Heat eq. and heat kernel on an arbitrary manifold.

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§ 0. Introduction.

M - Riem. manifold

Δ - Laplace-Beltrami operator on M.

In the local coordinates

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j}\right) \quad (1)$$

where (g_{ij}) - the Riemannian metric, $(g^{ij}) = (g_{ij})^{-1}$.

Hence, Δ is a second order elliptic diff. operator in divergence form. It is defined covariantly, that is, in another coordinate system it has the same form (1).

Our main object: the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{on } M \times (0, +\infty) \quad (2)$$

where $u = u(x,t)$ is a function of $x \in M$ and $t \in \mathbb{R}$.

The eq. (2) is usually need to be solved together with the initial condition

$$u(x,0) = f(x), \quad (3)$$

where f is a given function. The equality in (3) is usually understood ~~a~~ a limit:

$$u(\cdot,t) \longrightarrow f \quad \text{as } t \to 0$$

in an appropriate sense.

The problem (2)-(3) is called a <u>Cauchy problem</u>. Recall that for $M = \mathbb{R}^n$, and for $f \in C_b(\mathbb{R}^n)$, the Cauchy problem (2)-(3) has a solution

$$u(x,t) = \int_{\mathbb{R}^n} p_t(x,y) f(y) dy,$$

where

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

is the <u>heat kernel</u> in $\mathbb{R}^n$. It is also called the fundamental solution of the heat equation, or the Gauss-Weierstrass function.

Note also that solution of (2)-(3) is unique in the class of bounded functions, but not in general.

Our goal to investigate similar questions on an arbitrary manifold.

In particular, we'll construct (implicitly) the heat kernel on an arbitrary manifolds, descrébe uniqueness class for solutions for the Cauchy problem, and obtain some universal estimates of the heat kernel, which are true on arbitrary manifolds.

Of course, when one assume some additional conditions about the manifold M then more information can be obtained about the heat kernel. However, the purpose of this course to work anly with universal properties of the heat kernel, without any restriction on a manifold, except geodesic completeness.

§ 1. Laplace operator in L^2.

Denote by μ the Riemannian measure on a Riemannian manifold (M, g).
In the local coordinates we have

$$d\mu = \sqrt{\det g}\, dx_1 .. dx_n$$

The Laplace operator is symmetric w.r.t. μ in the following sense: if $u, v \in C^2(M)$ and one of $\operatorname{supp} u$, $\operatorname{supp} v$ is compact, then

$$(1) \qquad \int_M (\Delta u) v \, d\mu = - \int_M \langle \nabla u, \nabla v \rangle d\mu = \int_M u \Delta v \, d\mu,$$

where ∇u is the gradient, that is,

$$(\nabla u)^i = g^{ij} \frac{\partial u}{\partial x_j},$$

and $\langle \nabla u, \nabla v \rangle = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}$.

The identity (1) is proved by using partition of unity that allows to reduces it to the case when functions u, v are supported in a chart, and then by using integration by parts in $x_1, \ldots x_n$.

The identity (1) are also called <u>Green's formulas</u>.

Note also that $\Delta u = \operatorname{div}(\nabla u)$, where div of vector field is defined by $\operatorname{div} v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, v^i \right)$.

Now consider Δ as an operator in $L^2(M,\mu)$ with the domain $C_0^\infty(M)$.

We use the notion of "unbounded operator": this is a linear operator $A: D \to H$, where H is a Hilbert space, D is a subspace of H (usually dense). An unbounded operator A is called symmetric if

$$(Au, v) = (u, Av) \quad \forall u, v \in D.$$

In particular, Δ with the domain $D = C_0^\infty$ is symmetric in L^2 as follows from Green's f-la.

However, in order to use the spectral theory and functional calculus, one need <u>self-adjoint</u> operators.

