ON HOMOLOGY THEORIES OF CUBICAL DIGRAPHS

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Abstract. We prove the equivalence of the singular cubical homology and the path homology on the category of cubical digraphs. As a corollary we obtain a new relation between the singular cubical homology of digraphs and simplicial homology.

Contents

1. Introduction

The path homology theory and the singular cubical homology theory for the category of digraphs were introduced in [1, 2, 3, 4, ?]. In this category, there is a natural mapping of the cubical homology theory to the path homology theory, that induces an isomorphism of homology groups in dimensions 0 and 1. However, in [1] an example of a digraph was constructed, for which the path homology is trivial in dimension 2 while the singular cubical homology is non-trivial in this dimension. Hence, in general, these two theories give different homologies in dimensions ≥ 2. A natural question arises whether these two theories are equivalent on some subclass of digraphs.

In this paper we present a class of cubical digraphs and prove the equivalence of the singular cubical homology and the path homology theories on this class. As the main technical tool for that, we prove that the image of every map of a digraph cube to a cubical digraph is contractible.

The paper is organized as follows. In Section 2, we recall the basic definitions from graph theory and describe some properties of singular cubical homology H_*^c and the path homology H_* on the category of digraphs using the sources [1], [2], [3], and [4]. In Section 3, we recall the definition of cubical digraph from [4] and

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prove the contractibility of the image of a digraph cube in a cubical digraph for any digraph map. In Section 4, we prove the main result of the paper:

Theorem 1.1. *On the category of cubical digraphs, the singular cubical homology theory is equivalent to the path homology theory.*

In Corollary 4.6 we obtain a consequence about the relation between the singular cubical homology theory of digraphs and simplicial homology.

2. Singular cubical and path homology theories

In this Section we give necessary preliminary material about digraphs and homology theories on the category of finite digraphs.

Definition 2.1. *A digraph* G is a pair (V_G, E_G) of a set $V = V_G$ of *vertices* and a subset $E_G \subset \{V_G \times V_G \setminus \text{diagonal}\}$ of ordered pairs (v, w) of vertices which are called *arrows* and are denoted $v \to w$. The vertex $v = \text{orig } (v \to w)$ is called the *origin of the arrow* and the vertex $w = \text{end}(v \to w)$ is called the *end of the arrow.*

For two vertices $v, w \in V_G$, we write $v \,\overrightarrow{=}\, w$ if either $v = w$ or $v \to w$.

A *subgraph* H of a digraph G is a digraph whose set of vertices is a subset of that of G and the set edges of H is a subset of the set of edges of G. In this case we write $G \subset H$.

A subgraph H of G is called *induced* if the edges of H are all those edges of G whose adjacent vertices belong to H. In this case we write $G \sqsubset H$.

A *directed path* $p = (a_1, \alpha_1, a_2, \alpha_2, \ldots, \alpha_n, a_{n+1})$ in a digraph G is a sequence of vertices a_i and arrows α_i such that $\alpha_i = (a_i \to a_{i+1})$. The number n of arrows in path is called *length* of the path and is denoted by $|p|$. The vertex a_1 is called *the origin of the path* and the vertex a_{n+1} is called *the end of the path.*

Definition 2.2. A *digraph map* (or simply *map*) from a digraph G to a digraph H is a map $f\colon V_G \to V_H$ such that $v \,\overrightarrow{=}\, w$ in G implies $f(v) \,\overrightarrow{=}\, f(w)$ in H.

A digraph map f is *non-degenerate* if $v \to w$ in G implies $f(v) \to f(w)$ in H.

The set of all digraphs with digraph maps form the *category of digraphs* that will be denoted by $\mathcal{D}$.

Definition 2.3. For two digraphs G and H, the *box product* $\Pi = G \Box H$ is defined as a digraph with a set of vertices $V_\Pi = V_G \times V_H$ and a set of arrows E_Π given by the rule

$$(x, y) \to (x', y') \;\text{ if } x = x' \text{ and } y \to y', \text{ or } x \to x' \text{ and } y = y',$$

where $x, x' \in V_G$ and $y, y' \in V_H$.

Fix $n \geq 0$. Denote by I_n any digraph with the set of vertices $V = \{0, 1, \ldots, n\}$ such that, for $i = 0, 1, \ldots n-1$, there is exactly one arrow $i \to i+1$ or $i+1 \to i$ and there are no other arrows. Such a digraph is called a *line* digraph. It is called a *direct line* digraph if, additionally, all arrows have the form $i \to i+1$. We denote the digraph $0 \to 1$ by I.

For any $n \geq 0$, define a *standard n-cube digraph* I^n as follows. For $n = 0$ we put $I^0 = \{0\}$ that is an one-vertex digraph. For $n \geq 1$, the set of vertices of I^n consists of all 2^n binary sequences $a = (a_1, \ldots, a_n)$, and there is an arrow $a \to b$

between two such vertices if and only if the sequence $b = (b_1, \dots, b_n)$ is obtained from $a = (a_1, \dots, a_n)$ by replacing a digit 0 by 1 at exactly one position. It is easy to see that

$$I^n = \underbrace{I\Box I\Box I\Box \dots \Box I}_{n \text{ times}}.$$

For example, the digraph $0 \to 1$ is an 1-cube. Any digraph that is isomorphic to I^2 will be referred to as a *square*. Any digraphs that is isomorphic to I_n will be referred to as an *n-cube digraph* any digraph that is isomorphic to the standard n-cube.

Let us recall the notion of homotopy in the category of digraphs that was introduced in [2].

Definition 2.4. Two digraph maps $f, g\colon G \to H$ are called *homotopic* if there exists a line digraph I_n with $n \geq 1$ and a digraph map

$$F\colon G\Box I_n \to H,$$

such that

$$F|_{G\Box\{0\}} = f \quad \text{and} \quad F|_{G\Box\{n\}} = g,$$

where we identify $G\Box\{0\}$ and $G\Box\{n\}$ with G in a natural way. In this case we shall write $f \simeq g$. The map F is called a *homotopy* between f and g.

In the case $n = 1$ we refer to the map F as an *one-step homotopy.*

Definition 2.5. Digraphs G and H are called *homotopy equivalent* if there exist digraph maps

$$f : G \to H, \quad g : H \to G$$

such that

$$f \circ g \simeq \mathrm{id}_H, \qquad g \circ f \simeq \mathrm{id}_G.$$

In this case we shall write $H \simeq G$ and the maps f and g are called *homotopy inverses* of each other.

A digraph G is called *contractible* if $G \simeq \{*\}$ where $\{*\}$ is an one-vertex digraph.

Definition 2.6. [2, Def. 3.4] Let G be a digraph and H be its subgraph.

(i) A *retraction* of G onto H is a map $r\colon G \to H$ such that $r|_H = \mathrm{id}_H$.

