$.

An induced subgraph $H$ of a digraph $G$ is a digraph whose set of vertices is a subset of that of $G$ and the edges of $H$ are all those edges of $G$ whose adjacent vertices belong to $H$. In this case we write $G \sqsubset H$.

A directed path $p = (a_1, \alpha_1, a_2, \alpha_2, \ldots, \alpha_n, a_{n+1})$ in a digraph $G$ is a sequence of vertices $a_i$ and arrows $\alpha_i$ such that $\alpha_i = (a_i \rightarrow a_{i+1})$. The number of arrows fitting in path is called length of the path and is denoted by $|p|$. The vertex $a_1$ is the origin of the path and the vertex $a_{n+1}$ is the end of the path.

Definition 2.2. A digraph map (or simply map) from a digraph $G$ to a digraph $H$ is a map $f : V_G \rightarrow V_H$ such that $v \equiv w$ in $G$ implies $f(v) \equiv f(w)$ in $H$.

A digraph map $f$ is non-degenerate if $v \rightarrow w$ on $G$ implies $f(v) \rightarrow f(w)$ on $H$.

The set of all digraphs with digraph maps form the category of digraphs that will be denoted by $D$.

Definition 2.3. For digraphs $G, H$ define their Box product $\Pi = G \sqcap H$ as a digraph with a set of vertices $V_\Pi = V_G \times V_H$ and a set of arrows $E_\Pi$ given by the rule

$$(x, y) \rightarrow (x', y') \quad \text{if} \quad x = x' \text{ and } y = y', \text{ or } x \rightarrow x' \text{ and } y = y',$$

where $x, x' \in V_G$ and $y, y' \in V_H$.

Fix $n \geq 0$. Denote by $I_n$ a digraph with the set of vertices $V = \{0, 1, \ldots, n\}$ and, for $i = 0, 1, \ldots, n - 1$, there is exactly one arrow $i \rightarrow i + 1$ or $i + 1 \rightarrow i$ and there are no others arrows. Such digraph we call a line digraph and a direct line digraph if additionally all arrow have the form $i \rightarrow i + 1$. There are only two line digraphs with two vertices. We denote the digraph $0 \rightarrow 1$ by $I$.

For $n \geq 0$, a standard $n$-cube digraph $I^n$ is defined as follows. For $n = 0$ we put $I^0 = \{0\}$ — one-vertex digraph. For $n \geq 1$, $I^n$ is given by a set $V$ of $2^n$ vertices such that any vertex $a \in V$ can be identified with a sequence $a = (a_1, \ldots, a_n)$ of binary digits so that $a \rightarrow b$ if and only if the sequence $b = (b_1, \ldots, b_n)$ is obtained from $a = (a_1, \ldots, a_n)$ by replacing a digit 0 by 1 at exactly one position. The digraph $0 \rightarrow 1$ is an 1-cube and we call a square any digraph that is isomorphic the standard 2-cube digraph.
We shall call an \( n \)-cube digraph any digraph that is isomorphic to the standard \( n \)-cube. Note an \( n \)-cube digraph is isomorphic to the digraph
\[
I^n = \underbrace{\square \square \square \ldots \square}_{n \text{-times}}.
\]

The notion of homotopy in the category of digraphs was introduced in [2]. Now we recall several definitions which we shall use in the paper.

**Definition 2.4.** Two digraph maps \( f, g : G \to H \) are called homotopic if there exists a line digraph \( I_n \) with \( n \geq 1 \) and a digraph map \( F : G \square I_n \to H \) such that
\[
F|_{G \square \{0\}} = f \quad \text{and} \quad F|_{G \square \{n\}} = g
\]
where we identify \( G \square \{0\} \) and \( G \square \{n\} \) with \( G \) by the natural way. In this case we shall write \( f \simeq g \). The map \( F \) is called a homotopy between \( f \) and \( g \).

In the case \( n = 1 \) we refer to the map \( F \) as an one-step homotopy.

**Definition 2.5.** Digraphs \( G \) and \( H \) are called homotopy equivalent if there exist maps
\[
f : G \to H, \quad g : H \to G
\]
such that
\[
f \circ g \simeq \text{id}_H, \quad g \circ f \simeq \text{id}_G.
\]
In this case we shall write \( H \simeq G \) and the maps \( f \) and \( g \) are called homotopy inverses of each other.

A digraph \( G \) is called contractible if \( G \simeq \{*\} \) where \( \{*\} \) is a one-vertex digraph.

**Definition 2.6.** [2, Def. 3.4] Let \( G \) be a digraph and \( H \) be its subgraph.

(i) A retraction of \( G \) onto \( H \) is a map \( r : G \to H \) such that \( r|_H = \text{id}_H \).

(ii) A retraction \( r : G \to H \) is called a deformation retraction if \( i \circ r \simeq \text{id}_G \), where \( i : H \to G \) is the natural inclusion.

**Proposition 2.7.** [2, Corollary 3.7] Let \( r : G \to H \) be a retraction of a digraph \( G \) onto a sub-digraph \( H \) and
\[
x \equiv r(x) \quad \text{for all} \quad x \in V_G \quad \text{or} \quad r(x) \equiv x \quad \text{for all} \quad x \in V_G.
\]
Then \( r \) is a deformation retraction, the digraphs \( G \) and \( H \) are homotopy equivalent, and \( i, r \) are their homotopy inverses.

