Homology of digraphs

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Abstract

In the paper we define the singular cubical homology theory of digraphs and describe its basic properties. In particular, we prove the functoriality and homotopy invariance of theses homologies and compare the theory with the path homology theory that was introduced in our previos papers. Then we describe the transfer of the homology theory to the category of graphs.

Contents

1 Introduction

In this paper we construct a *singular cubical homology theory of digraphs* and describe its basic properties, including functoriality and homotopy invariance. This approach is motivated by the construction of singular homology theories of graphs in [3, 4] and [14], which were based on various types of graph cubes. We work in the category of digraphs defined in [7], where the homotopy theory of digraphs was constructed.

Here we compare the singular cubical homology theory with the *path homology theory* of digraphs that was constructed in the previous papers of the authors (see, for example, [7], [8], [9]), and show that these theories are not equivalent.

Considering the category of (undirected) graphs as the full subcategory of digraphs we transfer the results from our category of digraphs to the category of graphs and obtain the singular cubical homology theory for graphs that coincides with the discrete homology theory of [3, 4]. Such an approach, in particular, gives new properties of singular homology groups of (undirected) graphs.

In the paper we consider a commutative ring R with unity as a ring of coefficients unless otherwise stated. Let G be a finite digraph. Denote by $\Omega_*(G)$ the chain complex of ∂-invariant paths on G over R (see Section 2). Our main results are as follows.

(i) Construction of the (normalized) singular cubical chain complex $\Omega_*^c(G)$ on G and proof of functoriality and homotopy invariance of its homology groups $H_*^c(G)$ (Section 4).

(ii) Construction of a natural morphism of chain complexes

$$\tau_*\colon \Omega_*^c(G) \to \Omega_*(G) \tag{1.1}$$

such that the homomorphism τ_n is an isomorphism for $n = 0, 1$ and an epimorphism for $n = 2$ (Propositions 5.2, 5.3, 5.4).

(iii) The homomorphism $H_n^c(G) \longrightarrow H_n(G)$ of homology groups induced by τ_* is an isomorphism for $n = 0, 1$ and an epimorphism for $n = 2$ (Proposition 5.4).

(iv) We construct a connected digraph G for which the groups $H_n(G, \mathbb{Z})$ are trivial for $n \geq 1$, but the group $H_2^c(G, \mathbb{Z})$ is non-trivial. Hence, in general the groups $H_n(G)$ and $H_n^c(G)$ are not isomorphic for $n \geq 2$ (Theorems 5.6 and 5.9).

The paper is organized as follows. In Section 2, we give preliminary definitions and cite some results of the homotopy theory of digraphs and cite the necessary results about the path homology theory (see [7], [8], [9]).

In Section 3, we describe the properties of cubical digraphs which we need in the next sections.

In Section 4, we define the singular cubical homology theory of digraphs and prove its basic properties.

In Section 5, we compare the singular cubical homology theory with the path homology theory.

In Section 6, we transfer the above theory from the category of digraphs to that of graphs.

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2 Homotopy and path homology of digraphs

In this Section we cite the necessary material about homotopy of digraphs following [7], [8], [9].

Definition 2.1 *A directed graph (digraph)* $G = (V_G, E_G)$ is a set $V = V_G$ of *vertices* and a subset $E_G \subset \{V_G \times V_G \setminus \text{diagonal}\}$ of ordered pairs that are called *arrows.* If $(v, w) \in E_G$ then we shall write $v \to w$.

For two vertices $v, w \in V_G$, we write $v \overset{=}{\to} w$ if either $v = w$ or $v \to w$.

Definition 2.2 A *digraph map* (or simply *map*) from a digraph G to a digraph H is a map $f\colon V_G \to V_H$ such that $v \to w$ on G implies $f(v) \overset{=}{\to} f(w)$ on H.

The set of all digraphs with digraph maps forms a category that will be denoted by $\mathcal{D}$.

Definition 2.3 For digraphs G, H let define their *Cartesian product* $\Pi = G \Box H$ as a digraph with a set of vertices $V_\Pi = V_G \times V_H$ and a set of arrows E_Π given by the rule

$$(x, y) \to (x', y') \text{ if } x = x' \text{ and } y \to y', \text{ or } x \to x' \text{ and } y = y',$$

where $x, x' \in V_G$ and $y, y' \in V_H$.

Fix $n \geq 0$. Define a line digraph I_n as a digraph with the set of vertices $V = \{0, 1, \dots, n\}$ and such that, for any $i = 0, 1, \dots n - 1$, there is exactly one arrow $i \to i + 1$ or $i + 1 \to i$, and there are no other arrows. There are only two line digraphs with two vertices. One of them with the only arrow $0 \to 1$ will be denoted by I.

Definition 2.4 Two digraph maps $f, g\colon G \to H$ are called *homotopic* if there exists a line digraph I_n with some $n \geq 0$ and a digraph map $F\colon G \Box I_n \to H$ such that

$$F|_{G\Box\{0\}} = f, \quad F|_{G\Box\{n\}} = g,$$

where we identify $G \Box \{i\}$ with G. In this case we shall write $f \simeq g$. The map F is called a *homotopy* between f and g.

The relation $\simeq$ is an equivalence relation on the set of digraph maps from G to H. Thus, we obtain the category $\mathcal{D}'$ with the same objects as in $\mathcal{D}$ and morphisms are given by the classes of homotopic maps.

Definition 2.5 Two digraphs G and H are called *homotopy equivalent* if there exist digraph maps

$$f\colon G \to H, \quad g\colon H \to G$$

such that

$$f \circ g \simeq \mathrm{Id}_H, \quad g \circ f \simeq \mathrm{Id}_G.$$

In this case, we write $H \simeq G$, and the maps f and g are called *homotopy inverses* to each other.

Now we recall the definition of path homology groups of a digraph from [9] with coefficients in R. Let V be a finite set, whose elements will be called vertices. An *elementary p-path* on a finite set V is any (ordered) sequence $i_0, ..., i_p$ of $p+1$ vertices of V that will be denoted by $e_{i_0 \dots i_p}$. Denote by $\Lambda_p = \Lambda_p(V)$ the free R-module generated by all elementary p-paths $e_{i_0 \dots i_p}$. The elements of Λ_p are called *p-paths*. Thus each p-path $v \in \Lambda_p$ has the form

$$v = \sum_{i_0, \dots, i_p \in V} v^{i_0 i_1 \dots i_p} e_{i_0 i_1 \dots i_p},$$

where $v^{i_0 i_1 \dots i_p} \in R$ are the coefficients of v.

For $p \geq 1$, define the *boundary* operator $\partial \colon \Lambda_p \to \Lambda_{p-1}$ on basic elements by

$$\partial e_{i_0 \dots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \dots \widehat{i_q} \dots i_p}, \tag{2.1}$$

where $\widehat{k}$ means deleting of the corresponding index, and extend it to Λ_p by linearity. Let $\Lambda_{-1} = 0$, and define $\partial \colon \Lambda_0 \to \Lambda_{-1}$ by $\partial v = 0$ for all $v \in \Lambda_0$. It follows from this definition that $\partial^2 v = 0$. for any p-path v.

An elementary p-path $e_{i_0 \dots i_p}$ $(p \geq 1)$ is called *regular* if $i_k \neq i_{k+1}$ for all k. For $p \geq 1$, let $\mathcal{I}_p$ be the submodule of Λ_p that is spanned by all irregular $e_{i_0 \dots i_p}$ and we set $\mathcal{I}_0 = \mathcal{I}_{-1} = 0$. Then $\partial(\mathcal{I}_{p+1}) \subset \mathcal{I}_p$ for $p \geq -1$. Consider the quotient chain complex $\mathcal{R}_*$ with

$$\mathcal{R}_p = \mathcal{R}_p\,(V) = \Lambda_p / \mathcal{I}_p$$

and with the chain map that is induced by ∂.

Now we define paths on a digraph $G = (V, E)$. Let $e_{i_0 \dots i_p}$ be a regular elementary p-path on V. It is called *allowed* if $i_{k-1} \to i_k$ for any $k = 1, \dots, p$, and *non-allowed* otherwise. For $p \geq 1$, denote by $\mathcal{A}_p = \mathcal{A}_p\,(G)$ the submodule of $\mathcal{R}_p$ spanned by the allowed elementary p-paths, that is,

$$\mathcal{A}_p = \operatorname{span}\left\{ e_{i_0 \dots i_p} : i_0 \dots i_p \text{ is allowed} \right\}.$$

and set $\mathcal{A}_{-1} = 0$. The elements of $\mathcal{A}_p$ are called *allowed* p-paths.