(ii) A retraction $r : G \to H$ is called a *deformation retraction* if $i \circ r \simeq \mathrm{id}_G$, where $i\colon H \to G$ is the natural inclusion.

Proposition 2.7. [2, Corollary 3.7] *Let $r : G \to H$ be a retraction of a digraph G onto a sub-digraph H and*

$$x \overrightarrow{=} r(x) \ \text{ for all } x \in V_G \quad \text{or} \quad r(x) \overrightarrow{=} x \ \text{ for all } x \in V_G. \tag{2.1}$$

Then r is a deformation retraction, the digraphs G and H are homotopy equivalent, and i, r are the homotopy inverses of each other..

Now we recall the definitions of path homology groups from [4] with the group of coefficients $\mathbb{Z}$. An *elementary p-path* on a finite set V is any (ordered) sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted by $e_{i_0 \dots i_p}$. Denote by $\Lambda_p = \Lambda_p(V)$ the free abelian group generated by all elementary p-paths $e_{i_0 \dots i_p}$. The elements of Λ_p are called *p-paths.* Thus, each p-path $v \in \Lambda_p$ has the form

$$v = \sum_{i_0, \dots, i_p \in V} v^{i_0 i_1 \dots i_p} e_{i_0 i_1 \dots i_p},$$

where $v^{i_0 i_1 \dots i_p} \in \mathbb{Z}$ are the coefficients of v.

For $p \geq 0$, define the *boundary* operator $\partial\colon \Lambda_{p+1} \to \Lambda_p$ on basic elements by

$$\partial e_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \, e_{i_0 \dots \widehat{i_q} \dots i_{p+1}}, \tag{2.2}$$

where $\widehat{k}$ means omission of the corresponding index, and extend ∂ to Λ_{p+1} by linearity. Set also $\Lambda_{-1} = \{0\}$ and define $\partial\colon \Lambda_0 \to \Lambda_{-1}$ by $\partial v = 0$ for all $v \in \Lambda_0$. It follows from this definition that $\partial^2 v = 0$ for any p-path v.

An elementary p-path $e_{i_0 \dots i_p}$ $(p \geq 1)$ is called *regular* if $i_k \neq i_{k+1}$ for all k. For $p \geq 1$, let $\mathcal{I}_p$ be the subgroup of Λ_p that is spanned by all irregular $e_{i_0 \dots i_p}$ and we set $\mathcal{I}_0 = \mathcal{I}_{-1} = 0$. Then $\partial \mathcal{I}_{p+1} \subset \mathcal{I}_p$ for $p \geq -1$. Consider the chain complex $\mathcal{R}_*$ with

$$\mathcal{R}_p = \mathcal{R}_p(V) = \Lambda_p / \mathcal{I}_p$$

and with the chain map that is induced by ∂.

Now we define allowed paths on a digraph $G = (V, E)$. A regular elementary path $e_{i_0 \dots i_p}$ in V is called *allowed* if $i_{k-1} \to i_k$ for any $k = 1, ..., p$, and *non-allowed* otherwise. For $p \geq 1$, denote by $\mathcal{A}_p = \mathcal{A}_p(G)$ the subgroup of $\mathcal{R}_p$ spanned by the allowed elementary p-paths, that is,

$$\mathcal{A}_p = \operatorname{span}\left\{ e_{i_0 \dots i_p} : i_0 ... i_p \text{ is allowed} \right\}.$$

and set $\mathcal{A}_{-1} = 0$. The elements of $\mathcal{A}_p$ are called *allowed* p-paths.

Consider the following subgroup of $\mathcal{A}_p$ $(p \geq 0)$

$$\Omega_p = \Omega_p(G) = \{ v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1} \}. \tag{2.3}$$

The elements of Ω_p are called *∂-invariant* p-paths. It is easy ti see that $\partial \Omega_{p+1} \subset \Omega_p$ so that we obtain a chain complex

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{2.4}$$

The *path homology groups* $H_*(G)$ of the digraph G are defined as the homology groups of the chain complex (2.4), that is,

$$H_p(G) := \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

In what follows we will also need a natural augmentation $\varepsilon\colon \Omega_0 \to \mathbb{Z}$ that is defined by

$$\varepsilon\left(\sum k_i e_i\right) = \sum k_i, \quad k_i \in \mathbb{Z}.$$

Clearly, ε is an epimorphism and $\varepsilon \circ \partial = 0$.

Now we recall from [1] the construction of the cubical singular homology theory of digraphs.

Definition 2.8. *A singular n-cube* in a digraph G is a digraph map $\phi\colon I^n \to G$.

Fix $n \geq 1$. For any $1 \leq j \leq n$ and $\epsilon = 0, 1$, define the inclusion $F_{j\epsilon}^{n-1}\colon I^{n-1} \to I^n$ of digraphs as follows: if $n \geq 2$ then

$$F_{j\epsilon}^{n-1}(c_1, \dots, c_{n-1}) = \begin{cases} (\epsilon, c_1, \dots, c_{n-1}) & \textit{for } j = 1, \\ (c_1, \dots, c_{j-1}, \epsilon, c_j, \dots c_{n-1}) & \textit{for } 1 < j < n, \\ (c_1, \dots, c_{n-1}, \epsilon) & \textit{for } j = n, \end{cases} \tag{2.5}$$

and if $n = 1$ then $F^{n-1}_{1\epsilon}(0) = (\epsilon)$. We shall write shortly $F_{j\epsilon}$ instead of $F^{n-1}_{j\epsilon}$ if the dimension $n-1$ is clear from the context. Denote by $I^{n-1}_{j\epsilon}$ the image of $F^{n-1}_{j\epsilon}$. We shall write $I_{j\epsilon}$ instead $I^{n-1}_{j\epsilon}$ if the dimension is clear from the context.

Let $Q_{-1} = 0$. For $n \geq 0$, denote $Q_n = Q_n(G)$ the free abelian group generated by all singular n-cubes in G, and denote $\phi^{\square}$ the singular n-cube ϕ as the element of the group Q_n. For $n \geq 1$ and $1 \leq p \leq n$, denote

$$\phi^{\square}_{p\epsilon} = (\phi \circ F_{p\epsilon})^{\square} \in Q_{n-1}. \tag{2.6}$$

For any $n \geq 1$, define a homomorphism $\partial^c \colon Q_n \to Q_{n-1}$ on the basis elements $\phi^{\square}$ by the rule

$$\partial^c \phi^{\square} = \sum_{p=1}^{n} (-1)^p \left(\phi^{\square}_{p0} - \phi^{\square}_{p1}\right), \tag{2.7}$$

and $\partial^c = 0$ for $n = 0$. Then $(\partial^c)^2 = 0$ and the groups $Q_n(G)$ form a chain complex that we denote $Q_* = Q_*(G)$.