Now we recall the definitions of path homology groups from [4] and singular cubical homology groups from [1] on digraphs with the group of coefficients \( \mathbb{Z} \). Let \( V \) be a finite set, whose elements will be called vertices. An \textit{elementary} \( p \)-path on a finite set \( V \) is any (ordered) sequence \( i_0, \ldots, i_p \) of \( p + 1 \) vertices of \( V \) that will be denoted by \( e_{i_0 \ldots i_p} \). Denote by \( \Lambda_p = \Lambda_p(V) \) the free abelian group generated by all elementary \( p \)-paths \( e_{i_0 \ldots i_p} \). The elements of \( \Lambda_p \) are called \( p \)-paths. Thus each \( p \)-path \( v \in \Lambda_p \) has the form
\[
v = \sum_{i_0, \ldots, i_p \in V} v^{i_0 \ldots i_p} e_{i_0 \ldots i_p},
\]
where \( v^{i_0 \ldots i_p} \in \mathbb{Z} \) are the coefficients of \( v \).
For $p \geq 0$, define the boundary operator $\partial: \Lambda_{p+1} \to \Lambda_p$ on basic elements by

$$\partial e_{i_0 \ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{\hat{k}i_0 \ldots \hat{k}i_q \ldots i_{p+1}},$$

(2.2)

where $\hat{k}$ means deleting of the corresponding index, and extend it to $\Lambda_{p+1}$ by linearity. Let $\Lambda_{-1} = 0$, and define $\partial: \Lambda_0 \to \Lambda_{-1}$ by $\partial v = 0$ for all $v \in \Lambda_0$. It follows from this definition that $\partial^2 v = 0$, for any $p$-path $v$.

An elementary $p$-path $e_{i_0 \ldots i_p}$ ($p \geq 1$) is called regular if $i_k \neq i_{k+1}$ for all $k$. For $p \geq 1$, let $I_p$ be the subgroup of $\Lambda_p$ that is spanned by all irregular $e_{i_0 \ldots i_p}$, and we set $I_0 = I_{-1} = 0$. Then $\partial(I_{p+1}) \subset I_p$ for $p \geq -1$. Consider the chain complex $\mathcal{R}_p$ with

$$\mathcal{R}_p = \mathcal{R}_p(V) = \Lambda_p/I_p$$

and with the chain map that is induced by $\partial$.

Now we define paths on a digraph $G = (V, E)$. Let $e_{i_0 \ldots i_p}$ be a regular elementary $p$-path on $V$. It is called allowed if $i_{k-1} \to i_k$ for any $k = 1, \ldots, p$, and non-allowed otherwise. For $p \geq 1$, denote by $\mathcal{A}_p = \Lambda_p(G)$ the subgroup of $\mathcal{R}_p$ spanned by the allowed elementary $p$-paths, that is,

$$\mathcal{A}_p = \text{span}\{e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed}\}.$$

and set $\Lambda_{-1} = 0$. The elements of $\mathcal{A}_p$ are called allowed $p$-paths.

Consider the following subgroup of $\mathcal{A}_p$ ($p \geq 0$)

$$\Omega_p = \Omega_p(G) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}.$$

(2.3)

The elements of $\Omega_p$ are called $\partial$-invariant $p$-paths, and we obtain a chain complex

$$0 \leftarrow \Omega_0 \xrightarrow{\partial} \Omega_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_{p-1} \xrightarrow{\partial} \Omega_p \xrightarrow{\partial} \cdots$$

(2.4)

The homology groups of the digraph $G$ are defined as

$$H_p(G) := \ker \partial|_{\Omega_p}/\text{Im} \partial|_{\Omega_{p+1}}.$$

In what follows, we will refer to $H_p(G)$ as the path homology groups of a digraph $G$.

We can define a natural augmentation

$$\varepsilon: \Omega_0 \to \mathbb{Z} \text{ by } \varepsilon\left(\sum k_i e_i\right) = \sum k_i, \ k_i \in \mathbb{Z}$$

which is an epimorphism and $\varepsilon \circ \partial = 0$.

Now we recall the construction of the cubical singular homology theory of digraphs from [1].

Let $I^n$ be the standard $n$-cube digraph. A singular $n$-cube in a digraph $G$ is a digraph map $\phi: I^n \to G$.

Fix $n \geq 1$. For any $1 \leq j \leq n$ and $\epsilon = 0, 1$, consider the following inclusion of cuboids:

$$F_{je}^{n-1}: I^{n-1} \to I^n,$$

$$F_{je}^{n-1}(c_1, \ldots, c_{n-1}) = (c_1, \ldots, c_j-1, \epsilon, c_{j+1}, \ldots, c_n)$$

(2.5)

for $n \geq 2$, and $F_{1\epsilon}^{n-1}(0) = (\epsilon)$ for $n = 1$. We shall write shortly $F_{je}$ instead of $F_{je}^{n-1}$ if the dimension $n - 1$ is clear from the context. Denote by $I_{je}^{n-1}$ the image of $F_{je}^{n-1}$. We shall write $I_{je}$ instead $I_{je}^{n-1}$ if the dimension is clear from the context.
Let $Q_{-1} = 0$. For $n \geq 0$, denote $Q_n = Q_n(G)$ the free abelian group generated by all singular $n$-cubes in $G$, and denote $\phi^\square$ the singular $n$-cube $\phi$ as the element of the group $Q_n$. For $n \geq 1$ and $1 \leq p \leq n$, and

$$\phi^\square_{ep} = (\phi \circ F_{ep})^\square \in Q_{n-1}. \quad (2.6)$$

For $n \geq 1$, define a homomorphism $\partial^c: Q_n \to Q_{n-1}$ on the basis elements $\phi^\square$ by the rule

$$\partial^c(\phi^\square) = \sum_{p=1}^{n} (-1)^p \left( \phi^\square_{p0} - \phi^\square_{p1} \right), \quad (2.7)$$

and $\partial^c = 0$ for $n = 0$. Then $(\partial^c)^2 = 0$ and the groups $Q_n(G)$ form a chain complex which we denote $Q^* = Q_*(G)$.