If A is an unbounded operator, then define its <u>adjoint</u> operator (A^*, D^*) by

$$D^* = \{u \in H : \exists w \in H \ \forall v \in D \ (Av, u) = (v, w)\}$$

(2) $$= \{u \in H : v \mapsto (Av, u) \text{ is a bounded linear functional in } v \in D\},$$

and $A^* u = w$ for all $u \in D^*$.

In particular, we have

(2a) $$(Av, u) = (v, A^* u) \quad \forall u \in D^* \ \forall v \in D.$$

In fact, D^* is chosen to be a maximal domain with this property.

Op. A is called <u>self-adjoint</u> if $A^* = A$, which included also the identity $D^* = D$. Clearly, sa operator is symmetric, as follows from (2a). However, the converse is not true. For a symmetric operator (A, D), one can only claim that $D^* \supset D$: Indeed, if $u \in D$ then $\forall v \in D$ we have

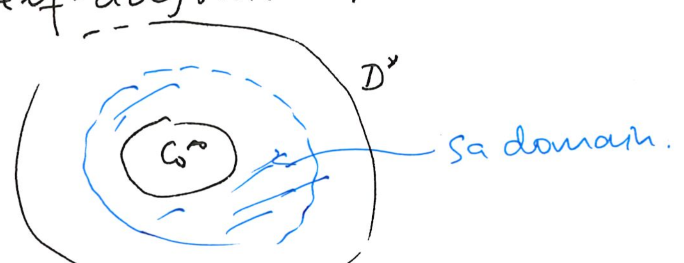
$$(Av, u) = (v, Au),$$

so that we can take in (2) $w = Au$. It follows that $A^* u = Au$, that is, $A^*|_D = A$. Hence, for a symmetric A, the adjoint A^* is an extension of A.

The operator (Δ, C_0^∞) is symmetric, but not self-adjoint, because D^* contains all $u \in C^2 \cap L^2$ s.t. $\Delta u \in L^2$. Indeed, for any such u and for any $v \in C_0^\infty$, we have ~~by def of~~

by Green' formula $\quad (\Delta v, u) = (v, \Delta u)$ ~~distr. Δ~~, which implies $u \in D^*$.

The first question to discuss is how to extend the domain C_0^∞ of Δ in order to obtain a self-adjoint operator.



Define first the notion of a distribution on M, similarly to that on $\mathbb{R}^n$. Consider the space $\mathcal{D} = C_0^\infty(M)$ of test functions and the dual space $\mathcal{D}'$ w.r.t. the usual topology of $\mathcal{D}$. The elements of $\mathcal{D}'$ are called distributions. Similarly define ~~vecto~~ test vectorfields $\vec{\mathcal{D}} = \vec{C_0^\infty}(M)$ and distributional vector fields – elements of $\vec{\mathcal{D}}'$.

The distributional (weak) Laplacian is defined as operator in $\mathcal{D}'$ by

$$(\Delta f, \varphi) = (f, \Delta \varphi) \quad \forall f \in \mathcal{D}', \forall \varphi \in \mathcal{D}.$$

The distributional gradient is defined as operator from $\mathcal{D}'$ to $\vec{\mathcal{D}}'$ by

$$(\nabla f, v) = -(f, \operatorname{div} v) \quad \forall f \in \mathcal{D}', \forall v \in \vec{\mathcal{D}}'$$

Any function $f \in L^1_{loc}(M)$ is identified as a distribution by

$$(f, \varphi) = \int_M f\varphi \, d\mu \quad \forall \varphi \in \mathcal{D}(M).$$

Similarly, any $L^1_{loc}(M)$-vector field ~~is~~ v identified as element of $\vec{\mathcal{D}}'$ by

$$(v, w) = \int_M \langle v, w \rangle \, d\mu \quad \forall w \in \vec{\mathcal{D}}(M).$$

<u>Def.</u> Define Sobolev space $W^1(M,\mu)$ by

$$W^1 = \{ f \in L^2(M) : \underset{\text{distr.}}{\nabla} f \in \vec{L}^2(M) \}.$$

Then W^1 is a Hilbert space with the inner product

$$(f,g)_{W^1} = \int_M fg \, d\mu + \int_M (\nabla f, \nabla g) \, d\mu$$

$$= (f,g)_{L^2} + (\nabla f, \nabla g)_{\vec{L}^2}.$$

Define $W_0^1 :=$ closure of C_0^∞ in W^1.

$$W_0^2 := \{ f \in W_0^1 : \Delta f \in L^2 \}.$$

Clearly, $C_0^\infty \subset W_0^2 \subset W_0^1 \subset W^1 \subset L^2$.