Consider the following submodule of $\mathcal{A}_p$ $(p \geq 0)$

$$\Omega_p = \Omega_p\,(G) = \{ v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1} \}. \tag{2.2}$$

The elements of Ω_p are called *∂-invariant* p-paths, and we obtain a chain complex

$$0 \;\leftarrow\; \Omega_0 \;\overset{\partial}{\leftarrow}\; \Omega_1 \;\overset{\partial}{\leftarrow}\; \dots \;\overset{\partial}{\leftarrow}\; \Omega_{p-1} \;\overset{\partial}{\leftarrow}\; \Omega_p \;\overset{\partial}{\leftarrow} \dots \tag{2.3}$$

The *path homology groups of the digraph G with the coefficients in the ring R* are defined as

$$H_p\,(G, R) := \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

In what follows, we will refer to $H_p(G) = H_p(G, R)$ as the *path homology groups of a digraph G*.

Let G and G' be two digraphs, and $f \colon G \to G'$ be a digraph map. Then map f induces a chain map (see [7])

$$f_* \colon \Omega_*(G) \longrightarrow \Omega_*(G') \tag{2.4}$$

Note also, that the path homology groups are homotopy invariant [7].

3 Cubical digraphs

Recall that I denotes the digraph ${}^0\bullet \to \bullet^1$. and $I_0 = \{0\}$ – the one-point digraph. For any $n \geq 0$, define a *standard* *n-cube digraph I^n* by

$$I^n = \begin{cases} I_0 & \text{for } n = 0, \\ \underbrace{I \Box I \Box \dots . \Box I}_{n \text{ times}} & \text{for } n \geq 1. \end{cases}$$

Equivalently, for any $n \geq 1$, I^n has 2^n vertices such that any vertex a of I^n can be identified with a sequence $a = (a_1, \ldots, a_n)$ of binary digits so that $a \to b$ if and only if the sequence $b = (b_1, \ldots, b_n)$ is obtained from a by replacing a digit 0 by 1 at exactly one position.

The standard 2-cube is also referred to as a *square* and is shown on the diagram:

$$\begin{array}{ccc} {}^{(0,1)}\bullet & \longrightarrow & \bullet^{(1,1)} \\ \uparrow & & \uparrow \\ {}^{(0,0)}\bullet & \longrightarrow & \bullet^{(1,0)} \end{array} \tag{3.1}$$

Any digraph that is isomorphic to the standard n-cube is called an *n-cube digraph.*

For any vertex $a = (a_i)_{i=1}^n$ of I^n, set

$$N(a) := \sum_{i=1}^{n} 2^i a_i.$$

If $b = (b_i)_{i=1}^n$ is another vertex of I^n such that $a \to b$ then the sequence (b_i) is obtained from (a_i) by replacing 0 by 1 exactly at one position i, which implies

$$N(b) - N(a) = 2^i.$$

Let $\alpha = \{\alpha_k\}_{k=0}^m$ be an allowed m-path on I^n, that is, $\alpha_0 \to \alpha_1 \to \ldots \to \alpha_m$. Define $\sigma(\alpha)$ as the number of inversions in the sequence

$$N(\alpha) := \{N(\alpha_k) - N(\alpha_{k-1})\}_{k=1}^m .$$

In other words, if α_k is obtained from α_{k-1} by replacing 0 by 1 at position i_k then $\sigma(\alpha)$ is the number of inversions in the sequence $\{i_1, \ldots, i_m\}$.

Given two vertices a and b of I^n, we write $a \prec b$ if for the corresponding binary sequences (a_i) and (b_i) holds $a_i \leq b_i$ for all i. In other words, $a \prec b$ if and only if there is an allowed path on I^n that starts at a and ends at b.

Assuming that $a \prec b$, set $m = |b| - |a|$, where $|\cdot|$ means the number of digits 1 in the binary sequence. Then the pair a, b determines the induced sub-digraph $D_{a,b}$ of I^n whose set of vertices is

$$\{c \in I^n : a \prec c \prec b\}.$$

We claim that $D_{a,b}$ is an m-cube digraph, where the digraph isomorphism

$$\chi_{a,b} : I^m \to D_{a,b}$$

is given as follows. There is exactly m values of the index $k \in \{1, \ldots, n\}$ such that $a_k < b_k$; denote them in increasing order by $k_1, \ldots, k_m$ so that

$$a_{k_i} = 0 \quad \text{and} \quad b_{k_i} = 1.$$

Then, for any vertex $z = (z_i)_{i=1}^m \in I^m$, define $\widetilde{z} \in I^n$ by

$$\widetilde{z}_k = \begin{cases} z_i & \text{if } k = k_i \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

and set

$$\chi_{a,b}(z) = a + \widetilde{z}.$$

For example, if $z = (0)_{i=1}^m$ then $\chi_{a,b}(z) = a$ and if $z = (1)_{i=1}^m$ then $\chi_{a,b}(z) = b$.

Denote by $P(a,b)$ the set of allowed paths between a and b. Clearly, every path in $P(a,b)$ has the length m. Define the following allowed m-path:

$$\omega_{a,b} = \omega(D_{a,b}) = \sum_{\alpha \in P(a,b)} (-1)^{\sigma(\alpha)} e_\alpha \tag{3.3}$$

that is the generator of $\Omega_m(D_{a,b})$ by [9]. In the case $a = (0, \ldots, 0)$ and $b = (1, \ldots, 1)$, the cube $D_{a,b}$ coincides with the cube I^n and we denote the allowed n-path $\omega_{a,b}$ by ω_n.

For example, if $a = (0,0)$ and $b = (1,1)$ (cf. the diagram (3.1)), then $m = 2$ and $P(a,b)$ consists of two 2-paths

$$\alpha = (0,0) \to (1,0) \to (1,1) \quad \text{and} \quad \beta = (0,0) \to (0,1) \to (1,1).$$

Since $N(\alpha) = \{2,2\}$ and $N(\beta) = \{4,2\}$, we have $\sigma(\alpha) = 0$ and $\sigma(\beta) = 1$, which implies

$$\omega_2 = \omega_{a,b} = e_\alpha - e_\beta.$$

Lemma 3.1 *Let $D_{a,b}$ be an m-cube digraph as above. Then*

$$\left(\chi_{a,b}\right)_* (\omega_m) = \omega_{a,b}.$$

Proof. Indeed, for any m-path z on I^m, we have $\chi_{a,b}(z) = \widetilde{z}$ where $\widetilde{z}$ is defined by (3.2). It follows that $\sigma(z) = \sigma(\widetilde{z})$. Hence, the result follows from the definition (3.3) of $\omega_{a,b}$. ■

Fix an m-cube $D = D_{a,b}$ as above and let $D' = D_{a',b'}$ be an $(m-1)$-cube such that $D' \subset D$. Then we have

$$a \prec a' \prec b' \prec b,$$

and since $|b| - |a| = |b'| - |a'| + 1$, there are only two possibilities:

(i) either $a \to a'$ and $b = b'$,

(ii) or $a = a'$ and $b' \to b$.

Define a number $\sigma(D, D')$ as follows. In the case (i), consider any m-path $\alpha = \{\alpha_k\}_{k=0}^m \in P(a,b)$ such that $\alpha_1 = a'$, and set

$$\sigma\left(D, D'\right) = \sigma(\alpha) - \sigma\left(\alpha'\right), \text{ where } \alpha' = \{\alpha_k\}_{k=1}^m. \tag{3.4}$$

In the case (ii), consider any m-path $\alpha = \{\alpha_k\}_{k=0}^m \in P(a,b)$ such that $\alpha_{m-1} = b'$, and set

$$\sigma\left(D, D'\right) = n + \sigma(\alpha) - \sigma\left(\alpha'\right), \text{ where } \alpha' = \{\alpha_k\}_{k=0}^{m-1}. \tag{3.5}$$

It is possible to prove that $\sigma(D, D')$ does not depend on the choice of α.

Example 3.2 Let $a = (0, ..., 0)$ and $b = (1, ..., 1)$ so that $D = D_{a,b} = I^n$. Let $a' = (0, ..., \overset{j}{\widehat{1}}, ..., 0)$ where the only digit 1 is at position j, and consider the $(n-1)$-cube $D_{j1} := D_{a',b}$. In order to compute $\sigma(D, D_{j1})$, consider the path $\alpha = \{\alpha_k\}_{k=0}^n$ from a to b such that $\alpha_1 = a'$ and that α_k with $k \geq 2$ is obtained from α_{k-1} by replacing 0 by 1 at the first available position from the left hand side. Let $\alpha' = \{\alpha_k\}_{k=1}^n$. Then $\sigma(\alpha') = 0$ while $\sigma(\alpha)$ is the number of inversion with j that is $j-1$. Therefore, we obtain by (3.4)

$$\sigma(D, D_{j1}) = \sigma(\alpha) - \sigma\left(\alpha'\right) = j - 1. \tag{3.6}$$

Consider now an example of the case (ii). Let a and b be as above. Set $b' = (1, ..., \overset{j}{\widehat{0}}, ..., 1)$ where the only digit 0 is at position j. Setting $D_{j0} := D_{a,b'}$, we obtain similarly that $\sigma(\alpha') = 0$, $\sigma(\alpha) = n - j$ and from (3.5)

$$\sigma(D, D_{j0}) = 2n - j. \tag{3.7}$$

Theorem 3.3 *[9] Set $D = D_{a,b}$ and $m = |b| - |a|$. Then the following identity holds*

$$\partial\omega(D) = \sum_{D' \subset D} (-1)^{\sigma(D,D')} \omega\left(D'\right), \tag{3.8}$$

where D' runs over all $(m-1)$-cubes such that $D' \subset D$. Consequently, ω is ∂-invariant.