For $n \geq 1$ and $1 \leq p \leq n$, consider the natural projection $T^p \colon I^n \to I^{n-1}$ on the p-face I^{n-1} defined as follows. For $n = 1$, T^1 is the unique digraph map $I^1 \to I^0$. For $n \geq 2$, we have on the set of vertices $T^p(i_1, \dots, i_n) = (i_1, \dots, i_{p-1}, i_{p+1}, \dots, i_n)$. The singular n-cube $\phi \colon I^n \to G$ is degenerate if there is $1 \leq p \leq n$ such that $\phi = \psi \circ T^p$ where $\psi \colon I^{n-1} \to G$ is a singular $(n-1)$-cube. Then an abelian group $B_n = B_n(G)$ that is generated by all degenerated n-cubes is a subgroup Q_n for $n \geq 1$. We put also $B_0 = 0, B_{-1} = 0$. Then the quotient group

$$\Omega^c_p(G) = Q_p(G)/B_p(G) \tag{2.8}$$

is defined for $p \geq 0$. We have $\partial(B_n) \subset B_{n-1}$ and, hence, $B_*(G) \subset Q_*(G)$. Hence the quotient complex $\Omega^c_*(G) = Q_*(G)/B_*(G)$ is defined. We continue to denote the boundary operator in this complex ∂^c. The homology group $H_k(\Omega^c_*(G))$ is called the *singular cubical homology group of digraph G in dimension k* and is denoted $H^c_k(G)$.

We have a natural augmentation homomorphism $\varepsilon \colon \Omega^c_0(G) \to \mathbb{Z}$, defined by

$$\varepsilon\left(\sum k_i \phi_i\right) = \sum k_i, \quad k_i \in \mathbb{Z}.$$

Then ε is an epimorphism and $\varepsilon \circ \partial^c = 0$.

Here are some basic properties of the path and the singular cubical homology groups from [4] and [1].

- The groups $H^c_*(X)$ and $H_*(X)$ are functors from the category $\mathcal{D}$ to the category of abelian groups.
- Let $f \simeq g \colon X \to Y$ be two homotopic digraph maps. Then the induced homomorphisms f_*, g_* of homology groups are equal for $k \geq 0$ for the both theories.

3. Maps from cube to cubical digraph

In this section we reformulate slightly the definition of a cubical digraph from [4] and prove Theorem 3.6 saying that an image of a cube in a cubical digraph is contractible.

Recall, that any vertex of a a cube I^n is given by a sequence of binary numbers $(a_1, \dots, a_n)$. For any arrow $a \to b$ in a digraph cube I^n we have also the arrow

$$\gamma_i = (0, \dots, 0) \to (b_1 - a_1, \dots, b_n - a_n) \tag{3.1}$$

in I^n where the right sequence represents a vertex in I^n that has only one non-trivial element 1 at some position i. We say that two arrows $\alpha = (a \to b)$ and $\beta = (c \to d)$ of I^n *are parallel* and write $\alpha || \beta$ if

$$(b_1 - a_1, \dots, b_n - a_n) = (d_1 - c_1, \dots, d_n - c_n).$$

In the opposite case we say that the arrows α and β are *orthogonal*.

An arrow $\alpha \in E_{I^n}$ defines two $(n-1)$ – faces of I^n: the face $I_0 = I_0^\alpha$ that contains the origin vertices of the arrows that are parallel to α and the face $I_1 = I_1^\alpha$ that contains the end vertices of the arrows that are parallel to α. Note that any arrow that is orthogonal to α lies in I_0 or in I_1.

For the digraph cube I^n, there is a natural partial order on the set of its vertices V_{I^n} that is defined as follows: we write $a \le b$ if there exists a path along the arrows with the origin vertex a and the end vertex b. Now we introduce the *distance* $\Delta(a, b)$ for a pair of vertex $a, b \in I^n$ that is defined only for comparable pair of vertices. Let a, b be two vertices of I^n such that $a \le b$. As it follows from the definition of I^n, the length of the path p from a to b does not depend on the choice of the path, and we set

$$\Delta(a, b) = \Delta(b, a) \colon = |p|.$$

We shall refer to the vertex $a = (0, \dots, 0)$ of a cube as the *origin vertex* and to the vertex $d = (1, \dots, 1)$ as the *end vertex*.

It follows immediately from the definition of I^n that, for any vertex x, the distances $\Delta(a, x)$ and $\Delta(x, d)$ are well defined. For an arrow $\alpha = (x \to y)$ we define $\Delta(\alpha, d) \colon = \Delta(y, d)$.

Let $a \le b$ be a pair of comparable vertices of I^n. Denote by $I_{a,b}$ the induced subgraph of I^n with the set of vertices $\{c \in V_{I^n} | a \le c \le b\}$. Clearly, $I_{a,b}$ is isomorphic to a digraph cube I^k, where $k = |p| = \Delta(a, b)$.

Definition 3.1. A subgraph G of I^n is called *cubical* if, for any two vertices $a, b \in V_G \subset V_{I^n}$ with $a \le b$, we have $I_{a,b} \sqsubset G$.

Note that the set of all paths from a to b in $I_{a,b}$ coincides with the set of all paths from a to b in G. It is easy to see that cubical digraphs with digraph maps form a category. Now we prove that the image of a cube I^n in any cubical digraph is contractible. Note, that this statement is not true for general digraphs.

Example 3.2. Consider a digraph map f presented on Fig. 1 that maps the cube I^3 onto the cycle digraph G and that is defined by $f(1) = f(8) = x,\ f(2) = f(3) = f(5) = y, f(4) = f(6) = f(7) = z$. Then the images of this map, that is, G, is non-contractible.

Now consider a digraph map $f \colon I^n \to G$ where G is a cubical digraph. The image $f(I^n)$ is connected as the image of a connected digraph. Let $s = (0, \dots, 0) \in V_{I^n}$ be the origin vertex and $z = (1, \dots, 1) \in V_{I^n}$ be the end vertex of I^n. Then $f(s) \in V_G, f(z) \in V_G$ and $f(I^n) \subset I_{f(s),f(z)} \subset G$ where $I_{f(s),f(z)}$ is isomorphic to

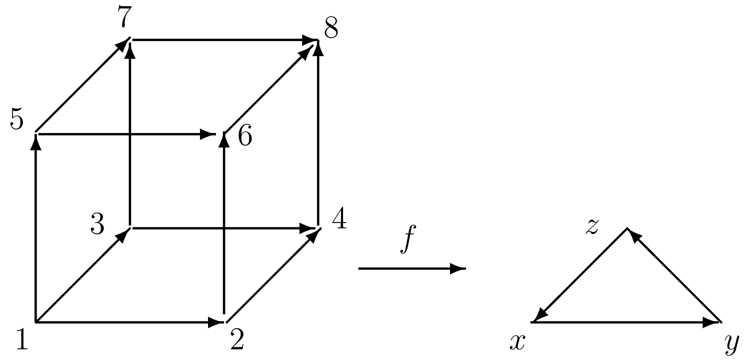

FIGURE 1. The map $f\colon I^3 \to G$ with non-contractible image.

a m-dimensional cube which we denote $J = J^m \cong I^m$ where $m = \Delta(f(s), f(z))$. Hence, without loss of generality, we can assume that $G = I_{f(s),f(z)} = J$, that is,

$$f(s) = (0, \dots, 0) \in V_J, \quad f(z) = d = (1, \dots, 1) \in V_J.$$

For $m = 0, 1, 2$ the image $f(I^n) \subset G$ is contractible since all connected subgraphs of the digraphs J^0, J^1, and J^2 are contractible.