<u>Thm 1.1.</u> (Δ, W_0^2) is self-adjoint and non-pos. definite.

Moreover, (Δ, W_0^2) is the only self-adjoint extension of (Δ, C_0^∞) with the domain $\subset W_0^1$.

Proof. Consider the quadratic form

$$\mathcal{E}(u,v) \cancel{\mathcal{E}(f,g)} = (\nabla u, \nabla v)_{L^2}$$

defined for $u, v \in W_0^1(M)$. This form is clearly symmetric. It is closed in L^2, that is, W_0^1 is complete w.r.t

$$(u,v)_{L^2} + \mathcal{E}(u,v),$$

which follows from completeness of W^1.

Hence, $\mathcal{E}$ has a self-adjoint generator $\mathcal{L}$, ~~#~~ in particular,

$$\forall u, v \in \mathrm{dom}(\mathcal{L}) \qquad \mathcal{E}(u,v) = (\mathcal{L}u, v). \qquad (3)$$

$\mathcal{L}$ is non-neg. definite because

$$(\mathcal{L}u, u) = \mathcal{E}(u,u) \geq 0.$$

$\mathrm{dom}(\mathcal{L})$ is dense in W_0^1 and is defined by

$$\mathrm{dom}(\mathcal{L}) = \{u \in W_0^1 : v \mapsto \mathcal{E}(u,v) \text{ is a } L^2\text{-bounded linear functional of } v \in W_0^1\}.$$

$$\Leftrightarrow \quad \exists!\, f \in L^2 \text{ s.t. } \quad \mathcal{E}(u,v) = (f,v)_{L^2} \quad \forall v \in W_0^1.$$

$$\Leftrightarrow \quad \mathcal{E}(u,v) = (f,v)_{L^2} \quad \forall v \in \mathcal{D}$$

On the other hand, for $u \in \mathrm{dom}(\mathcal{L})$ and $v \in \mathcal{D}$,

we have

$$\mathcal{E}(u,v) = (\nabla u, \nabla v)_{L^2} = (\nabla u, \nabla v)_{\text{distr.}}$$

$$= -(u, \operatorname{div} \nabla v) = -(u, \Delta v) = -(\Delta u, v).$$

Comparing with (3), we conclude $\Delta = -\mathcal{L}$,

which implies the first two claims.

<u>Def</u>. The operator (Δ, W_0^2) is called the <u>Dirichlet Laplace operator</u>.

§ 2. Heat semigroup.

Now we are going to solve the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

by considering function $u(t, x)$ as a path in L^2, that is, $u : \mathbb{R}_+ \longrightarrow L^2$, s.t. $u(t, \cdot)$ is for any t an element of L^2. The time derivative will be understood simply as derivative of this path: $\frac{d}{dt} u(t) = L^2\text{-}\lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$.

The Laplace operator will be understood as the Dirichlet Laplacian; for that, we need to ensure that $u(t) \in W_0^2$ for all t.

Finally, the initial condition $u(0, x) = f(x)$ will be understood as $u(t) \xrightarrow{L^2} f$ as $t \to 0$, where $f \in L^2$ is a given function.

The eq. $\frac{du}{dt} = \Delta u$, $u(0) = f$ is a particular case of such evolution equation in any

Hilbert space H:

$$\frac{du}{dt} = Au, \quad u(0) = u_0 \qquad (4)$$

where u is a path in H, A is an (unbounded) operator in H. If A is a bounded operator (in particular, $\mathrm{dom}(A) = H$), then (4) has trivially the unique solution

$$u(t) = e^{At} u_0,$$

where $e^{At} = \sum_{k=1}^{\infty} \frac{(At)^k}{k!}$ is the exponential function of operator, and the series converges due to boundedness of A.