Fix $m \geq 1$. For any $1 \leq j \leq m$ and $\epsilon = 0, 1$, consider the following inclusion of digraphs:

$$\begin{gathered} F_{j\epsilon}^{m-1} \colon I^{m-1} \to I^m \\ F_{j\epsilon}^{m-1}(c_1, \ldots, c_{m-1}) = (c_1, \ldots, c_{j-1}, \epsilon, c_j, \ldots, c_{m-1}) \end{gathered} \tag{3.9}$$

for $m \geq 2$, and $F_{1\epsilon}^{m-1}(0) = (\epsilon)$ for $m = 1$. We shall write shortly $F_{j\epsilon}$ instead of $F_{j\epsilon}^{m-1}$ if the dimension $m-1$ is clear from the context.

Proposition 3.4 *Let $D = D_{a,b}$ be an m-cube with $m \geq 1$, and $D' = D_{a',b'} \subset D$ be an $(m-1)$-cube as above. Let $k_1 < k_2 < ... < k_m$ be the sequence of indices such that $a_{k_i} < b_{k_i}$.*

- *In the case (i), define $j \in \{1, ..., m\}$ as the only index such that $a_{k_j} = 0$ and $a'_{k_j} = 1$. Then*
$$\chi_{a,b} \circ F_{j1} = \chi_{a',b}.$$

- *in the case (ii), define $j \in \{1, ..., m\}$ as the only index such that $b'_{k_j} = 0$ and $b_{k_j} = 1$. Then*
$$\chi_{a,b} \circ F_{j0} = \chi_{a,b'}.$$

Proof. For any $c = (c_1, ..., c_{m-1}) \in I^{m-1}$, set

$$c' = F_{j1}(c) = (c_1, .., c_{j-1}, 1, c_j, ..., c_{m-1})$$

so that

$$\chi_{a,b} \circ F_{j1}(c) = \chi_{a,b}\left(c'\right) = a + \widetilde{c'},$$

where $\widetilde{c'} \in I^n$ is defined by (3.2), that is, the components of $\widetilde{c'}$ at positions $k_1, ..., k_m$ are

$$c_1, .., c_{j-1}, 1, c_j, ..., c_{m-1},$$

and all other components of $\widetilde{c'}$ are zeros.

The sequence a' differs from b at positions k_i except for k_j because $a'_{k_j} = 1 = b_{k_j}$. Therefore,

$$\chi_{a',b}(c) = a' + \widetilde{c},$$

where the components of $\widetilde{c}$ at positions $k_1, .., k_{j-1}, k_{j+1}, ..., k_m$ are

$$c_1, ..., c_{j-1}, c_{j+1}, ..., c_{m-1},$$

and all other components of $\widetilde{c}$ are zeros. Since $a' - a$ has component 1 at position k_j and all other components 0, we conclude that

$$a + \widetilde{c'} = a' + \widetilde{c},$$

which finishes the proof in the case (i). The case (ii) is similar. ■

4 The singular cubical homology of digraphs

In this section we construct the cubical singular homology theory of digraphs and study its properties. This construction is similar to the corresponding construction in algebraic topology [11, §8.3].

Let I^n be the standard n-cube digraph defined in the previous section. *A singular n-cube in a digraph* G is a digraph map $\phi\colon I^n \to G$. In particular, the set of vertices V_G of a digraph G is in one-to-one correspondence with the set of singular 0-cubes in G. Similarly, it is sufficiently easy to describe all singular n-cubes in finite digraph for $n = 1, 2, 3$. It is easy to see, that for a finite digraph G there are only finite number of singular n-cubes for any $n \geq 0$.

For $n \geq 0$, let $Q_n = Q_n(G)$ denote the free R-module generated by all singular n-cubes in G. We denote $\phi^{\square}$ the singular n-cube $\phi\colon I^n \to G$ as the element of the module Q_n. For $n \geq 1$ and $1 \leq j \leq n$, let

$$\phi^{\square}_{j\epsilon} = (\phi \circ F_{j\epsilon})^{\square} \in Q_{n-1} \tag{4.1}$$

where the inclusions $F_{j\epsilon}$ are defined in (3.9). We put also $Q_{-1} = 0$.

For $n \geq 1$, define a homomorphism $\partial^c\colon Q_n \to Q_{n-1}$ on the basis elements $\phi^{\square}$ by the rule

$$\partial^c(\phi^{\square}) = \sum_{j=1}^{n} (-1)^j \left(\phi^{\square}_{j0} - \phi^{\square}_{j1}\right), \tag{4.2}$$

and $\partial^c = 0$ for $n = 0$.

Proposition 4.1 *We have $(\partial^c)^2 = 0$, and hence the groups $Q_n (n \geq -1)$ with the differential ∂^c give rise to a chain complex $Q_* = Q_*(G)$.*

Proof. The proof is similar to the proof in [11]. ■

Proposition 4.2 *Let I^0 be the one-point digraph. Then*

$$H_n(Q_*(I^0)) = R \ \ for \ \ n \geq 0.$$

Proof. Direct computing similarly to [11, Theorem 8.3.2]. ■

For $n \geq 1$ and $1 \leq j \leq n$, consider the natural projection $T^j\colon I^n \to I^{n-1}$ defined by the following way. For $n = 1$ the T^1 is the unique digraph map $I^1 \to I^0$. For $n \geq 2$ we define the map T^j on the set of vertices by

$$T^j(i_1, \ldots, i_n) = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n).$$

We shall call this map the *projection on the j-face*. We shall call the singular n-cube $\phi\colon I^n \to G$ *degenerate* if for an index j it can be presented as a composition $\psi \circ T^p$ where $\psi\colon I^{n-1} \to G$ is a singular $(n-1)$-cube in G. Denote by $B_n = B_n(G)$ the free submodule of the module Q_n that is generated by all degenerate n-cubes for $n \geq 1$ and put $B_0 = 0, B_{-1} = 0$.

Proposition 4.3 *We have* $\partial^c(B_n) \subset B_{n-1}$.

Proof. The proof is standard [11]. ■

Thus for any digraph G we have a chain subcomplex $B_*(G) \subset Q_*(G)$. Hence the quotient complex $\Omega^c_*(G) = Q_*(G)/B_*(G)$ is well defined. We shall call this complex the *(normalized) singular cubical chain complex of the digraph G*. The homology group $H_k(\Omega_*(G))$ is called the *singular cubical homology group of digraph G in dimension k* or simply *singular cubical homology group*. We shall denote this group by $H^c_k(G) = H^c_k(G, R)$.

Note that we can define a natural augmentation

$$\varepsilon\colon \Omega^c_0 \to R \ \text{ by } \ \varepsilon\left(\sum k_i\phi_i\right) = \sum k_i, \ k_i \in \mathbb{Z}$$

which is an epimorphism and $\varepsilon \circ \partial^c = 0$. In this case we obtain reduced cubical homology groups. However, in what follows we do not use them, although the main results can be stated also in the case of reduced homologies.

Theorem 4.4 *For the one-point digraph I^0, we have*

$$H^c_k(I^0) = \begin{cases} R, & \textit{for } \ k = 0 \\ 0, & \textit{for } \ k \geq 1. \end{cases}$$

Proof. The proof is similar to [11]. ■

From now we describe the basic properties of singular cubical homology groups. Any digraph map $f\colon X \to Y$ induces a chain map

$$f_*\colon Q_*(X) \to Q_*(Y), \ f_*(\phi^\square) = \begin{cases} (f \circ \phi)^\square, & \text{for } \phi^\square \in Q_n(X) \text{ and } n \geq 0 \\ f_*(0) = 0, & \text{for } n = -1. \end{cases}$$

It is easy to see that $f_*(B_n(X)) \subset B_n(Y)$, hence the chain map

$$\Omega^c_*(X) = Q_*(X)/B_*(X) \to \Omega^c_*(Y) = Q_*(Y)/B_*(Y)$$

is well defined.

It is clear, that $(f \circ g)_* = f_* \circ g_*$, and for the identity digraph map $\mathrm{Id}\colon X \to X$ we have $\mathrm{Id}_* = \mathrm{Id}_{\Omega^c_*(X)}$. Thus we obtain a functor from the category $\mathcal{D}$ of digraphs to the category $\mathcal{C}$ of chain complexes. It follows that, for $k \geq 0$, the digraph map f induces also homomorphisms

$$f_*\colon \ H^c_k(X) \to H^c_k(Y)$$

of homology groups. It is clear, that $(f\circ g)_* = f_*\circ g_*$ and for the identity map $\mathrm{Id}\colon X\to X$ we have $\mathrm{Id}_* = \mathrm{Id}_{H_k^c(X)}$. Thus for any $i\geq 0$, the groups $H_k^c(X)$ provide a functor from the category $\mathcal{D}$ to the category of abelian groups $\mathcal{A}$.