Consider the case $J = J^m$ where $m \geq 3$ and $d = (1, \dots, 1) \in V_J$ is the end vertex of the cube J. Since $d = f(z) \in \mathrm{Image}(f)$, there exists a nonempty set of arrows $\Gamma \subset E_J$ defined as follows

$$[\tau \in \Gamma] \ \Leftrightarrow \ [\mathrm{end}(\tau) = d \ \& \ \tau = f(\alpha), \alpha \in E_{I^n}].$$

The set Γ consists of arrows in E_J with the end vertex d that are lying in the image of the map f. Let $\gamma = (c \to d) \in \Gamma$ be an arrow such that

$$\begin{array}{cc} f(\alpha) = f(x \to y) = (c \to d) = \gamma & \text{and} \\ \Delta(\alpha, z) = \Delta(y, z) = k \geq 0 & \text{is minimal.} \end{array} \tag{3.2}$$

Note that α is not uniquely defined.

Lemma 3.3. *For every vertex $v \in V_{I^n}$ with $\Delta(v, z) \leq k$ we have $f(v) = d$. Hence the cube $I_{y,z} \sqsubset I^n$ is mapped by f into the vertex d.*

Proof. It follows immediately from definition of k in (3.2). ■

The arrow γ defines two $(m-1)$ – dimensional faces J_0 and J_1 of the cube J with $c \in V_{J_0}$, $d \in V_{J_1}$ and we have the natural projection $\pi\colon J \to J_0$ along the arrow γ. Let H be a subgraph of I^n. We define subgraphs $K_0, K_1, K \subset J$ that depend on the map $f\colon I^n \to J$ and $H \subset I^n$ as follows:

$$K\colon = f(H) \subset J, \quad K_0\colon = f(H) \cap J_0 \subset J_0, \quad \text{and} \quad K_1\colon = f(H) \cap J_1 \subset J_1. \tag{3.3}$$

It is easy to see that for an arrow $(v \to w) \in E_J$ we have:

$$[(v \to w) || \gamma] \ \Leftrightarrow \ [(v \in J_0) \ \& \ (w \in J_1)]. \tag{3.4}$$

For technical reasons we introduce the following definition.

Definition 3.4. Let H be a subgraph of I^n and $f\colon I^n \to J$ be a digraph map. Let the digraphs $K_0, K_1, K \subset J$ be defined as above using (3.2) and (3.3). We say that the subgraph H satisfies to the Π-*condition* if the following conditions are satisfied

$$\begin{array}{ll} (1) & \forall \;\; w\in V_{K_1} \text{ there is a vertex } v\in V_{K_0} \text{ such that } (v\to w)\in E_K. \\ (2) & \forall \;\; (w\to w')\in E_{K_1} \text{ we have } \; \pi(w\to w')\in E_{K_0}. \end{array} \tag{3.5}$$

The next statement is our key technical result,

Proposition 3.5. *Consider the map $f\colon I^n \to J = J^m$ with $m \geq 3$. Let k and γ are defined in (3.2). Then the cube I^n satisfies to the Π-condition.*

Proof. Using induction on $k \geq 0$.

The base of induction, $k = 0$. Hence $y = z = (1, \dots, 1) \in V_{I^n}$ is the end vertex of I^n and $n \geq m \geq 3$. The arrow $\alpha = (x \to z) \in E_{I^n}$ with

$$f(\alpha) = f(x \to z) = \gamma = (c \to d)$$

defines $(n-1)$-face $I_0 = I_{s,x}$ and opposite (n-1)-face I_1 of the cube I^n. Let $a = (0, \dots, 0)$ be the origin vertex of J (and hence origin vertex of J_0) and b be the origin vertex of J_1. Then $a \to b$ is parallel $\gamma = (c \to d)$. We have

$$f(I_0) = f(I_{s,x}) \subset I_{f(s),f(x)} = I_{a,c} = J_0 \tag{3.6}$$

and, hence, by (3.3) for $H = I^n$, we have $f(I_0) \subset K_0$. Let t be a vertex of I_1 such that $w = f(t) \notin V_{K_0}$ that is $w \in V_{K_1} \subset V_{J_1}$. There exists an unique vertex $r \in V_{I_0}$ such that $(r \to t) \in E_{I^n}$ is parallel to α and

$$f(r) = v \in K_0 \subset J_0$$

by (3.6). Thus $f(r \to t) = v \to w$ with $v \in V_{K_0}$ and condition (1) of (3.5) is satisfied.

Now let $\tau = (w \to w') \in E_{K_1}$ be an arrow such that $f(t \to t') = \tau$, that is

$$f(t) = w, \;\; f(t') = w', \;\; t, t' \in V_{I_1}.$$

The same line of arguments as above gives the vertices $r, r' \in V_{I_0}$ such that $(r \to t)$ and $r' \to t'$ are parallel to α and, hence, $\pi(\tau) = f(r \to r')$ since $f(r), f(r') \in V_{K_0}$. This proves condition (2) of (3.5). Thus Π-condition is satisfied for the cube I^n and $k = 0$.

The induction step. By inductive assumption we have that any map $f\colon I^n \to J$ satisfies the Π-condition if $\Delta(y,z) \leq k-1 \geq 0$. Consider the case $\Delta(y,z) = k \geq 1$ and, hence,

$$\Delta(x,z) = \Delta(y,z) + 1 = k + 1 \geq 2$$

where

$$z = (\underbrace{1, \dots, 1}_{n}) \in V_{I^n}.$$

Thus, without loss of generality, we can suppose that

$$x = (\underbrace{1, \dots, 1,}_{n-k-1}\underbrace{0, 0, \dots, 0,}_{k+1}), \;\; y = (\underbrace{1, \dots, 1,}_{n-k-1} 1, \underbrace{0, \dots, 0,}_{k}). \tag{3.7}$$

From now we put $y_0 = y \in V_{I^n}$ and let the vertex y_i is obtained from y by replacing the last coordinate "1" in y by "0", and i-th coordinate "0" of y by "1" for $1 \leq i \leq k$. For example,

$$y_2 = (\underbrace{1, \dots, 1,}_{n-k-1} 0, \underbrace{0, 1, 0 \dots, 0,}_{k}), \ \ y_k = (\underbrace{1, \dots, 1,}_{n-k-1} 0, \underbrace{0, 0 \dots, 0, 1}_{k}).$$