For $n \geq 1$ and $1 \leq p \leq n$, consider the natural projection $T^p: I^n \to I^{n-1}$ on the $p$-face $I^{n-1}$ defined as follows. For $n = 1$, $T^1$ is the unique digraph map $I^1 \to I^0$. For $n \geq 2$, we have on the set of vertices $T^p(i_1, \ldots, i_1) = (i_1, \ldots, i_{p+1}, i_{p-1}, \ldots, i_1)$. The singular $n$-cube $\phi: I^n \to G$ is degenerate if there is $1 \leq p \leq n$ such that $\phi = \psi \circ T^p$ where $\psi: I^{n-1} \to G$ is a singular $(n-1)$-cube. Then an abelian group $B_n = B_n(G)$ that is generated by all degenerated $n$-cubes is a subgroup $Q_n$ for $n \geq 1$. We put also $B_0 = 0, B_{-1} = 0$. Then the quotient group

$$Q^*_p(G) = Q_p(G)/B_p(G) \quad (2.8)$$

is defined for $p \geq 0$. We have $\partial(B_n) \subset B_{n-1}$ and, hence, $B_*(G) \subset Q_*(G)$. Hence the factor complex $Q^*_*(G) = Q_*(G)/B_*(G)$ is defined. We continue to denote the differential in this complex $\partial^c$. The homology group $H_k(Q^*_c(G))$ is called the singular cubical homology group of digraph $G$ in dimension $k$ and is denoted $H^c_k(G)$. We have a natural augmentation homomorphism

$$\varepsilon: Q^*_c(G) \to \mathbb{Z}, \quad \varepsilon \left( \sum_k k_i \phi_i \right) = \sum k_i, \quad k_i \in \mathbb{Z}$$

which is an epimorphism and $\varepsilon \circ \partial^c = 0$.

Recall the basic properties of the path and the singular cubical homology groups (see [4] and [1]).

- The groups $H^*_c(X)$ and $H_*(X)$ are functors from the category $\mathcal{D}$ to the category of abelian groups.

- Let $f \simeq g: X \to Y$ be two homotopic digraph maps. Then the induced homomorphisms $f_*, g_*$ of homology groups are equal for $k \geq 0$ for both theories.

## 3 Maps of cube to cubical digraph

In this section we reformulate slightly the definition of a cubical digraph from [4] and prove that the image of a cube in a cubical digraph is contractible. Then we prove Theorem 1.1.

Recall, that any vertex of a cube $I^n$ is given by a sequence of binary numbers $(a_1, \ldots, a_n)$. For any arrow $a \to b$ in a digraph cube $I^n$ we have also the arrow

$$\gamma_i = (0, \ldots, 0) \to (b_1 - a_1, \ldots, b_n - a_n) \quad (3.1)$$

in $I^n$ where right sequence of binary numbers presents a vertex in $I^n$ which has only one non-trivial element 1 on a place $i$. We say that two arrows $\alpha = (a \to b)$ and $\beta = (c \to d)$ of $I^n$ are parallel and write $\alpha \parallel \beta$ if

$$(b_1 - a_1, \ldots, b_n - a_n) = (d_1 - c_1, \ldots, d_n - c_n).$$
In the opposite case we shall call two arrows **orthogonal**.

An arrow \( \alpha \in E_{1^n} \) defines two \((n - 1) - \) faces of \( I^n \): the face \( I_0 = I_0^n \) that contains origin vertices of the arrows that are parallel to \( \alpha \) and the face \( I_1 = I_1^n \) that contains end vertices of the arrows that are parallel to \( \alpha \). Note that any arrow that is orthogonal to \( \alpha \) lies in \( I_0 \) or in \( I_1 \).

For the digraph cube \( I^m \) there is a natural partial order on the set of its vertices \( V_{I^m} \) that is defined as follows: we write \( a \leq b \) if there exists a directed path with the origin vertex \( a \) and the end vertex \( b \). Now we introduce a **distance** \( \Delta(a, b) \) for a pair of vertex \( a, b \in I^n \) that is defined only for comparable pair of vertices. Let \( a \leq b \) be two vertices then as follows from definition of the cube digraph the length of the path \( p \) from \( a \) to \( b \) does not depend on the choice of the path, and we put \( \Delta(a, b) = \Delta(b, a) : = |p| \). We shall call the vertex \( a = (0, \ldots, 0) \) of a cube **origin vertex** and the vertex \( d = (1, \ldots, 1) \) **end vertex**. It follows immediately from the definition of a cube digraph that the for any vertex \( x \) the distances \( \Delta(a, x) \) and \( \Delta(x, d) \) are well defined. For an arrow \( \alpha = (x \rightarrow y) \) we define \( \Delta(\alpha, d) : = \Delta(y, d) \) where \( d \) is end vertex of the cube. Let \( a \leq b \) be a pair of comparable vertices of \( I^n \) for which there is a direct path \( p \) from \( a \) to \( b \). Denote by \( I_{a,b} \) induced subgraph of \( I^n \) with the set of vertices \( \{ c \in V_{I^n} | a \leq c \leq b \} \). Clearly, \( I_{a,b} \) is isomorphic to a digraph cube \( I^k \), where \( k = |p| = \Delta(a, b) \).