Theorem 4.5 *Let $f\simeq g\colon X\to Y$ be two homotopic digraph maps. Then*

$$f_* = g_*\colon H_k^c(X)\to H_k^c(Y) \quad \textit{for any} \quad k\geq 0.$$

Proof. Similar to [11, Theorem 8.3.8].
■

Corollary 4.6 *If $f\colon X\to Y$ is a homotopy equivalence of digraphs, then, for any $k\geq 0$,*

$$f_*\colon H_k^c(X)\to H_k^c(Y)$$

is an isomorphism.

Now we recall standard definitions of *an union* and *an intersection* of a family of digraphs.

Definition 4.7 Let $\{G_i\}_{i\in A}$ be a family of sub-digraphs of a digraph, where A is any index set.

i) The *union $G=\bigcup_{i\in A} G_i$ of digraphs G_i* is a digraph G such that

$$V_G = \bigcup_{i\in A} V_{G_i},\quad E_G = \bigcup_{i\in A} E_{G_i}$$

ii) The *intersection $G=\bigcap_{i\in A} G_i$ of digraphs G_i* is a digraph G such that

$$V_G = \bigcap_{i\in A} V_{G_i},\quad E_G = \bigcap_{i\in A} E_{G_i}.$$

Definition 4.8 Let $G=G_1\cup G_2$. We shall say that the sub-digraphs G_1 and G_2 provide a *special cover* of the digraph G if the image $\phi(I^n)$ of any singular non-degenerated n-dimensional cube $\phi\colon I^n\to G$ lies at least in one of the sub-digraphs G_1 or G_2.

Example 4.9 We give here several examples of special covers of digraphs.

i) $G=G_1\cup G_2$ with $G_1\cap G_2=\emptyset$.

ii) $G=G_1\cup G_2$ with $G_1\cap G_2 = *$ — is a vertex, and all arrows that are incident to the vertex $*$ has the form $*\to v, v\in V_G$ or the form $v\to *, v\in V_G$.

iii) $G=G_1\cup G_2$, $G_1\cap G_2 = v\to w$, and there are no outcoming arrows from w and incoming arrows to v in G.

iv) $G=G_1\cup G_2$, $G_1\cap G_2 = v\to w\leftarrow s$, and there are no arrows in G except $v\to w, s\to w$ that are incident to v or s and all arrows in G that are incident to the vertex w has the form $*\to w, w\in V$.

Let $G=G_1\cup G_2$. Then we can write down the following commutative diagram of the natural inclusions of the digraphs:

$$\begin{array}{ccc} G_1\cap G_2 & \overset{i^1}{\to} & G_1 \\ \downarrow i^2 & & \downarrow j^1 \\ G_2 & \overset{j^2}{\to} & G \end{array} \tag{4.3}$$

Lemma 4.10 *Let $G = G_1 \cup G_2$ be a special cover of a digraph G. Then diagram (4.3) induces the following short exact sequence of chain complexes:*

$$0 \longrightarrow \Omega^c_*(G_1 \cap G_2) \xrightarrow{\delta} \Omega^c_*(G_1) \oplus \Omega^c_*(G_2) \xrightarrow{d} \Omega^c_*(G) \longrightarrow 0, \tag{4.4}$$

where $\delta = (i^1_*, i^2_*)$ and $d(a,b) = j^1_*(a) - j^2_*(b)$.

Proof. The map $\delta = (i^1_*, i^2_*)$ is evidently a monomorphism. From the commutative diagram (4.3) we obtain that $d \circ \delta = 0$. Let $\phi \in \Omega^c_n(G_1)$ and $\psi \in \Omega^c_n(G_2)$ be non-degenerate singular n-cubes for which $d(\phi, \psi) = 0$. Then the maps ϕ and ψ provide a map $I^n \to G_1 \cap G_2$, which we denote by γ, such that $\delta(\gamma) = (\phi, \psi)$. Hence the sequence (4.4) is exact in $\Omega^c_*(G_1) \oplus \Omega^c_*(G_2)$. The map d is an epimorphism, as follows from Definition 4.8 of a special cover. ■

Theorem 4.11 *Under assumptions of Lemma 4.10, diagram (4.3) induces the long exact sequences of homology groups:*

$$\ldots \longrightarrow H^c_n(G_1 \cap G_2) \xrightarrow{\delta_*} H^c_n(G_1) \oplus H^c_n(G_2) \xrightarrow{d_*} H^c_n(G) \longrightarrow H^c_{n-1}(G_1 \cap G_2) \longrightarrow \ldots$$

where $\delta_* = (i^1_*, i^2_*)$ and $d_*(a,b) = j^1_*(a) - j^2_*(b)$.

Proof. It follows from Lemma 4.10 by zig-zag Lemma. ■

Let $X \subset Y$ be a sub-digraph of a digraph H. Then the chain complexes $\Omega^c_*(X)$ and $\Omega^c_*(Y)$ are defined, and we have the natural inclusion $\Omega^c_*(X) \subset \Omega^c_*(Y)$. Hence we can define a quotient complex $\Omega^c_*(Y)/\Omega^c_*(X)$ which we denote by $\Omega^c_*(Y,X)$. It fits in the short exact sequence of chain complexes

$$0 \longrightarrow \Omega^c_*(X) \longrightarrow \Omega^c_*(Y) \longrightarrow \Omega^c_*(Y,X) \longrightarrow 0. \tag{4.5}$$

The homology groups of this complex are called *the relative cubical singular homology groups* and is denoted by $H^c_*(Y,X)$. Now we transfer on the category of digraphs the standard results of singular cubical homology for topological spaces (see, for example,[10, Chpt. 2.1] and [12, Chpt. 1.3]).

Proposition 4.12 *Under assumptions above, there is the relative homology long exact sequence*

$$\ldots \longrightarrow H^c_n(X) \longrightarrow H^c_n(Y) \longrightarrow H^c_n(Y,X) \longrightarrow H^c_{n-1}(X) \longrightarrow \ldots$$

Proof. It follows from (4.4) by zig-zag Lemma. ■

Corollary 4.13 *Let $X \subset Y$ be a connected sub-digraph of a connected digraph Y. Then*

$$H^c_0(Y,X) = 0.$$

Let $X_1 \subset Y_1$ and $X_2 \subset Y_2$ be two digraph pairs. A digraph map $f\colon Y_1 \to Y_2$ with $f(X_1) \subset X_2$ we shall call a *digraph map of pairs* and write $f\colon (Y_1, X_1) \to (Y_2, X_2)$. This map induces a homomorphism

$$f_*\colon \Omega^c_*(Y_1, X_1) \to \Omega^c_*(Y_2, X_2)$$

of chain complexes and hence homomorphisms of homology groups

$$f_*\colon H^c_n(Y_1, X_1) \to H^c_n(Y_2, X_2), \quad n \geq 0.$$

Proposition 4.14 *Let $f\colon (Y_1, X_1) \to (Y_2, X_2)$ be a digraph map of pairs, such that the digraph maps*

$$f|_{Y_1}\colon Y_1 \to Y_2 \quad and \quad f|_{X_1}\colon X_1 \to X_2$$

are homotopy equivalences. Then the induced map

$$f_*\colon H_n^c(Y_1, X_1) \to H_n^c(Y_2, X_2)$$

is an isomorphism for all $n \geq 0$.

Proof. The result follows from consideration the natural map of relative homology long exact sequences by Five-lemma. ∎

Proposition 4.15 *Let $f, g\colon (Y_1, X_1) \to (Y_2, X_2)$ be two digraph maps of pairs, which are homotopic through homotopy of pairs. Then*

$$f_* = g_*\colon H_n^c(Y_1, X_1) \to H_n^c(Y_2, X_2).$$

Proof. Similarly to [10, Proposition 2.19]. ∎

The next result is similar to [6, Theorem 3.21] where it was proved for path homology groups.