We define also

$$\alpha_i = (x \to y_i) \in E_{I^n} \ \text{ for } \ 0 \leq i \leq k.$$

By Lemma 3.3 we have

$$f(\alpha_i) = f(x \to y_i) = (c \to d) = \gamma \quad \text{for} \quad 0 \leq i \leq k.$$

Let $I_0 = I_{s,x}$ be $(n-k-1)$-dimensional subcube of I^n. Then, as before,

$$f(I_0) \subset K_0 \subset J_0.$$

Consider a vertex $t \in V_{I^n}$ and $t \notin V_{I_0}$ that has the form

$$t = (a_1, \dots, a_{n-k-1}, b_0, \dots, b_k) \notin I_0 \ \text{ where } a_i, b_j \in \{0,1\}$$

where at least one coordinate b_j is "1". If at least one coordinate b_j is zero we obtain that $t \in I_{s,z_j} \sqsubset I^n$ where

$$z_j = (\underbrace{1, \dots, 1,}_{n-k-1} \underbrace{1, \dots, \overset{j}{\widehat{0}}, \dots, 1}_{k+1}).$$

The $(n-1)$-dimensional subcube $I_{s,z_j} \subset I^n$ contains the vertices x and t. Moreover $\Delta(x, z_j) = k$ and there is an arrow

$$\alpha_i = (x \to y_i) \in E_{I_{s,z_j}}$$

with

$$f(\alpha_i) = \gamma \quad \text{and} \quad \Delta(\alpha_i, z_j) = k-1.$$

Hence, by the inductive assumption, the map

$$f|_{I_{s,z_j}} : I_{s,z_j} \to J$$

satisfies the Π-condition. Hence the conditions (1) and (2) of (3.5) are satisfied for every $(n-1)$-dimensional subcube $I_{s,z_j} \subset I^n$.

Now consider a vertex t for which all $(k+1)$-coordinates b_j are equal "1" such that $t \notin I_{x,z}$. This means that at least one of the first $(n-k-1)$-coordinates a_i is "0". Recall that $(k+1) \geq 2$. Thus consider the vertices

$$t = (a_1, \dots, a_{n-k-1}, \underbrace{1, \dots, 1}_{k+1}) \notin I_0, \quad r = (a_1, \dots, a_{n-k-1}, \underbrace{0, \dots, 0}_{k+1}) \in I_0 \tag{3.8}$$

where $a_i \in \{0,1\}$. Consider a directed path p in the digraph I_0 from the vertex $r \in V_{I_0}$ to the vertex $x \in V_{I_0}$ of the length $l = |p| \geq 1$ (since $t \notin I_{x,z}$). Write this path in the following form

$$p = (r \to x_1 \to x_2 \to \dots \to x_{l-1} \to x_l = x) \subset I_{r,x} \subset I_0.$$

Consider a directed path q from the vertex $r \in V_{I_0}$ to the vertex t of the length

$$k+1 = |q| \geq 2.$$

Note that q lies in the digraph $I_{r,t}$ of dimension $k+1$. Write this path in the following form

$$q = (r \to r^1 \to r^2 \to \dots \to r^k \to r^{k+1} = t) \subset I_{r,t}.$$

Any such two paths p and q defines a unique subgraph of the digraph I^n that has the following form

$$\begin{array}{ccccccccccc}
t = r^{k+1} & \longrightarrow & r_1^{k+1} & \longrightarrow & r_2^{k+1} & \longrightarrow & \dots & \longrightarrow & r_l^{k+1} & = z \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r^k & \longrightarrow & r_1^k & \longrightarrow & r_2^k & \longrightarrow & \dots & \longrightarrow & r_l^k & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r^1 & \longrightarrow & r_1^1 & \longrightarrow & r_2^1 & \longrightarrow & \dots & \longrightarrow & r_l^1 & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r & \longrightarrow & x_1 & \longrightarrow & x_2 & \longrightarrow & \dots & \longrightarrow & x_l & = x
\end{array} \tag{3.9}$$

Now we prove, using induction in the length $l = |p| \geq 1$ the following statement.

(L): For every path q and every path p, as above, there is a path

$$p' = (r \to x'_1 \to x'_2 \to \dots x'_{l-1} \to x'_l = x) \subset I_{r,x} \subset I_0,$$

(that may be equal to p) such that q and p' defines the subgraph (similarly above)

$$\begin{array}{ccccccccccc}
t = r^{k+1} & \longrightarrow & r_1^{k+1'} & \longrightarrow & r_2^{k+1'} & \longrightarrow & \dots & \longrightarrow & r_l^{k+1'} & = z \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r^k & \longrightarrow & r_1^{k'} & \longrightarrow & r_2^{k'} & \longrightarrow & \dots & \longrightarrow & r_l^{k'} & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r^1 & \longrightarrow & r_1^{1'} & \longrightarrow & r_2^{1'} & \longrightarrow & \dots & \longrightarrow & r_l^{1'} & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
r & \longrightarrow & x'_1 & \longrightarrow & x'_2 & \longrightarrow & \dots & \longrightarrow & x'_l & = x
\end{array} \tag{3.10}$$

and at least one of the following conditions is satisfied

$$\begin{array}{rl}
(i) & f(t) = f(r^k), \\
(ii) & f(t) = f(r_1^k), \\
(iii) & f(t) = f(r_1^{k'}).
\end{array} \tag{3.11}$$

The base of induction for (L), the case $l = 1$. Consider the unique path $p = (r \to x) \subset I_0$ of the length $l = 1$ an a path q as above. We have the following subgraph of the digraph I^n::

$$\begin{array}{cccc}
t = r^{k+1} & \longrightarrow & r_1^{k+1} & = z \\
\uparrow & & \uparrow & \\
r^k & \longrightarrow & r_1^k & \\
\uparrow & & \uparrow & \\
\dots & \longrightarrow & \dots & \\
\uparrow & & \uparrow & \\
r^1 & \longrightarrow & r_1^1 & \\
\uparrow & & \uparrow & \\
r & \longrightarrow & x_1 & = x
\end{array} \tag{3.12}$$

where

$$r, x \in V_{I_0} \quad \text{and} \quad f(r), f(x) \in V_{K_0},$$

and

$$f(r_1^i) = d \quad \text{for} \quad 1 \leq i \leq k+1$$

since $k \geq 1$. Hence,

$$f(r_1^k) = f(r_1^{k+1}) = d$$

and thus at least one of the conditions (i) or (ii) in (3.11) is satisfied because there are no triangles in the digraph J. We put in this case $p' = p$, and the base of induction $l = 1$ is proved.