**Definition 3.1.** A subgraph \( G \) of \( I^n \) is called **cubical** if for any two vertices \( a, b \in V_G \subset V_{I^n} \) with \( a \leq b \) we have \( I_{a,b} \subset G \).

Note that the set of all paths from \( a \) to \( b \) in \( I_{a,b} \) coincides with the set of all paths from \( a \) to \( b \) in \( G \). It is easy to see that cubical digraphs with digraph maps form a category. Now we prove that the image of a cube \( I^n \) in any cubical digraph is contractible. Note, that this statement is not true in general case.

**Example 3.2.** Consider the nondegenerate map \( f \) presented on Fig. 1 of the cube \( I^3 \) to the cycle digraph \( G \) given on the set of vertices by \( f(1) = f(8) = x, f(2) = f(3) = f(5) = y, f(4) = f(6) = f(7) = z \).

![Diagram](image)

**Theorem 3.3.** Let \( f : I^n \rightarrow G \) be a digraph map to a cubical digraph. Then the image \( f(I^n) \subset G \) is contractible.

**Proof.** The image \( f(I^n) \) is connected as the image of the connected graph. Let \( s = (0, \ldots, 0) \in V_{I^n} \) be the origin vertex and \( z = (1, \ldots, 1) \in V_{I^n} \) be the end vertex of \( I^n \). Then \( f(s) \in V_G, f(z) \in V_G \) and \( f(I^n) \subset I_{f(s), f(z)} \subset G \) where \( I_{f(s), f(z)} \) is isomorphic to
a \(m\)-dimensional cube which we denote \(J = J^m \cong I^m\) where \(m = \Delta(f(s), f(z))\). Hence, without loss of generality, we can suppose that \(G = I_{f(s), f(z)} = J\) that is \(f(s) = (0, \ldots, 0) \in V_J, f(z) = d = (1, \ldots, 1) \in V_J\). We prove the statement of the Theorem using induction on dimension \(m\).

The base of induction by \(m\). For \(m = 0, 1, 2\) the statement is trivial since any connected subgraph of the digraphs \(J^0, J^1,\) and \(J^2\) is contractible.

The step of induction by \(m\). Suppose that the statement of the Theorem is proved for every map \(I^n \to J^{m-1}\). Consider the case \(J = J^m\) where \(m \geq 3\) and \(d = (1, \ldots, 1) \in V_J\) is the end vertex of the cube \(J\). Since \(d = f(z) \in \text{Image}(f)\), there exists a nonempty set of arrows \(\Gamma \subset E_J\) defined as follows

\[
[\tau \in \Gamma] \Leftrightarrow \\text{end}(\tau) = d \& \tau = f(\alpha), \alpha \in \text{Image}(f).
\]

The set \(\Gamma\) consists of arrows in \(E_J\) with the end vertex \(d\) that are lying in the image of the map \(f\). Let \(\gamma = (c \to d) \in \Gamma\) be an arrow such that

\[
f(\alpha) = f(x \to y) = (c \to d) = \gamma \quad \text{and} \quad \Delta(\alpha, z) = \Delta(y, z) = k \geq 0\]

is minimal. (3.2)

Note that \(\alpha\) is defined may be by a non unique way. For for ease of references we formulae the following result.

**Lemma 3.4.** For every vertex \(v \in V_I^n\) with \(\Delta(v, z) \leq k\) we have \(f(v) = d\). Hence the cube \(I_{y, z} \subset I^n\) is mapped by \(f\) into the vertex \(d\).

**Proof.** Follows immediately from definition of \(k\) in (3.2). \(\blacksquare\)

The arrow \(\gamma\) defines two \((m - 1)\) - dimensional faces \(J_0\) and \(J_1\) of the cube \(J\) with \(c \subset V_{J_0}\), \(d \subset V_{J_1}\) and we have the natural projection \(\pi: J \to J_0\) along the arrow \(\gamma\). Let \(H\) be a subgraph of \(I^n\). We define subgraphs \(K_0, K_1, K \subset J\) which depend on the map \(f: I^n \to J\) and \(H \subset I^n\) as follows:

\[
K : = f(H) \subset J, \quad K_0 : = f(H) \cap J_0 \subset J_0, \quad \text{and} \quad K_1 : = f(H) \cap J_1 \subset J_1.
\]

(3.3)

It is easy to see that for an arrow \((v \to w) \in E_J\) we have:

\[
[(v \to w)|\gamma] \Leftrightarrow [(v \in J_0) \& (w \in J_1)].
\]

(3.4)

For technical reasons we introduce the following definition.

**Definition 3.5.** Let \(H \subset I^n, f: I^n \to J, \gamma\) is defined in (3.2), and the digraphs \(K, K_0, K_1 \subset J\) are defined in (3.3). We say that a subgraph \(H \subset I^n\) satisfies to the \(\Pi - \text{condition}\) if the following properties are satisfied

\[
\begin{align*}
(1) & \quad \forall \ w \in V_{K_1} \text{ there is a vertex } v \in V_{K_0} \text{ such that } (v \to w) \in E_K, \\
(2) & \quad \forall \ (w \to w') \in E_{K_1} \text{ we have } \pi(w \to w') \in E_{K_0}.
\end{align*}
\]

(3.5)

**Proposition 3.6.** Consider the map \(f: I^n \to J = J^m\) with \(m \geq 3\). Let \(k\) and \(\gamma\) are defined in (3.2) and let us consider the same designations as above. Then the cube \(I^n\) satisfies to \(\Pi\)-condition.