Theorem 4.16 *For a triple of graphs $Z \subset Y \subset X$ there is the commutative braid of groups and homomorphisms*

$$\begin{array}{ccccccccc}
...H_{n+1}^c(X) & & \longrightarrow & & H_{n+1}^c(X,Y) & & \longrightarrow & & H_n^c(Y,Z) & & \longrightarrow & & H_{n-1}^c(Z)... \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & H_{n+1}^c(X,Z) & & & & H_n^c(Y) & & & & H_n^c(X,Z) & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
...H_{n+1}^c(Y,Z) & & \longrightarrow & & H_n^c(Z) & & \longrightarrow & & H_n^c(X) & & \longrightarrow & & H_n^c(X,Y)...
\end{array}$$

consisting of the following long exact sequences

$$\cdots \to H_{n+1}^c(X,Y) \longrightarrow H_n^c(Y) \longrightarrow H_n^c(X) \longrightarrow H_n^c(X,Y) \to \dots$$

$$\cdots \to H_{n+1}^c(X,Z) \longrightarrow H_n^c(Z) \longrightarrow H_n^c(X) \longrightarrow H_n^c(X,Z) \to \dots$$

$$\cdots \to H_{n+1}^c(Y,Z) \longrightarrow H_n^c(Z) \longrightarrow H_n^c(Y) \longrightarrow H_n^c(Y,Z) \to \dots$$

$$\cdots \to H_{n+1}^c(X,Y) \longrightarrow H_n^c(Y,Z) \longrightarrow H_n^c(X,Z) \longrightarrow H_n^c(X,Y) \to \dots$$

Proof. We have the natural inclusions of chain complexes

$$\Omega_*^c(Z) \longrightarrow \Omega_*^c(Y) \longrightarrow \Omega_*^c(X).$$

By Noether isomorphism theorem (see [13, Chpt. 4] we obtain a short exact sequence

$$0 \longrightarrow \Omega_*^c(Y)/\Omega_*^c(Z) \longrightarrow \Omega_*^c(X)/\Omega_*^c(Z) \longrightarrow \Omega_*^c(X)/\Omega_*^c(Y) \longrightarrow 0.$$

and, hence, the commutative diagram of chain complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_*(Z) & \longrightarrow & \Omega_*(Y) & \longrightarrow & \Omega_*(Y)/\Omega_*(Z) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_*(Z) & \longrightarrow & \Omega_*(X) & \longrightarrow & \Omega_*(X)/\Omega_*(Z) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega_*(X)/\Omega_*(Y) & \overset{\cong}{\longrightarrow} & \Omega_*(X)/\Omega_*(Y) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{4.6}
$$

in which the rows and columns are short exact sequences. The homology long exact sequences of the short exact sequences from (4.6) give the commutative diagram as in the statement. ■

Corollary 4.17 *Let for a triple of digraphs $Z \subset Y \subset X$ one of the inclusions $Y \to X$ or $Z \to Y$ is a homotopy equivalence. Then in the first case we have an isomorphism $H^c_*(Y,Z) \cong H^c_*(X,Z)$, and in the second case $H^c_*(X,Z) \cong H^c_*(X,Y)$.*

5 Comparison of the singular cubical homology theory with the path homology theory

In this section we compare the singular cubical homology theory of digraphs with the path homology theory. We shall consider the homology theories with the same ring of coefficient R in the both cases.

For any finite digraph G, we have the path chain complex $\Omega_* = \Omega_*(G)$ and the cubical chain complex $\Omega^c_* = \Omega^c_*(G)$. Note that all free R-modules Ω^c_i and Ω_i are finitely generated. Now we define a morphism $\tau_* : \Omega^c_* \to \Omega_*$ of chain complexes.

Let $n \geq 0$. We define a homomorphism $\tau_n : \Omega^c_n \to \Omega_n$ on any singular non-degenerated n-cube and then extend it by linearity to Ω^c_n.

The module $\Omega_n(I^n)$ is generated by a single element ω_n defined as in (3.3). By (2.4), any singular n-cube $\phi\colon I^n \to G$, as a digraph map, induces a chain map $\phi_*\colon \Omega_*(I^n) \to \Omega_*(G)$. We define a homomorphism $\tau_n\colon \Omega^c_n(G) \to \Omega_n(G)$ on any basic element $\phi^\square \in \Omega^c_n(G)$, as follows

$$\tau_n(\phi^\square) := \phi_*(\omega_n). \tag{5.1}$$

Lemma 5.1 *For $n \geq 0$, consider a diagram*

$$
\begin{array}{ccc}
\Omega^c_n(I^n) & \overset{\tau_n}{\longrightarrow} & \Omega_n(I^n) \\
\downarrow \partial^c & & \downarrow \partial \\
\Omega^c_{n-1}(I^n) & \overset{\tau_{n-1}}{\longrightarrow} & \Omega_{n-1}(I^n).
\end{array}
$$

Let $\mathrm{Id}^\square \in \Omega^c_n(I^n)$ be the singular cube given by the identity map $\mathrm{Id}\colon I^n \to I^n$. Then

$$\tau_{n-1}\partial^c(\mathrm{Id}^\square) = \partial\tau_n(\mathrm{Id}^\square). \tag{5.2}$$

Proof. The case $n = 0$ is trivial. By (4.1), (4.2) and (5.1), we can rewrite the left hand side of (5.2) in the following form:

$$\begin{aligned}\tau_{n-1}\partial^c(\mathrm{Id}^\square) &= \tau_{n-1}\left(\sum_{j=1}^{n}(-1)^j\left(F_{j0}^\square - F_{j1}^\square\right)\right) \qquad (5.3)\\ &= \sum_{j=1}^{n}(-1)^j\left([F_{j0}]_*(\omega_{n-1}) - [F_{j1}]_*(\omega_{n-1})\right).\end{aligned}$$

By (5.1), the right hand side of (5.2) is equal to

$$\partial\tau_n(\mathrm{Id}^\square) = \partial(\mathrm{Id}_*(\omega_n)) = \partial\omega_n. \qquad (5.4)$$

By Theorem 3.3 for the cube $D = I^n$, we have

$$\partial\omega\left(D\right) = \sum_{D'\subset D}(-1)^{\sigma(D,D')}\,\omega\left(D'\right). \qquad (5.5)$$

where D' is any $(n-1)$-subcube of D. Using the notation from Example 3.2, observe that D' has the form D_{j1} or D_{j0} where $j = 1, ..., n$. Using (3.6) and (3.7), we rewrite (5.5) in the form

$$\begin{aligned}\partial\omega\left(D\right) &= \sum_{j=1}^{n}(-1)^{j-1}\omega\left(D_{j1}\right) + \sum_{j=1}^{n}(-1)^{2n-j}\omega\left(D_{_{j0}}\right)\\ &= \sum_{j=1}^{n}(-1)^j\left(\omega\left(D_{_{j0}}\right) - \omega\left(D_{_{j1}}\right)\right).\end{aligned}$$

In order to finish the proof of (5.2), it remains to verify two identities:

$$[F_{j1}]_*(\omega_{n-1}) = \omega\left(D_{_{j1}}\right) \quad\text{and}\quad [F_{j0}]_*(\omega_{n-1}) = \omega\left(D_{_{j0}}\right).$$

For example, let us prove the first of these identities. Using the notation of Example 3.2, Lemma 3.1 and the case (i) of Proposition 3.4, we obtain

$$\begin{aligned}\omega\left(D_{_{j1}}\right) &= \omega_{a',b} = \left(\chi_{a',b}\right)_*(\omega_{n-1}) = \left(\chi_{a,b}\circ F_{j1}\right)_*(\omega_{n-1})\\ &= \left(\chi_{a,b}\right)_*(F_{j1})_*(\omega_{n-1}) = (F_{j1})_*(\omega_{n-1}),\end{aligned}$$

since $\chi_{a,b}$ is the identity map of I^n. ■

Proposition 5.2 *Let $f\colon G \to G'$ be a digraph map. Then we have a commutative diagram*

$$\begin{array}{ccc}\Omega_n^c(G) & \xrightarrow{\tau_n} & \Omega_n(G)\\ \downarrow f_* & & \downarrow f_*\\ \Omega_n^c(G') & \xrightarrow{\tau_n} & \Omega_n(G').\end{array}$$

Proof. It is sufficient to check commutativity only for a non-degenerated singular n-cube $\psi\colon I^n \to G$. By definition of ψ_* and (5.1), we have

$$f_*\tau_n(\psi^\square) = f_*\psi_*(\omega_n) = (f\psi)_*(\omega_n)$$

and

$$\tau_n f_*(\psi^\square) = \tau_n\left((f\psi)^\square\right) = (f\psi)_*(\omega_n),$$

which finishes the proof. ■

Proposition 5.3 *For any digraph G and $n \geq 0$, the following diagram*

$$\begin{array}{ccc} \Omega_n^c(G) & \overset{\tau_n}{\longrightarrow} & \Omega_n(G) \\ \downarrow \partial^c & & \downarrow \partial \\ \Omega_{n-1}^c(G) & \overset{\tau_{n-1}}{\longrightarrow} & \Omega_{n-1}(G) \end{array} \tag{5.6}$$

is commutative. Hence, the homomorphisms τ_n with $n \geq 0$ provide a chain map $\tau_*\colon \Omega_*^c(G) \to \Omega_*(G)$.