Inductive step of induction for (L). Consider vertices $t, r \in V_J$ given in (3.8) where

$$\Delta(t, r) = k + 1 \geq 2 \quad \text{and} \quad \Delta(r, x) \geq 2.$$

Let p be a path from r to x and q be a path from r to t as the above. Recall that

$$|q| = k + 1 \geq 2 \quad \text{and} \quad |p| = l \geq 2.$$

These paths define the subgraph of I^n given on (3.9). By the inductive assumption, for the vertex r_1^{k+1} at least one of the conditions

$$\begin{array}{cc} (i) & f(r_1^{k+1}) = f(r_1^k), \\ (ii) & f(r_1^{k+1}) = f(r_2^k), \\ (iii) & f(r_1^{k+1}) = f(r_2^{k\prime\prime}), \end{array} \tag{3.13}$$

that is similar to (3.11) is realized. In (3.13) we have a path

$$r^k \to r_1^k \to r_2^{k\prime\prime} \to \cdots \to r_l^k$$

that is similar to the path

$$r^k \to r_1^k \to r_2^k \to \cdots \to r_l^k$$

from (3.9).

If condition (i) is realized, that is $f(r_1^{k+1}) = f(r_1^k)$, then for $f(t)$ at least one of the conditions (i) or (ii) in (3.11) is satisfied since there are no triangles in the digraph J (similarly to the case $l = 1$).

If condition (ii) is realized and condition (i) is not realized, that is

$$f(r_1^{k+1}) = f(r_2^k) \quad \text{and} \quad f(r_1^k) \neq f(r_2^k),$$

we can consider the subcube of I^n given on Fig. 2 that is defined by the subgraph of (3.9) given below in (3.14):

$$\begin{array}{ccccc} t = r^{k+1} & \longrightarrow & r_1^{k+1} & \longrightarrow & r_2^{k+1} \\ \uparrow & & \uparrow & & \uparrow \\ r^k & \longrightarrow & r_1^k & \longrightarrow & r_2^k. \end{array} \tag{3.14}$$

We have

$$f(r_1^{k+1}) = f(r_2^k) \quad \text{and} \quad f(r_1^k) \neq f(r_2^k),$$

that is

$$f(r_1^k \to r_1^{k+1}) = f(r_1^k \to r_2^k) \in E_J$$

is an arrow. If $f(r^k) = f(r_1^k)$ then the same line of above gives that

$$f(t) = f(r_1^k) \quad \text{or} \quad f(t) = f(r_2^k)$$

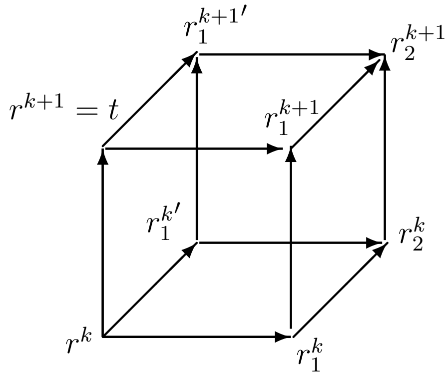

FIGURE 2. The subcube of I^n that is defined by the digraph on (3.13).

and the step of induction is proved. Let $f(r_k) \neq f(r_1^k)$ then

$$f\left(I_{r^k,\ r_2^k}\right) \subset f\left(I_{f\left(r^k\right),\ f\left(r_2^k\right)}\right) \quad \text{and} \quad f\left(I_{r^k,\ r_1^{k+1}}\right) \subset f\left(I_{f\left(r^k\right),\ f\left(r_2^k\right)}\right)$$

where $I_{f\left(r^k\right), f\left(r_2^k\right)}$ is the digraph square. Hence at least one of conditions

$$f(r^{k+1}) = f(r_1^k) \quad \text{or} \quad f(r^{k+1}) = f(r_1^{k'})$$

is satisfied and the inductive assumption is proved.

Consider the case when condition (iii) is realized and conditions (i) and (ii) are not realized. This case is the same as the case (ii). We must to start the consideration from the path

$$r^k \to r_1^k \to r_2^{k''} \to \cdots \to r_l^k$$

on the place of the path

$$r^k \to r_1^k \to r_2^k \to \cdots \to r_l^k$$

from (3.9). This finishes the proof of the inductive step and, statement (L) is proved.

Since each of the vertices r^k, r_1^k, $r_1^{k'}$ lies in the image of a subcube I_{r,z_j} it follows from the statement (L) that image $w = f(t)$ lies in the image of a subcube I_{r,z_j} with

$$\Delta(x, z_j) = \Delta(r, z_j) = k$$

which satisfies to Π-condition by the inductive assumption in k. Hence the condition (1) of (3.5) is satisfied for every subcube $I_{r,t} \subset I^n$. By a similar way, it follows from the statement (L) that the image of every arrow with end or origin t lies in the image of a subcube I_{r,z_j} which satisfies to Π-condition by the inductive assumption in k. Hence the condition (2) of (3.5) is satisfied for every subcube $I_{r,t} \subset I^n$. Hence every cube $I_{r,t}$ satisfies to the Π-condition and, hence, the cube I^n satisfies to the Π-condition. The Proposition is proved. ■

Theorem 3.6. *Let $f\colon I^n \to G$ be a digraph map to a cubical digraph. Then the image $f(I^n) \subset G$ is contractible.*

Proof. By the above the image $f(I^n)$ lies in the digraph $J = J^m$. Now we use the induction in m. For $m = 0, 1, 2$ the image $f(I^n)$ is contractible since all connected subgraphs of J are contractible. For $m \geq 3$ the digraph I^n satisfies the Π-condition, then Proposition 2.7 and (3.5) imply that restriction $\pi|_K$ of the projection $\pi\colon J^m \to J_0^{m-1}$ to the image K of the map f is well defined deformation retraction to K_0. But K_0 is contractible by the inductive assumption in m. Thus the theorem is proved. ■

4. Equivalence to homology theories on cubical digraphs

In this section we prove our main result – Theorem 1.1, that is stated below as Theorem 4.5. For that we use the Acyclic Carrier Theorem from homology theory (see, for example, [5, §3.4] and [6, §1.2.1]). Recall that a chain complex C_* is called *non-negative* if $C_p = 0$ for $p < 0$ and is called *free* if C_p are finitely generated free abelian groups for all p. We say that C_* is a *geometric chain complex* if it is non-negative, free, and if a basis $\mathcal{B}_p$ is chosen in the group C_p for any $p \geq 0$. For example, any finite simplicial complex gives rise to a geometric chain complex, where $\mathcal{B}_p$ consists of all p-simplexes.