**Proof.** Induction in \(k \geq 0\).

The base of induction, \(k = 0\). Hence \(y = z = (1, \ldots, 1) \in V_{I^n}\) is the end vertex of \(I^n\) and \(n \geq m \geq 3\). The arrow \(\alpha = (x \to z) \in E_{I^n}\) with \(f(\alpha) = f(x \to z) = \gamma = (c \to d)\) defines \((n - 1)\)-face \(I_0 = I_{k,x}\) and opposite \((n-1)\)-face \(I_1\) of the cube \(I^n\). Let \(a = (0, \ldots, 0)\) be
Then, as before, generality, we can suppose that this proves condition (2) of (3.5). Thus \( \Pi \)-condition is satisfied for the cube \( J \).

The \((n-1)\)-dimensional subcube \( I_{s,z_j} \subset I^n \) contains the vertices \( x \) and \( t \). Moreover \( \Delta(x,z_j) = k \) and there is an arrow \( \alpha_i = (x \rightarrow y_i) \in E_{I_{s,z_j}} \) with \( f(\alpha_i) = \gamma \) and \( \Delta(\alpha_i,z_j) = k - 1 \). Hence, by the inductive assumption, the map

\[
 f|_{I_{s,z_j}} : I_{s,z_j} \to J
\]

satisfies the \( \Pi \)-condition.

Now consider a vertex \( t \) for which all \( (k+1) \)-coordinates \( b_j \) are equal "1" such that \( t \notin I_{s,z} \). This means that at least one of the first \( (n-k-1) \)-coordinates \( a_i \) is "0". Recall that \( (k+1) \geq 2 \). Thus consider the vertices

\[
 t = (a_1, \ldots, a_{n-k-1}, 1, \ldots, 1) \notin I_0, \quad r = (a_1, \ldots, a_{n-k-1}, 0, \ldots, 0) \in I_0 \quad (3.8)
\]
where \( a_i \in \{0,1\} \). Consider a directed path \( p \) in the digraph \( I_0 \) from the vertex \( r \in V_{I_0} \) to the vertex \( x \in V_{I_0} \) of the length \( l = |p| \geq 1 \) (since \( t \notin I_{x,z} \)). Write this path in the following form

\[
p = (r \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{l-1} \rightarrow x_l = x) \subset I_{r,x} \subset I_0.
\]

Consider a directed path \( q \) from the vertex \( r \in V_{I_0} \) to the vertex \( t \) of the length \( k + 1 = |q| \geq 2 \). Note that \( q \) lies in the digraph \( I_{r,t} \) of dimension \( k + 1 \). Write this path in the following form

\[
q = (r \rightarrow r_1 \rightarrow r_2 \rightarrow \ldots r_k \rightarrow r_{k+1} = t) \subset I_{r,t}.
\]

Any such two paths \( p \) and \( q \) defines an unique subgraph of the graph \( I^n \) that has the following form

\[
t = r^{k+1} \rightarrow r_1^{k+1} \rightarrow r_2^{k+1} \rightarrow \ldots \rightarrow r_l^{k+1} = z
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
r^k \rightarrow r_1^k \rightarrow r_2^k \rightarrow \ldots \rightarrow r_l^k
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
\ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
r \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_l = x
\]

(3.9)

Now we prove, using induction in the length \( l = |q| \geq 1 \) the following statement.

\( (L) \): For every path \( q \) and every path \( p \), as above, there is a path

\[
p' = (r \rightarrow x_1' \rightarrow x_2' \rightarrow \ldots x_{l-1}' \rightarrow x_l' = x) \subset I_{r,x} \subset I_0.
\]

(that may be is equal to \( p \)) such that \( q \) and \( p' \) defines the subgraph (similarly above)

\[
t = r^{k+1} \rightarrow r_1^{k+1'} \rightarrow r_2^{k+1'} \rightarrow \ldots \rightarrow r_l^{k+1'} = z
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
r^k \rightarrow r_1^k \rightarrow r_2^k \rightarrow \ldots \rightarrow r_l^k
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
\ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots \rightarrow \ldots
\]

\[
\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow
\]

\[
r \rightarrow x_1' \rightarrow x_2' \rightarrow \ldots \rightarrow x_l' = x
\]

(3.10)

and at least one of the following conditions is satisfied

\[
(i) \quad f(t) = f(r^k),
\]

\[
(ii) \quad f(t) = f(r_1^k),
\]

\[
(iii) \quad f(t) = f(r_1^l).
\]

(3.11)

The base of induction for \( (L) \), the case \( l = 1 \). Consider the unique path \( p = (r \rightarrow x) \subset I_0 \)
of the length \( l = 1 \) an a path \( q \) as above. We have the following subgraph of the digraph \( I^n \):

\[
\begin{array}{c}
t = r^{k+1} \\
\uparrow \\
r^k \\
\uparrow \\
\vdots \\
\uparrow \\
r^1 \\
\uparrow \\
r \\
\end{array}
\rightarrow
\begin{array}{c}
t = r^{k+1} = z \\
\uparrow \\
r^k = r_1^k \\
\uparrow \\
\vdots \\
\uparrow \\
r^1 = r_1^1 \\
\uparrow \\
x_1 \\
\end{array}
\]  

(3.12)

where \( r, x \in V_0, \ f(r), f(x) \in V_{K_0} \), and \( f(r_1^i) = d \) for \( 1 \leq i \leq k + 1 \) since \( k \geq 1 \). Hence \( f(r_1^k) = f(r_1^{k+1}) = d \) and thus at least one of the conditions (i) or (ii) in (3.11) is satisfied since there are no triangles in the digraph \( J \). We put in this case \( p' = p \), and the base of induction \( l = 1 \) is proved.