Proof. The case $n = 0$ is trivial. For $n \geq 1$, it is sufficient to prove the commutativity of (5.6) for any $\psi^{\square} \in \Omega_n^c(G)$ where $\psi\colon I^n \to G$ is a non-degenerated singular n-cube. Consider the diagram

$$\begin{tikzcd} \Omega_n^c(I^n) \arrow[rrr, "\tau_n"] \arrow[dr, "\psi_*"] \arrow[ddd, "\partial^c"'] & & & \Omega_n(I^n) \arrow[dl, "\psi_*"'] \arrow[ddd, "\partial"] \\ & \Omega_n^c(G) \arrow[r, "\tau_n"] \arrow[d, "\partial^c"] & \Omega_n(G) \arrow[d, "\partial"] & \\ & \Omega_{n-1}^c(G) \arrow[r, "\tau_{n-1}"] & \Omega_{n-1}(G) & \\ \Omega_{n-1}^c(I^n) \arrow[rrr, "\tau_{n-1}"] \arrow[ur, "\psi_*"] & & & \Omega_{n-1}(I^n). \arrow[ul, "\psi_*"'] \end{tikzcd} \tag{5.7}$$

The upper and bottom trapeziums are commutative by Proposition5.2. The left and right trapeziums are commutative by the functoriality of chain complexes Ω_*^c and Ω_*, respectively. By Lemma 5.1, the external square in (5.7) is commutative for the element $\mathrm{Id}^{\square} \in \Omega_n^c(I^n)$. Hence, the interior square is commutative for the element $\psi_*(\mathrm{Id}^{\square}) = \psi^{\square} \in \Omega_n^c(G)$, which was to be proved. ■

Proposition 5.4 *(i) The homomorphism $\tau_n\colon \Omega_n^c(G) \to \Omega_n(G)$ is an isomorphism for $n = 0, 1$ and an epimorphism for $n = 2$.*

(ii) The homomorphism $H_n^c(G) \longrightarrow H_n(G)$ of homology groups induced by τ_* is an isomorphism for $n = 0, 1$ and an epimorphism for $n = 2$.

Proof. It follows immediately from definition that τ_n is an isomorphism for $n = 0, 1$. Now we prove, that τ_2 is an epimorphism. Then there exists a basis in $\Omega_2(G)$ such that any element of the basis has one of the following three forms (see [7, Proposition 2.9]).

1. e_{iji} with $i \to j \to i$ (a double edge in G);
2. e_{ijk} with $i \to j \to k$ and $i \to k$ (a triangle as a sub-digraph of G);
3. $e_{ijk} - e_{imk}$ with $i \to j \to k$, $i \to m \to k$, $i \not\to k$, $i \neq k$ (a square as a sub-digraph of G).

It is sufficient to check that any such element lies in the image of τ_2. Let I^2 be the square (3.1) where we denote the vertices as integers instead of binary sequences:

$$\begin{array}{ccc} {}^1\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ {}^0\bullet & \longrightarrow & \bullet^2 \end{array}$$

Then $\omega = e_{013} - e_{023} \in \Omega_2(I^2)$. In the first case, consider the singular 2-cube

$$\phi\colon I^2 \to G; \quad \phi(0) = \phi(2) = \phi(3) = i, \phi(1) = j,$$

for which we have $\tau_2(\phi^{\square}) = \phi_*(\omega) = e_{iji} - e_{iii} = e_{iji}$.

In the second case, consider the singular 2-cube

$$\phi\colon I^2 \to G; \quad \phi(0) = i, \phi(1) = j, \phi(2) = \phi(3) = k,$$

for which we have $\tau_2(\phi^{\square}) = \phi_*(\omega) = e_{ijk} - e_{ikk} = e_{ijk}$.

In the third case, consider the singular 2-cube

$$\phi\colon I^2 \to G; \quad \phi(0) = i, \phi(1) = j, \phi(2) = m, \phi(3) = k,$$

for which we have $\tau_2(\phi^{\square}) = \phi_*(\omega) = e_{ijk} - e_{imk}$. Thus, τ_2 is an epimorphism and the statement (i) of the Proposition is proved.

The statement (ii) of the Proposition follows from (i) by the diagram chasing. ■

The fundamental group $\pi_1(G^*)$ of a digraph G with a based vertex $* \in V_G$ is defined in [7]. The statement (ii) of Proposition 5.4 implies the following result.

Corollary 5.5 *For any connected based digraph G^* there is an isomorphism*

$$H_1^c(G) \cong \pi_1(G^*)/[\pi_1(G^*), \pi_1(G^*)]$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is the commutator subgroup of $\pi_1(G^*)$.

Proof. It is proved in [7], that $H_1(G) = \pi_1(G^*)/[\pi_1(G^*), \pi_1(G^*)]$. Now the statement follows from Proposition 5.4. ■

Now we give an example of a digraph for which the path homology groups and singular cubical homology groups are not isomorphic. From now to the end of this section the ring of coefficient R is $\mathbb{Z}$.

Theorem 5.6 *Consider the planar digraph $G = (V, E)$ in Fig 1. Then $H_0(G) \cong \mathbb{Z}$ and the groups $H_n(G)$ are trivial for $n \geq 1$.*

Proof. The digraph G is connected, hence $H_0(G) \cong \mathbb{Z}$. It is easy to see from results in [7, Theorem 4.13 and Example 4.15] that $\pi_1(G) = 0$, hence $H_1(G) = 0$. The digraph G has no paths of length greater than 3, hence $\Omega_k(G) = 0$ for $k \geq 4$ and $H_k(G) = 0$ for $k \geq 4$. The group $\mathcal{A}_3(G)$ is generated by the paths

$$e_{0122'},\ e_{011'2'},\ e_{0A1'2'},\ e_{0322'},\ e_{033'2'},\ e_{0B3'2'}.$$

For these elements we have

$$\begin{aligned}
\partial e_{0122'} &= e_{122'} - \underline{e_{022'}} + \underrightarrow{e_{012'}} - e_{012},\\
\partial e_{011'2'} &= e_{11'2'} - \underleftarrow{e_{01'2'}} + \underrightarrow{e_{012'}} - e_{011'},\\
\partial e_{0A1'2'} &= e_{A1'2'} - \underleftarrow{e_{01'2'}} + e_{0A2'} - e_{0A1'},\\
\partial e_{0322'} &= e_{322'} - \underline{e_{022'}} + \underleftrightarrow{e_{032'}} - e_{032},\\
\partial e_{033'2'} &= e_{33'2'} - \underbrace{e_{03'2'}} + \underleftrightarrow{e_{032'}} - e_{033'},\\
\partial e_{0B3'2'} &= e_{B3'2'} - \underbrace{e_{03'2'}} + e_{0B2'} - e_{0B3'}
\end{aligned}$$

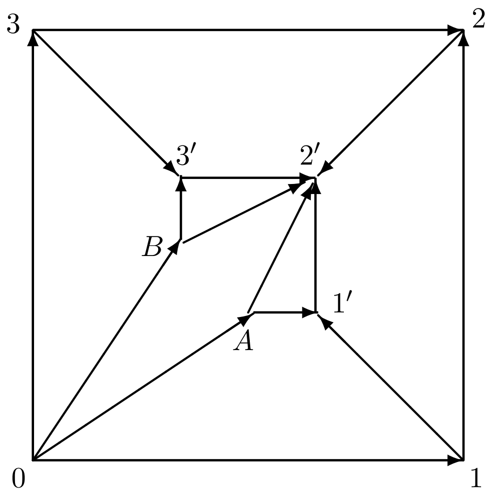

Figure 1: The planar digraph G from Proposition 5.6.

where we have underlined the non-allowed paths, equal paths being underlined similarly. From these relations and (2.2) we conclude that $\Omega_3(G)$ is generated by the single element

$$\tau = e_{0122'} - e_{011'2'} + e_{0A1'2'} - e_{0322'} + e_{033'2'} - e_{0B3'2'}$$

for which

$$\begin{aligned} \partial\tau &= e_{122'} - e_{012} - e_{11'2'} + e_{011'} + e_{A1'2'} + e_{0A2'} - e_{0A1'} \\ &- e_{322'} + e_{032} + e_{33'2'} - e_{033'} - e_{B3'2'} - e_{0B2'} + e_{0B3'} \in \mathcal{A}_2(G). \end{aligned}$$

Hence $\partial\tau \neq 0$ and $H_3(G) = 0$. The group $\mathcal{A}_2(G)$ is generated by the paths

$$e_{012},\ e_{032},\ e_{011'},\ e_{0A1'},\ e_{0A2'},\ e_{0B2'},\ e_{0B3'},\ e_{033'},\ e_{122'},\ e_{11'2'},\ e_{A1'2'},\ e_{B3'2'},\ e_{33'2'},\ e_{322'},$$

and, hence, the basis of $\Omega_2(G)$ is given by the elements

$$e_{012} - e_{032},\ e_{011'} - e_{0A1'}, e_{0A2'} - e_{0B2'}, e_{0B3'} - e_{033'}, e_{122'} - e_{11'2'}, e_{A1'2'}, e_{B3'2'}, e_{33'2'} - e_{322'}.$$