Let C_* be a geometric chain complex with fixed bases $\mathcal{B}_p$. For $b \in \mathcal{B}_{p-1}$ and $b' \in \mathcal{B}_p$, we write $b \prec b'$ if b enters with a non-zero coefficient into the expansion of $\partial b'$ in the basis $\mathcal{B}_{p-1}$. The *augmentation homomorphism* $\varepsilon\colon C_0 \to \mathbb{Z}$ is defined as

$$\varepsilon\left(\sum_i k_i b_i\right) = \sum_i k_i, \quad k_i \in \mathbb{Z},\ b_i \in \mathcal{B}_0,$$

and we denote $\widetilde{C}_*$ the augmented complex

$$0 \longleftarrow \mathbb{Z} \overset{\varepsilon}{\longleftarrow} C_0 \overset{\partial}{\longleftarrow} C_1 \overset{\partial}{\longleftarrow} \dots$$

A geometric chain complex C_* is called *acyclic* if all homology groups of the augmented complex $\widetilde{C}_*$ are trivial.

Let C_* and D_* be two geometric complexes with augmentation homomorphism ε and ε', respectively. A chain map $\phi_*\colon C_* \to D_*$ is called *augmentation preserving* if $\varepsilon'\phi_0(c) = \varepsilon(c)$ for any $c \in C_0$.

Definition 4.1. Let C_* and D_* be two geometric chain complexes.

(i) An *algebraic carrier* function from C_* to D_* is a mapping E that assigns to any basis element b in C_* a subcomplex $E_*(b) := E(b)$ of D_*, such that $b \prec b'$ implies $E_*(b) \subset E_*(b')$.

(ii) An algebraic carrier function E is called *acyclic* if each complex $E_*(b)$ is non-empty and acyclic.

(iii) A chain map $f_*\colon C_* \to D_*$ is *carried by* E if $f_n(b) \in E_*(b)$ for any basis element b in C_n.

We state the Acyclic Carrier Theorem in the following form.

Theorem 4.2. [5, §3.4], [6, §1.2.1] *Let C_* and D_* be two geometric chain complexes and E be an acyclic carrier function from C_* to D_*. If $f_*, g_*\colon C_* \to D_*$ are augmentation preserving chain maps that are carried by E, then f_* and g_* are chain homotopic.*

Before the proof of Theorem 1.1, we state and prove some technical results. We use the notations of [1, 4]. Let G be a cubical digraph. The free abelian groups $\Omega_p^c = \Omega_p^c(G)$ and $\Omega_p = \Omega_p(G)$ defined in (2.3) and (2.8) are finitely generated.

Let $I^0 = \{*\}$ be the one-vertex digraph. Any zero-dimensional singular cube $\phi\colon I^0 = \{*\} \to G$ is given by the vertex $\phi(*) \in V_G$ and thus we obtain the map $\tau_0\colon \Omega_0^c(G) \to \Omega_0(G)$ which preserve augmentation.

For any digraph cube I^n ($n \geq 1$) denote by P the set of all directed paths of the length n going from the origin vertex $(\underbrace{0,\dots,0}_{n})$ of the cube to the end vertex $(\underbrace{1,\dots,1}_{n})$. Every path $p \in P$ has the following form

$$p = (a_0 \to a_1 \to a_2 \to \cdots \to a_n), \quad a_i \in V_{I_n}. \tag{4.1}$$

In (4.1) for $1 \leq i \leq n$ the vertex a_i differs from a_{i-1} only by one coordinate $1 \leq \pi(i) \leq n$ that equals "0" for a_{i-1} and "1" for a_i. Let $\sigma(p)$ be a sign of the permutation

$$\pi(p) = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}.$$

Consider the path $w_n \in \Omega_n(I^n)$ given by

$$w_n = \sum_{p\in P} (-1)^{\sigma(p)} p \tag{4.2}$$

that is the generator of the group $\Omega_n(I^n)$ (see [1] and [4]). For any singular n-dimensional cube $\phi\colon I^n \to G$, which gives a basic element $\phi^\square \in \Omega_n^c(G)$, we have a morphism of chain complexes defined in [1]

$$\tau_* : \Omega_*^c(G) \to \Omega_*(G), \quad \tau_n(\phi^\square) := \phi_*(w_n) \tag{4.3}$$

where $\phi_*\colon \Omega_*(I^n) \to \Omega_*(G)$ is the induced of ϕ morphism of chain complexes.

For $n \geq 0$ consider the set K_n of all subcubes G of dimension n that have the form $I_{s,t}$ with $s,t \in V_G$. By [1, 4], for every cube $I_{s,t} \in K_n$ there is an isomorphism $\chi_{s,t}\colon I^n \to I_{s,t}$ such that the set of elements

$$\{(\chi_{s,t})_*(w_n) : \ I_{s,t} \in K_n\}$$

give the basis of $\Omega_n(G)$. For $n \geq 1$, define homomorphisms $\theta_n\colon \Omega_n(G) \to \Omega_n^c(G)$ on basic elements by

$$\theta_n((\chi_{s,t})_*(w_n)) = \chi_{s,t}^\square, \tag{4.4}$$

and then extend it by linearity. It is clear that θ_0 preserves the augmentation.

Proposition 4.3. *The homomorphisms θ_n define a morphism of chain complexes*

$$\theta_*\colon \Omega_*(G) \to \Omega_*^c(G) \tag{4.5}$$

that is a right inverse morphism to τ_*, that is

$$\tau_*\theta_* = \mathrm{Id}\colon \Omega_*(G) \to \Omega_*(G).$$

Proof. Let us first prove that $\tau_n\theta_n = \mathrm{Id}$. For $n = 0, 1$ this is trivial. Let $n \geq 2$ and $(\chi_{s,t})_*(w_n) \in \Omega_n(G)$ be a basic element. By (4.4) and (4.3) we have

$$\tau_n\theta_n\left((\chi_{s,t})_*(w_n)\right) = \tau_n(\chi_{s,t}^\square) = \chi_{s,t*}(w_n). \tag{4.6}$$