Inductive step of induction for \( (L) \). Consider vertices \( t, r \in V_I \) given in (3.8) where \( \Delta(t,r) = k + 1 \geq 2 \) and \( \Delta(r,x) \geq 2 \). Let \( p \) be a path from \( r \) to \( x \) and \( q \) be a path from \( r \) to \( t \) as the above. Recall that \( |p| = k + 1 \geq 2, \ |q| = l \geq 2 \). These paths define the subgraph of \( I^n \) given on (3.9). By the inductive assumption, for the vertex \( r_1^{k+1} \) at least one of the conditions

\[
\begin{align*}
(i) & \quad f(r_1^{k+1}) = f(r_1^k), \\
(ii) & \quad f(r_1^{k+1}) = f(r_2^k), \\
(iii) & \quad f(r_1^{k+1}) = f(r_2^k). \\
\end{align*}
\]  

(3.13)

that is similar to (3.11) is realized. In (3.13) we have a path \( r^k \rightarrow r_1^k \rightarrow r_2^k \rightarrow \cdots \rightarrow r_1^k \) that is similar to the path \( r^k \rightarrow r_1^k \rightarrow r_2^k \rightarrow \cdots \rightarrow r_1^k \) from (3.9).

If condition (i) is realized, that is \( f(r_1^{k+1}) = f(r_1^k) \), then for \( f(t) \) at least one of the conditions (i) or (ii) in (3.11) is satisfied since there are no triangles in the digraph \( J \) (similarly to the case \( l = 1 \)).

If condition (ii) is realized and condition (i) is not realized, that is \( f(r_1^{k+1}) = f(r_2^k) \) and \( f(r_1^i) \neq f(r_2^k) \), we can consider the subcube of \( I^n \) given on Fig. 2 of \( I^n \) that is defined by the subgraph of (3.9) given below in (3.14):

\[
\begin{array}{c}
t = r^{k+1} \\
\uparrow \\
r^k \\
\uparrow \\
\vdots \\
\uparrow \\
r^1 \\
\uparrow \\
r \\
\end{array}
\rightarrow
\begin{array}{c}
t = r^{k+1} \\
\uparrow \\
r^k \\
\uparrow \\
\vdots \\
\uparrow \\
r^1 \\
\uparrow \\
x_1 \\
\end{array}
\]  

(3.14)

We have \( f(r_1^{k+1}) = f(r_2^k) \) and \( f(r_1^k) \neq f(r_2^k) \), that is \( f(r_1^k \rightarrow r_1^{k+1}) = f(r_1^k \rightarrow r_2^k) \in E_J \) is an arrow. If \( f(r_k) = f(r_2^k) \) then the same line of above give that \( f(t) = f(r_1^k) \) or \( f(t) = f(r_2^k) \) and the step of induction is proved. Let \( f(r_k) \neq f(r_2^k) \) then

\[
\begin{align*}
f\left(I_{r^k, r_2^k}\right) & \subseteq f\left(I_{f(r^k), f(r_2^k)}\right) \quad \text{and} \quad f\left(I_{r^k, r_1^{k+1}}\right) \subseteq f\left(I_{f(r^k), f(r_1^k)}\right) \\
\end{align*}
\]

where \( I_{f(r^k), f(r_2^k)} \) is the digraph square. Hence at least one of conditions \( f(r_1^{k+1}) = f(r_1^k) \) or \( f(r_1^{k+1}) = f(r_2^k) \) is satisfied and the inductive assumption is proved.

Consider the case when condition (iii) is realized and conditions (i) and (ii) are not realized. This case is the same as the case (ii). We must to start the consideration from the
Figure 2: The subcube of $I^n$ that is defined by the digraph on (3.13).

path $r^k \rightarrow r^k_1 \rightarrow r^k_2 \rightarrow \cdots \rightarrow r^k_1$ on the place of the path $r^k \rightarrow r^k_1 \rightarrow r^k_2 \rightarrow \cdots \rightarrow r^k_1$ from (3.9). This finishes the proof of the inductive step and, statement (L) is proved.

It follows from the statement (L) that image $w = f(t)$ and images of all arrows with end or origin $t$ lay in the image of the subcube $I_{r,z_j}$ with $\Delta(x, z_j) = \Delta(r, z_j) = k$ which satisfies to II-condition by the inductive assumption in $k$. Hence the cube $I^n$ satisfies to II-condition and the Proposition is proved.

Now we finish the proof of Theorem 3.3. Since digraph $I^n$ satisfies the II-condition then Proposition 2.7 and (3.5) implies that restriction $\pi|_K$ of the projection $\pi: J^m \rightarrow J^{m-1}_0$ to the image $K$ of the map $f$ is well defined deformation retraction to $K_0$. But $K_0$ is contractible by the inductive assumption in $m$. Thus Theorem 3.3 is proved.