Computing the differentials of these elements, we can check directly that $\mathrm{Ker}\{\partial\colon \Omega_2(G) \to \Omega_1(G)\}$ is generated by $\partial\tau$ and, hence, $H_2(G) = 0$. ∎

Now we construct a non trivial cycle in $\Omega_2^c(G)$ of the digraph G. Consider eight induced sub-digraphs S_i, $i = 1, ..., 8$, of the digraph G where the set of vertices V_i of S_i is given as follows:

$$V_1 = \{0,1,2,3\},\ V_2 = \{1,1',2',2\},\ V_3 = \{0,A,1',1\},\ V_4 = \{A,2',1'\},$$

$$V_5 = \{0,3,3',B\}, V_6 = \{3,2,2',3'\},\ V_7 = \{0,B,2',A\},\ V_8 = \{B,3',2'\}.$$

Denote by $V = \{00, 01, 11, 10\}$ the set of vertices of the standard square I^2 as in (3.1). Define for $i = 1,2,3,5,6,7$ the singular square $\phi_i\colon I^2 \to G$ as the digraph map given by the order preserving mapping from V onto V_i. Then define the singular squares $\phi_i\colon I^2 \to G$ for $i = 4, 8$ by the following maps:

$$\phi_4(00) = A,\ \phi_4(01) = 2',\ \phi_4(11) = 2',\ \phi_4(10) = 1',$$

and

$$\phi_8(00) = B,\ \phi_8(01) = 3',\ \phi_8(11) = 2',\ \phi_8(10) = 2'.$$

Finally, define the chain $\phi^{\square} \in \Omega_2^c(G)$ as follows:

$$\phi^{\square} = \sum_{i=1}^{8} \phi_i^{\square} \ . \tag{5.8}$$

Lemma 5.7 *The chain $\phi^{\square}$ in (5.8) is a cycle.*

Proof. An easy computation shows that $\partial^c \phi^{\square} = 0 \in \Omega_1^c(G)$, so the chain $\phi^{\square}$ is a cycle. Indeed, let us denote a non-degenerate one-dimensional singular cube $\alpha\colon I \to G$ with $\alpha(0) = a$ and $\alpha(1) = b$ by $e_{ab}^{\square}$. Then by (4.2) we have the following equations:

$$\begin{array}{rcl}
\partial^c \phi_1^{\square} & = & -e_{01}^{\square} - e_{12}^{\square} + e_{03}^{\square} + e_{32}^{\square}, \\
\partial^c \phi_2^{\square} & = & -e_{11'}^{\square} - e_{1'2'}^{\square} + e_{12}^{\square} + e_{22'}^{\square}, \\
\partial^c \phi_3^{\square} & = & -e_{0A}^{\square} - e_{A1'}^{\square} + e_{01}^{\square} + e_{11'}^{\square}, \\
\partial^c \phi_4^{\square} & = & -e_{A2'}^{\square} - 0 \ + e_{A1'}^{\square} + e_{1'2'}^{\square}, \\
\partial^c \phi_5^{\square} & = & -e_{03}^{\square} - e_{33'}^{\square} + e_{0B}^{\square} + e_{B3'}^{\square}, \\
\partial^c \phi_6^{\square} & = & -e_{32}^{\square} - e_{22'}^{\square} + e_{33'}^{\square} + e_{3'2'}^{\square}, \\
\partial^c \phi_7^{\square} & = & -e_{0B}^{\square} - e_{B2'}^{\square} + e_{0A}^{\square} + e_{A2'}^{\square}, \\
\partial^c \phi_8^{\square} & = & -e_{B3'}^{\square} - e_{3'2'}^{\square} + e_{B2'}^{\square} + 0.
\end{array}$$

Summing up these equations we obtain $\partial^c \phi^{\square} = 0 \in \Omega_1^c(G)$. ∎

Now we prove a technical lemma that we need for the proof of the fact that $\phi^{\square}$ is not a boundary.

Lemma 5.8 *The boundary $\partial \psi^{\square}$ of any non-degenerate singular cube $\psi\colon I^3 \to G$ contains an even number of singular squares that are isomorphisms $I^2 \to S_1$, where S_1 is a subdigraph of G with the vertices $\{0, 1, 2, 3\}$.*

Proof. Let $\psi\colon I^3 \to G$ be a non-degenerate singular cube. It is clear that there is only a finite number of such maps. Denote by $\Delta(\psi)$ the number of faces of I^3 that maps isomorphically onto $S_1 \subset G$. In the case $\Delta(\psi) = 0$ there is nothing to prove. Assume that $\Delta(\psi) \geq 1$. This means that there is at least one two-face $D \subset I^3$ of the cube I^3 such that $\psi|_D\colon D \to S_1$ is an isomorphism. Let us represent I^3 as the planar digraph on Fig. 2 so that D is one of its faces. Without loss of generality, it suffices to consider the following two different cases for D:

(i) D has vertices $\{a, b, c, d\}$,

(ii) D has vertices $\{x, y, z, t\}$.

Consider the case (i), that is, ψ maps the vertices a, b, c, d of I^3 onto the vertices $0, 1, 2, 3$ of G. By the symmetry, we can assume without loss of generality that

$$\psi\,(a) = 0,\ \psi\,(b) = 1,\ \psi\,(c) = 2,\ \psi\,(d) = 3.$$

Consider separately all possible cases for the image of the vertex $x \in I^3$ under the map ψ:

1. $\psi(x) = 0$. 2. $\psi(x) = A$. 3. $\psi(x) = B$. 4. $\psi(x) = 1$. 5. $\psi(x) = 3$.

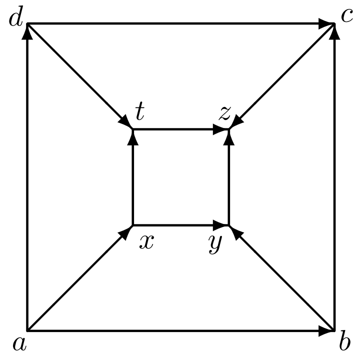

Figure 2: The planar digraph isomorphic to I^3.

1. $\psi(x) = 0$: since $0 = \psi(x) \overrightarrow{=} \psi(y)$ and $1 = \psi(b) \overrightarrow{=} \psi(y)$, we see that $\psi(y) = 1$. Similarly, we obtain $\psi(z) = 2$ and $\psi(t) = 3$. Hence, in this case the map ψ is degenerated.

2. $\psi(x) = A$: by inspection $\psi(y) = 1'$ since there is only one vertex $1'$ in G that has incoming arrows from A and 1. Similarly, $\psi(z) = 2'$ and $\psi(t) = 3'$. However, this ψ is not a digraph map since the condition $A = \psi(x) \overrightarrow{=} \psi(t) = 3'$ is not satisfied in G.

3. $\psi(x) = B$: the same argument as in the case 2 shows that there are no a digraph map ψ in this case.

4. $\psi(x) = 1$: since $x \to t \leftarrow d$ in I^3, we must have in G that $1 = \psi(x) \overrightarrow{=} \psi(t) \overleftarrow{=} \psi(d) = 3$, which is only possible for $\psi(t) = 2$. Then there are three possible images of y: $\psi(y) = 1, 2, 1'$ and two possible images of z: $\psi(z) = 2, 2'$. Consider the following possible cases.

   (a) $\psi(y) = 1$, then necessarily $\psi(z) = 2$.

   (b) $\psi(y) = 2$, $\psi(z) = 2$ or $\psi(z) = 2'$.

   (c) $\psi(y) = 1'$, then necessarily $\psi(z) = 2'$.

   In all cases (a), (b), (c) ψ maps the squares $\{a, b, c, d\}$ and $\{a, x, t, d\}$ isomorphically onto S_1. Hence $\Delta(\psi) = 2$.

5. $\psi(x) = 3$: similarly to the case 4, ψ maps the squares $\{a, b, c, d\}$ and $\{y, b, c, z\}$ isomorphically onto S_1. Hence $\Delta(\psi) = 2$.

Now we consider the case (ii), when the internal square of I^3 with the vertices $\{x, y, z, t\}$ on Fig. 2 is mapped isomorphically onto S_1. Without loss of generality, we can assume that

$$\psi(x) = 0, \ \psi(y) = 1, \ \psi(z) = 2, \ \psi(t) = 3.$$

Then $\psi(a) = 0$ since $a \to x$ in I^3 and there are no incoming arrows to $0 = \psi(x)$ in G. Also, $\psi(b) = 0$ or $\psi(b) = 1$ since $b \to y$ in I^3 and there is only one incoming arrow $0 \to 1 = \psi(y)$. Similarly, $\psi(d) = 0$ or $\psi(d) = 3$. Next, consider the following four cases.