Consider the diagram

$$\begin{array}{ccccc} \Omega_n(G) & \xrightarrow{\theta_n} & \Omega^c_n(G) & \xrightarrow{\tau_n} & \Omega_n(G) \\ \partial\downarrow & & \partial^c\downarrow & & \partial\downarrow \\ \Omega_{n-1}(G) & \xrightarrow{\theta_{n-1}} & \Omega^c_{n-1}(G) & \xrightarrow{\tau_{n-1}} & \Omega_{n-1}(G) \end{array} \tag{4.7}$$

where the horizontal compositions are identity homomorphisms by (4.6), the right square is commutative and the large square is evidently commutative. Now we prove that the left square is commutative. It follows from [4, Lemma 4] that, for

$$(\phi_{s,t})_*(w_n) \in \Omega_n(G),$$

we have

$$\begin{aligned} \theta_{n-1}\left(\partial\left((\phi_{s,t})_*(w_n)\right)\right) &= \theta_{n-1}\left(\sum_{I_{s',t'}\subset I_{s,t}} (-1)^{\sigma(I,I')}(\phi_{s',t'})_*(w_{n-1})\right) \\ &= \sum(-1)^{\sigma(I,I')}\phi^{\square}_{s',t'} \end{aligned} \tag{4.8}$$

where the sum is taken over all $(n-1)$-cubes $I' = I_{s',t'} \subset I_{s,t} = I$. By (2.7) and (4.4) we have. for

$$(\phi_{s,t})_*(w_n) \in \Omega_n(G),$$

that

$$\partial^c\left(\theta((\phi_{s,t})_*(w_n))\right) = \partial^c\left(\phi^{\square}_{s,t}\right) = \sum_{p=1}^{n}(-1)^p\left((\phi^{\square}_{s,t})_{p,0} - (\phi^{\square}_{s,t})_{p,1}\right) \tag{4.9}$$

where the sum consists of all singular $(n-1)$-subcubes of the cube I^n with coefficients. Since bottom row in (4.7) is the identity homomorphism we conclude from (4.3), (4.8) and (4.9) that the left square in (4.7) is commutative, which finishes the proof. ■

Proposition 4.4. *There is a chain homotopy between* $\theta_*\tau_*\colon \Omega^c_*(G) \to \Omega^c_*(G)$ *and the identity map* $\mathrm{Id}\colon \Omega^c_*(G) \to \Omega^c_*(G)$.

Proof. The chain complex $\Omega^c_*(G)$ is geometric and the chain maps $\theta_*\tau_*$ and Id evidently preserve augmentation. For a singular cube $\phi\colon I^n \to G$ consider the subgraph $G_\phi \subset G$ that is image of ϕ. This is a contractible cubical digraph by Theorem 3.6. Thus we assign to every basic element $\phi^{\square} \in \Omega^c_*(G)$ the subcomplex

$$E_*\left(\phi^{\square}\right) \overset{def}{=} \Omega^c_*(G_\phi) \subset \Omega^c_*(G) \tag{4.10}$$

which is acyclic since G_ϕ is contractible.

Now we check that E is an algebraic carrier function, that is condition (i) of Definition 4.1 is satisfied. Let $\phi^{\square} \in \Omega^c_*(G)$ be a basic element given by a singular cube $\phi\colon I^n \to G$ with $n \geq 0$. By (2.6) and (2.7), the element $\partial(\phi^{\square})$ is given by the sum of the basic elements $(\phi\circ V_{p\epsilon})^{\square}$ with coefficients (± 1) where the maps $V_{p\epsilon}\colon I^{n-1} \to I^n$ are the inclusions. Hence the digraph $G_{\phi\circ V_{p\epsilon}}$ is a subgraph of G_ϕ and, hence, the chain complex

$$E_*\left((\phi\circ V_{p\epsilon})^{\square}\right) = \Omega^c_*\left(G_{\phi\circ V_{p\epsilon}}\right)$$

is a subcomplex of $E_*(\phi^\square)$. Thus for the basic singular cube $b \in \Omega^c_{n-1}(G)$ and $b \prec \phi^\square$ we obtain that $b = (\phi \circ V_{p\epsilon})^\square$,

$$E_*(b) = E_*\left((\phi \circ V_{p\epsilon})^\square\right) \prec E_*(\phi^\square).$$

Hence we have the algebraic acyclic carrier function E from $\Omega^c_*(G)$ to itself.

Now we prove, that the chain maps $\theta_*\tau_*$ and Id from $\Omega^c_*(G)$ to itself are carried by the function E. Consider a basic element $\phi^\square \in \Omega^c_n(G)$. Then

$$\mathrm{Id}\left(\phi^\square\right) = \phi^\square \in \Omega^c_*(G_\phi) = E_*(\phi^\square) \tag{4.11}$$

since image of ϕ is the digraph G_ϕ. Hence the chain map

$$\mathrm{Id}\colon \Omega^c_n(G) \to \Omega^c_n(G)$$

is carried by the algebraic carrier function E.

By (4.3) and (4.4), we have

$$\theta_n\tau_n\left(\phi^\square\right) = \theta_n\left(\phi_*(w_n)\right), \quad \phi\colon I^n \to G. \tag{4.12}$$

We have only two different possibilities for the $\phi_*(w_n)$. In the first case, ϕ is an isomorphism on its image $G_\phi = I_{s,t} \cong I^n$ with

$$s = \phi(0,\dots,0),\ t = \phi(1,\dots,1),$$

where $(0,\dots,0) \in V_{I^n}$ is the origin vertex, and $(1,\dots,1) \in V_{I^n}$ is the end vertex of the cube I^n. Note that for any isomorphism $\psi\colon I^n \to I^n$ we have $\psi_*(w_n) = \pm w_n$. Hence in this case subgraph $G_\phi \subset G$ coincides with the subgraph cube $G_{\chi_{s,t}} \subset G$ and by (4.4) we have

$$\theta_n\tau_n(\phi^\square) = \theta_n\left(\phi_*(w_n)\right) = \theta_n\left(\pm(\chi_{s,t})_*(w_n)\right) = \pm\chi^\square_{s,t} \tag{4.13}$$

where

$$\chi_{s,t}\colon I^n \to D_{s,t} = G_\phi.$$

That is,

$$\theta_n\tau_n\left(\phi^\square\right) \in \Omega^c_n\left(G_{\chi_{s,t}}\right) = \Omega^c_n(G_\phi) = E_n\left(\phi^\square\right).$$

In the second case, the image of ϕ does not contain any cube of dimension n and, hence $\phi_*(w_n) = 0$. Consequently, we have

$$\theta_n\phi_*(w_n) = 0 \in E_*(\phi^\square).$$

Then the claim follows from the Acyclic Carriers Theorem 4.2. ■

Theorem 4.5. *For any finite cubical digraph G, the chain maps τ_* and θ_* are homotopy inverses and, hence, induce isomorphisms of homology groups*

$$H^c_*(G) \cong H_*(G).$$

Proof. Indeed, It follows from Propositions 4.3 and 4.4 that the chain maps τ_* and θ_* are homotopy inverses. ■

Corollary 4.6. *Let Δ be a finite simplicial complex. Consider a digraph G_Δ (see [4]) with the set of vertices given by the set of all simplexes from Δ, and*

$$s \to t\ (t, s \in \Delta) \quad \textit{iff} \quad s \supset t \quad \textit{and}\ \dim s = \dim t + 1.$$

Then the graph G_Δ is a cubical digraph and

$$H^c_*(G_\Delta) \cong H_*(\Delta)$$

where $H_*(\Delta)$ *are the simplicial homology groups of* Δ*.*

Proof. Indeed, it is proved in [4] that path homology groups $H_*(G_\Delta)$ are isomorphic to simplicial homology groups $H_*(\Delta)$. ■

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