If A is unbounded then the exp series does not make sense, but e^{At} still can be defined using the spectral theory, provided A is self-adjoint.

Indeed, spectral theory ensures that any unbounded self-adjoint operator admits the following spectral decomposition:

$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda \qquad (5)$$

where $\{E_\lambda\}$ is a <u>resolution of identity</u> that is, a family of operators in H with the following properties:

- E_λ is a orthoprojector on H (in particular, bdd sa)
- E_λ is monotone increasing in λ, that is,
  $\operatorname{Im} E_{\lambda_1} \subset \operatorname{Im} E_{\lambda_2}$ if $\lambda_1 \le \lambda_2$
- E_λ is strongly left-cont in λ, and
  $\operatorname{s.lim}_{\lambda \to -\infty} E_\lambda = 0$, $\quad \operatorname{s-lim}_{\lambda \to +\infty} E_\lambda = Id.$

The integral (5) is a Lebesgue-Stieltjes integral that is understood as follows.

End of lec 1.

<u>Lemma</u>. For any resolution of id $\{E_\lambda\}$ and for any Borel function $\varphi: \mathbb{R} \to \mathbb{R}$, the space

$$D = \{u \in H : \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda u\|^2 < \infty\}$$

is dense in H, and there exists a unique sa operator B in H with domain D s.t.

$$(Bu, v) = \int_{-\infty}^{\infty} \varphi(\lambda) \, d(E_\lambda u, v) \quad \forall u \in D, v \in H.$$

For $\varphi = 1$:

$$Id = \int_{-\infty}^{\infty} dE_\lambda$$

and $\|u\|^2 = \int_{-\infty}^{\infty} d\|E_\lambda u\|^2$

In this case we write

$$B = \int_{-\infty}^{\infty} \varphi(\lambda) \, dE_\lambda$$

and

$$Bu = \int_{-\infty}^{\infty} \varphi(\lambda) \, dE_\lambda u.$$

Moreover, $\|Bu\|^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\|E_\lambda u\|^2 \quad \forall u \in D.$

$\|E_\lambda u\|^2 = (E_\lambda u, u)$

~~Moreover,~~

$$\cancel{\|Bu\|^2 = \int_{-\infty}^{\infty} |\varphi(x)|^2 \, d\|E_x u\|^2}.$$

<u>Example</u> If A is a linear operator in $\mathbb{R}^n$, that is symmetric, then it admits diagonalization, that is, $\exists$ an orthonormal basis $e_1, \dots, e_n$ in $\mathbb{R}^n$,

s.t. $\quad A e_k = \lambda_k e_k$.

denote by P_k the orthoprojector onto the space spanned by ~~all~~ e_k ~~with $\lambda_k < \lambda$~~,

s.t. $\quad A = \sum_{k=1}^{n} \lambda_k P_k$.

Set $E_k = P_1 + \dots + P_k$ then

$$A = \sum_{k=1}^{n} \lambda_k \left(E_k - E_{k-1}\right)$$

that is analogous of (5). In fact, this sum can be represented as integral.

Clearly, we have $E_0 = 0$, $E_n = Id$, $\operatorname{Im} E_k \subset \operatorname{Im} E_{k+1}$

In this case, we can define any function φ of A

by $\varphi(A) = \sum \varphi(\lambda_k) P_k$.

For a general self-adjoint operator A in H, and for any Borel function φ on $\mathbb{R}$, ~~we have~~ define operator $\varphi(A)$ by:

$$\operatorname{dom} \varphi(A) = \{u \in H : \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 \, d\|E_\lambda u\|^2 < \infty\}$$

$$\varphi(A)u = \int_{-\infty}^{\infty} \varphi(\lambda) \, dE_\lambda u$$

We have also

$$\|\varphi(A)u\|^2 = \int_{-\infty}^{\infty} \varphi(\lambda)^2 \, d\|E_\lambda u\|^2.$$

Moreover, all integrations here can be restricted to the spectrum $\sigma(A)$:

Recall that

$$\sigma(A) = \mathbb{C} \setminus \rho(A),$$

where $\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ has a bounded inverse}\}$.