4 Equivalence of homology theories on cubical digraphs

In this section we prove our main result – Theorem 1.1, that is stated below as Theorem 4.5. For that we use the Acyclic Carrier Theorem from homology theory (see, for example, [5, §3.4] and [6, §1.2.1]). Recall that a chain complex $C_\ast$ is called non-negative if $C_p = 0$ for $p < 0$ and is called free if $C_p$ are finitely generated free abelian groups for all $p$. We say that $C_\ast$ is a geometric chain complex if it is non-negative, free, and if a basis $B_p$ is chosen in the group $C_p$ for any $p \geq 0$. For example, any finite simplicial complex gives rise to a geometric chain complex, where $B_p$ consists of all $p$-simplexes.

Let $C_\ast$ be a geometric chain complex with fixed bases $B_p$. For $b \in B_{p-1}$ and $b' \in B_p$, we write $b \prec b'$ if $b$ enters with a non-zero coefficient into the expansion of $\partial b'$ in the basis $B_{p-1}$.

The augmentation homomorphism $\varepsilon: C_0 \rightarrow \mathbb{Z}$ is defined as

$$\varepsilon \left( \sum_i k_i b_i \right) = \sum_i k_i, \quad k_i \in \mathbb{Z}, \quad b_i \in B_0,$$

and we denote $\tilde{C}_\ast$ the augmented complex

$$\mathbb{Z} \xrightarrow{\varepsilon} C_0 \xrightarrow{\partial} C_1 \xrightarrow{\partial} \cdots$$

A geometric chain complex $C_\ast$ is called acyclic if all homology groups of the augmented complex $\tilde{C}_\ast$ are trivial.

Let $C_\ast$ and $D_\ast$ be two geometric complexes with augmentation homomorphism $\varepsilon$ and $\varepsilon'$, respectively. A chain map $\phi_\ast: C_\ast \rightarrow D_\ast$ is called augmentation preserving if $\varepsilon' \phi_0(c) = \varepsilon(c)$ for any $c \in C_0$. 

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Definition 4.1. Let $C_*$ and $D_*$ be two geometric chain complexes.

(i) An algebraic carrier function from $C_*$ to $D_*$ is a mapping $E$ that assigns to any basis element $b$ in $C_*$ a subcomplex $E_*(b) := E(b)$ of $D_*$, such that $b < b'$ implies $E_*(b) \subset E_*(b')$.

(ii) An algebraic carrier function $E$ is called acyclic if each complex $E_*(b)$ is non-empty and acyclic.

(iii) A chain map $f_* : C_* \to D_*$ is carried by $E$ if $f_*(b) \in E_*(b)$ for any basis element $b$ in $C_n$.

We state the Acyclic Carrier Theorem in the following form.

Theorem 4.2. Let $C_*$ and $D_*$ be two geometric chain complexes and $E$ be an acyclic carrier function from $C_*$ to $D_*$. If $f_*, g_* : C_* \to D_*$ are augmentation preserving chain maps that are carried by $E$, then $f_*$ and $g_*$ are chain homotopic.

Before the proof of Theorem 1.1, we state and prove some technical results. We use the notations of [1, 4]. Let $G$ be a cubical digraph. The free abelian groups $\Omega^i_p = \Omega^i_p(G)$ and $\Omega_p = \Omega_p(G)$ defined in (2.3) and (2.8) are finitely generated.

Let $I^0 = \{ * \}$ be the one-vertex digraph. Any zero-dimensional singular cube $\phi : I^0 \to G$ is given by the vertex $\phi(*) \in V_G$ and we thus obtain the map $\tau_0 : \Omega^0_0(G) \to \Omega_0(G)$ which preserve augmentation.

For any digraph cube $I^n$ ($n \geq 1$) denote by $P$ the set of all directed paths of the length $n$ going from the origin vertex $(0, \ldots, 0)_n$ of the cube to the end vertex $(1, \ldots, 1)_n$. Every path $p \in P$ has the following form

$$p = (a_0 \to a_1 \to a_2 \to \cdots \to a_n), \quad a_i \in V_{I^n}. \tag{4.1}$$

In (4.1) for $1 \leq i \leq n$ the vertex $a_i$ differs from $a_{i-1}$ only by one coordinate $1 \leq \pi(i) \leq n$ that equals "0" for $a_{i-1}$ and "1" for $a_i$. Let $\sigma(p)$ be a sign of the permutation

$$\pi(p) = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{array} \right).$$

Consider the path $w_n \in \Omega_n(I^n)$ given by

$$w_n = \sum_{p \in P} (-1)^{\sigma(p)} p \tag{4.2}$$

that is the generator of the group $\Omega_n(I^n)$ (see [1] and [4]). For any singular $n$-dimensional cube $\phi : I^n \to G$, which gives a basic element $\phi^\square \in \Omega^*_n(G)$, we have a morphism of chain complexes defined in [1]

$$\tau_* : \Omega^*_0(G) \to \Omega_*(G), \quad \tau_n(\phi^\square) := \phi_*(w_n) \tag{4.3}$$

where $\phi_* : \Omega_*(I^n) \to \Omega_*(G)$ is the induced of $\phi$ morphism of chain complexes.