1. $\psi(b) = 0$ and $\psi(d) = 0$. Then by inspection $\psi(c) = 1$ or $\psi(c) = 2$. In the first case ψ maps isomorphically onto S_1 the squares $\{x, y, z, t\}$ and $\{d, c, z, t\}$, and in the second case – the squares $\{x, y, z, t\}$ and $\{b, y, z, c\}$. Hence, $\Delta(\psi) = 2$.

2. $\psi(b) = 0$ and $\psi(d) = 3$. Since $d \to c \leftarrow b$ in I^3, we have $\psi(d) = 3 \overrightarrow{=} \psi(c) \overleftarrow{=} 0 = \psi(b)$, which is satisfied only for $\psi(c) = 3$. Then ψ maps the squares $\{x, y, z, t\}$ and $\{b, y, z, c\}$ isomorphically onto S_1. Hence, $\Delta(\psi) = 2$.

3. $\psi(b) = 1$ and $\psi(d) = 0$. This case is similar to the case 2 and $\Delta(\psi) = 2$.

4. $\psi(b) = 1$ and $\psi(d) = 3$. By inspection, $\psi(c) = 2$ but in this case ψ is degenerate.

■

Theorem 5.9 *Let G be the digraph on Fig 1. Then the group $H_2^c(G)$ is nontrivial.*

Proof. By Lemma 5.7, $\phi^\square \in \Omega_2^c(G)$ is a cycle. The sum (5.8) contains exactly one map $\phi_1 : I^2 \to G$ that is an isomorphism of I^2 to the sub-digraph $S_1 \subset G$. Hence, by Lemma 5.8, $\phi^\square$ is not a boundary. ■

6 Singular cubical homology theory for undirected graphs

In this section we apply the results from the previous sections to the category of (undirected) graphs. Using the isomorphism between the category of graphs and the full subcategory of symmetric digraphs (see [7]), we transfer the singular cubical homology theory to the category of undirected graphs. For a graph G, the groups $H_*^c(G)$ coincide with the homology groups of [3, 4] but differs from [14]. In our approach, we obtain a more developed homology theory including various exact sequences of homology groups for pairs and triples of graphs. This theory gives, in particular, the homology theory for the Atkins connectivity graph of a simplicial complex (see [1], [2], and [5]).

Fix a commutative ring R with a unity as a ring of coefficients. To denote (undirected) graph and morphisms of graphs we shall use a bold font similarly to [7, §6], for example, $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G}), \mathbf{f} : \mathbf{G} \to \mathbf{H}$.

Definition 6.1 i) A *graph* $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G})$ is a couple of a set $\mathbf{V_G}$ of vertices and a subset $\mathbf{E_G} \subset \{\mathbf{V_G} \times \mathbf{V_G} \setminus \mathbf{diag}\}$ of non-ordered pairs of different vertices that are called edges. Any edge $(v, w) \in \mathbf{E_G}$ will be also denoted by $v \sim w$.

ii) A *morphism* from a graph $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G})$ to a graph $\mathbf{H} = (\mathbf{V_H}, \mathbf{E_H})$ is a map $\mathbf{f} : \mathbf{V_G} \to \mathbf{V_H}$ such that for any edge $v \sim w$ on $\mathbf{G}$ we have either $\mathbf{f}(v) = \mathbf{f}(w)$ or $\mathbf{f}(v) \sim \mathbf{f}(w)$. We will refer to morphisms of graphs as *graph maps.*

The set of all graphs with graph maps forms a category $\mathcal{G}$. We can associate to each graph $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G})$ a symmetric digraph $G = \mathcal{O}(\mathbf{G}) = (V_G, E_G)$ where $V_G = \mathbf{V_G}$ and E_G is defined by the condition $v \leftrightarrows w \Leftrightarrow v \sim w$. Thus we obtain a functor $\mathcal{O}$ that provides an isomorphism of the category $\mathcal{G}$ and the full subcategory of symmetric digraphs of the category $\mathcal{D}$.

Definition 6.2 For any graph $\mathbf{G} = (\mathbf{V_G}, \mathbf{E_G})$ define the singular cubical homology groups in dimension $n \geq 0$ by

$$H_n^c(\mathbf{G}) = H_n^c(\mathcal{O}(\mathbf{G})).$$

Theorem 6.3 *For any pair of graphs $\mathbf{G} \subset \mathbf{H}$, the relative singular cubical homology groups $H_i^c(\mathbf{G}, \mathbf{H})$ are defined. These groups satisfy the following properties.*

i) There is a relative long homology exact sequence

$$\dots \to H_i^c(\mathbf{G}) \to H_i^c(\mathbf{H}) \to H_i^c(\mathbf{H}, \mathbf{G}) \to H_{i-1}^c(\mathbf{G}) \to \dots$$

ii) A graph map of pairs $\mathbf{f}\colon (\mathbf{H_1}, \mathbf{G_1}) \to (\mathbf{H_2}, \mathbf{G_2})$ induces a homomorphism of abelian groups

$$\mathbf{f}_*\colon H_i^c(\mathbf{H_1}, \mathbf{G_1}) \to H_i^c(\mathbf{H_2}, \mathbf{G_2})$$

and, additionally, if

$$\mathbf{f}|_{\mathbf{H_1}} \simeq \mathbf{g}|_{\mathbf{H_1}} \text{ and } \mathbf{f}|_{\mathbf{G_1}} \simeq \mathbf{g}|_{\mathbf{G_1}} \quad \text{then } \mathbf{f}_* = \mathbf{g}_*.$$

iii) Let $\mathbf{G} \subset \mathbf{H}$ and $\mathbf{G}$, $\mathbf{H}$ are connected. Then

$$H_0^c(\mathbf{H}, \mathbf{G}) = 0.$$

iv) For any connected based graph $\mathbf{G}^*$ we have an isomorphism

$$\pi_1(\mathbf{G}^*)/[\pi_1(\mathbf{G}^*), \pi_1(\mathbf{G}^*)] \cong H_1^c(\mathbf{G})$$

where $\pi_1(\mathbf{G}^*)$ is the fundamental group of the graph $\mathbf{G}$ defined in [7], and $[\pi_1(\mathbf{G}^*), \pi_1(\mathbf{G}^*)]$ is a commutator subgroup.

Proof. Follows from results obtained above for digraphs and results about fundamental group of digraphs obtained in [7]. ■

Theorem 6.4 *For a triple of graphs $\mathbf{Z} \subset \mathbf{Y} \subset \mathbf{X}$ there is the commutative braid of abelian groups and homomorphisms*

$$\begin{array}{ccccccccccc}
\dots \to & H_{n+1}^c(\mathbf{X},\mathbf{Y}) & & \longrightarrow & & H_n^c(\mathbf{Y},\mathbf{Z}) & & \longrightarrow & & H_{n-1}^c(\mathbf{Z}) & \to \dots \\
& \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \\
& & & H_n^c(\mathbf{Y}) & & & & H_n^c(\mathbf{X},\mathbf{Z}) & & & \\
& \searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow & \\
\dots \to & H_n^c(\mathbf{Z}) & & \longrightarrow & & H_n^c(\mathbf{X}) & & \longrightarrow & & H_n^c(\mathbf{X},\mathbf{Y}) & \to \dots
\end{array} \tag{6.1}$$

consisting of the following long exact sequences

$$\dots \to H_{n+1}^c(\mathbf{X},\mathbf{Y}) \longrightarrow H_n^c(\mathbf{Y}) \longrightarrow H_n^c(\mathbf{X}) \longrightarrow H_n^c(\mathbf{X},\mathbf{Y}) \to \dots$$

$$\dots \to H_{n+1}^c(\mathbf{X},\mathbf{Z}) \longrightarrow H_n^c(\mathbf{Z}) \longrightarrow H_n^c(\mathbf{X}) \longrightarrow H_n^c(\mathbf{X},\mathbf{Z}) \to \dots$$

$$\dots \to H_{n+1}^c(\mathbf{Y},\mathbf{Z}) \longrightarrow H_n^c(\mathbf{Z}) \longrightarrow H_n^c(\mathbf{Y}) \longrightarrow H_n^c(\mathbf{Y},\mathbf{Z}) \to \dots$$

$$\dots \to H_{n+1}^c(\mathbf{X},\mathbf{Y}) \longrightarrow H_n^c(\mathbf{Y},\mathbf{Z}) \longrightarrow H_n^c(\mathbf{X},\mathbf{Z}) \longrightarrow H_n^c(\mathbf{X},\mathbf{Y}) \to \dots$$

Proof. Follows from Definition 6.2 and Theorems 4.16 and 6.3. ■

Corollary 6.5 *Under assumptions of Theorem 6.4, let one of the inclusions $Y \to X$ or $Z \to Y$ be a homotopy equivalence. Then in the first case $H_*^c(\mathbf{Y}, \mathbf{Z}) \cong H_*^c(\mathbf{X}, \mathbf{Z})$, and in the second case $H_*^c(\mathbf{X}, \mathbf{Z}) \cong H_*^c(\mathbf{X}, \mathbf{Y})$.*

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