For s.a. operator, we have $\sigma(A) \subset \mathbb{R}$.

In particular, if A is non-negative definite, then $\sigma(A) \subset [0, +\infty)$.

It follows, that $\|\varphi(A)u\|^2 \leq \sup_{\sigma(A)} |\varphi(\lambda)|^2 \cdot \|u\|^2$.

In part. if φ is bounded on $\sigma(A)$ $\Rightarrow$ $\varphi(A)$ is bdd!

Since $-\Delta$ is a non-negative def. sa operator, it admits a spectral decomposition

$$-\Delta = \int_0^\infty \lambda \, dE_\lambda .$$

Since the function $\lambda \mapsto e^{-t\lambda}$ is bounded on $\lambda \in [0,+\infty)$, assuming $t \geq 0$, we obtain that the operator $e^{-t(-\Delta)} = e^{t\Delta}$ is bounded in L^2,

$$e^{t\Delta} = \int_0^\infty e^{-t\lambda} \, dE_\lambda ,$$

more precisely, $\|e^{t\Delta}\| \leq 1$.

<u>Def</u>. The family $\{e^{t\Delta}\}_{t \geq 0}$ is called the <u>heat semigroup</u>.

<u>Th 2.1</u>. (i) for any $t \geq 0$, $e^{t\Delta}$ is a bdd sa operator in L^2, $\|e^{t\Delta}\| \leq 1$.

(ii) $e^{t\Delta} e^{s\Delta} = e^{(t+s)\Delta}$ (the semigroup property).

(iii) $\{e^{t\Delta}\}_{t \geq 0}$ is strongly contin, that is,

$$\underset{t \to s}{\text{s-lim}}\, e^{t\Delta} = e^{s\Delta} .$$

In particular, $\forall f \in L^2$, $e^{t\Delta} f \xrightarrow[t \to 0]{L^2} f$.

(iv) For any $f \in L^2$ and $t > 0$, $e^{t\Delta} f \in W_0^2$

and $\frac{d}{dt}(e^{t\Delta} f) = \Delta(e^{t\Delta} f)$.

Hence, $u(t) = e^{t\Delta} f$ is L^2-solution to the Cauchy problem $\frac{du}{dt} = \Delta u$, $u|_{t=0} = f$.

<u>Proof</u>. (i) already proved

(ii) Follows from general properties of functional calculus:

$$\varphi \cdot \psi(A) = \varphi(A)\psi(A).$$

(iii) We need to prove that for any $f \in L^2$

$$e^{t\Delta} f \underset{t \to s}{\longrightarrow} e^{s\Delta} f. \Leftrightarrow e^{t\Delta} \underset{t \to s}{\longrightarrow} e^{s\Delta} \text{ strongly}.$$

We have $\|e^{t\Delta} f - e^{s\Delta} f\|^2 = \int_0^\infty \underbrace{(e^{-t\lambda} - e^{-s\lambda})^2}_{\substack{\downarrow \\ 0 \text{ as } t \to s}} \underset{\leq 1}{d\|E_\lambda f\|^2}$,

and RHS $\to 0$ by the ~~dominated~~ bounded conv. theorem.

(iv). Since the function $\lambda \mapsto \lambda e^{-\lambda t}$ is bounded on $\lambda \in [0, +\infty)$ for any $t > 0$, we obtain

that the operator
$$\int_0^\infty \lambda e^{-\lambda t} dE_\lambda$$
is bounded. Since this operator is equal to $(-\Delta)e^{t\Delta}$, we see that $e^{t\Delta}f \in \operatorname{dom}(\Delta) = W_0^2$.