For $n \geq 0$ consider the set $K_n$ of all subcubes $G$ of dimension $n$ that have the form $I_{s,t}$ with $s, t \in V_G$. By [1, 4], for every cube $I_{s,t} \in K_n$ there is an isomorphism $\chi_{s,t} : I^n \to I_{s,t}$ such that the set of elements $\{(\chi_{s,t})_*(w_n) : I_{s,t} \in K_n\}$ give the basis of $\Omega_n(G)$. For $n \geq 1$, define homomorphisms $\theta_n : \Omega_n(G) \to \Omega^*_n(G)$ on basic elements by

$$\theta_n((\chi_{s,t})_*(w_n)) = \chi^\square_{s,t}, \tag{4.4}$$

and then extend it by linearity. It is clear that $\theta_0$ preserves the augmentation.
Proposition 4.3. The homomorphisms $\theta_n$ define a morphism of chain complexes

$$\theta_n: \Omega_n(G) \to \Omega^c_n(G)$$

(4.5)

that is a right inverse morphism to $\tau_n$, that is

$$\tau_n \theta_n = \text{Id}: \Omega_n(G) \to \Omega_n(G).$$

Proof. Let us first prove that $\tau_n \theta_n = \text{Id}$. For $n = 0, 1$ this is trivial. Let $n \geq 2$ and $(\xi_{s,t})_n(w_n) \in \Omega_n(G)$ be a basic element. By (4.4) and (4.3) we have

$$\tau_n \theta_n \big((\xi_{s,t})_n(w_n)\big) = \tau_n(\xi_{s,t}) = \xi_{s,t}(w_n).$$

(4.6)

Now consider the commutative diagram

$$
\begin{array}{ccc}
\Omega_n(G) & \overset{\theta_n}{\longrightarrow} & \Omega^c_n(G) \\
\downarrow \partial & & \downarrow \partial^c \\
\Omega_{n-1}(G) & \overset{\theta_{n-1}}{\longrightarrow} & \Omega^c_{n-1}(G)
\end{array}
$$

(4.7)

where the horizontal compositions are identity homomorphisms by (4.6) and the right square is commutative. It follows from [4, Lemma 4] that, for $(\xi_{s,t})_n(w_n) \in \Omega_n(G)$, we have

$$\theta_{n-1} \big(\partial \big((\xi_{s,t})_n(w_n)\big)\big) = \theta_{n-1} \left(\sum_{I_{s',t'} \subset I_{s,t}} (-1)^{\sigma(I,I')} (\phi^{I,I'}_{s',t'})_n(w_{n-1})\right)$$

$$= \sum (-1)^{\sigma(I,I')} (\phi^\square_{s',t'})$$

(4.8)

where the sum is taken over all $(n-1)$-cubes $I' = I_{s',t'} \subset I_{s,t} = I$. By (2.7) and (4.4) we have for $(\phi_{s,t})_n(w_n) \in \Omega_n(G)$

$$\partial^c \left(\theta \big((\phi_{s,t})_n(w_n)\big)\right) = \partial^c \left(\phi^\square_{s,t} \sum_{p=1}^n (-1)^p \left((\phi_{s,t})_{p,0} - (\phi^\square_{s,t})_{p,1}\right)\right)$$

(4.9)

where the sum is taken over all $(n-1)$-subcubes of the cube $I^n$. Since bottom row in (4.7) is the identity homomorphism we conclude from (4.3), (4.8) and (4.9) that the left square in (4.7) is commutative, which finishes the proof. \qed

Proposition 4.4. There is a chain homotopy between $\theta_n \circ \tau_n: \Omega^c_n(G) \to \Omega^c_n(G)$ and the identity map $\text{Id}: \Omega^c_n(G) \to \Omega^c_n(G)$.

Proof. The chain complex $\Omega^c_n(G)$ is geometric and the chain maps $\theta_n \circ \tau_n$ and $\text{Id}$ evidently preserve augmentation. For a singular cube $\phi: I^n \to G$ consider the subgraph $G_\phi \subset G$ that is image of $\phi$. This is a contractible cubical digraph by Theorem 3.3. Thus we assign to every basic element $\phi^\square \in \Omega^c_n(G)$ the subcomplex

$$E_n \left(\phi^\square\right) \overset{\text{def}}{=} \Omega^c_n(G_\phi) \subset \Omega^c_n(G)$$

(4.10)

which is acyclic since $G_\phi$ is contractible.

Now we check that $E$ is an algebraic carrier function, that is condition (i) of Definition 4.1 is satisfied. Let $\phi^\square \in \Omega^c_n(G)$ be a basic element given by a singular cube $\phi: I^n \to G$ with $n \geq 0$. By (2.6) and (2.7), the element $\partial(\phi^\square)$ is given by the sum of the basic elements $(\phi \circ V_{pe})^\square$
simplicial homology groups

That

E

function

on its image

G

the set of vertices given by the set of all simplexes from

φ

subcomplex of

E

the Acyclic Carriers Theorem 4.2.

Corollary 4.6.

Let

Δ

be a finite simplicial complex. Consider a digraph

G

(see [4]) with the set of vertices given by the set of all simplexes from

Δ,

and

s → t (t, s ∈ Δ) iff s ⊇ t and dim s = dim t + 1.

Then the graph

GΔ

is a cubical digraph and

Hc∗(GΔ) ≃ Hs(Δ)

where

H∗(Δ)

are the simplicial homology groups of

Δ.

Proof.

Indeed, it is proved in [4] that path homology groups

H∗(GΔ)

are isomorphic to simplicial homology groups

H∗(Δ).
References


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