It remains to show that
$$\frac{d}{dt}(e^{t\Delta}f) = \Delta e^{t\Delta}f,$$
that is,
$$\frac{e^{(t+s)\Delta} - e^{t\Delta}}{s} \xrightarrow[s\to 0]{} \Delta e^{t\Delta} \text{ strongly}.$$

We have
$$\frac{e^{(t+s)\Delta} - e^{t\Delta}}{s} = \int_0^\infty \frac{e^{-s\lambda}-1}{s} e^{-t\lambda} dE_\lambda .$$

As $s \to 0$, we have $\frac{e^{-s\lambda}-1}{s} \to -\lambda$. In order to be able to interchange this limit with the integration, it suffices to show that the function
$$\lambda \longmapsto \frac{e^{-s\lambda}-1}{s} e^{-t\lambda} \qquad (6)$$
is ~~uniformly~~ bounded uniformly in $s \in [-\varepsilon, \varepsilon]$, for some $0 < \varepsilon < t$. Take $\varepsilon = t/2$.

Observe that for any $\theta \in \mathbb{R}$

$$|e^{\theta}-1| \leq |\theta|\, e^{|\theta|}$$

as it follows from $e^{\theta}-1 = \int_0^{\theta} e^{\xi}\, d\xi$.

Setting $\theta = -\lambda s$, we obtain

$$|e^{-\lambda s}-1| \leq \lambda |s| e^{\lambda |s|}$$

$$\Rightarrow \qquad \left| \frac{e^{-\lambda s}-1}{s} e^{-t\lambda} \right| \leq \lambda\, e^{\lambda|s|} e^{-t\lambda} \leq \lambda e^{-\frac{t}{2}\lambda}.$$

Since $\lambda e^{-t/2 \lambda}$ is bounded, we obtain that the function (6) is uniformly bounded.

Hence,

$$\frac{e^{(t+s)\Delta} - e^{t\Delta}}{s} \xrightarrow[s\to 0]{} -\int_0^{\infty} \lambda e^{-t\lambda}\, dE_s \quad \text{strongly},$$

$$\qquad\qquad\qquad\qquad \| \; \Delta e^{t\Delta}$$

which finishes the proof.

T 2.2. The L^2 Cauchy problem has a unique solution.

Proof. It suffices to prove the following:

If $\quad \dfrac{du}{dt} = \Delta u(t) \quad \forall t>0$

and $u(t) \xrightarrow[t\to 0]{L^2} 0$

Then $u(t) = 0$.

~~Proof~~. Consider

$$J(t) = \|u(t)\|_2^2 = (u(t), u(t)).$$

We have $\frac{d}{dt} J(t) = 2\left(\frac{du}{dt}, u\right) = 2(\Delta u, u)$.

Since $u(t) \in dom(\Delta)$, we have $(\Delta u, u) \leq 0$

$\Rightarrow \frac{d}{dt} J(t) \leq 0$.

Since $J(t) \searrow$ and $J(t) \xrightarrow[t \to 0+]{} 0$, it follows $J \equiv 0$.

and, hence, $u \equiv 0$.

§3 Markovian properties of heat semigroup

Set $P_t = e^{t\Delta}$. So far we know some properties of P_t as operators in the Hilbert space $L^2(M,\mu)$. Now we use that elements of this space are functions.

Th 3.1. Let $f \in L^2(M)$.

(a) $f \geq 0 \Rightarrow P_t f \geq 0$ (positivity preserving)

(b) $f \leq 1 \Rightarrow P_t f \leq 1$ (Sub-Markovian)

The proof will be based on the notion of resolvent: for any $\alpha > 0$, define

$$R_\alpha = (-\Delta + \alpha)^{-1} = \int_0^\infty \frac{1}{\lambda + \alpha} \, dE_\lambda .$$

Since the function $\lambda \mapsto \frac{1}{\lambda+\alpha}$ is bounded on $[0, +\infty)$, the resolvent R_α is a bounded operator, $\operatorname{dom} R_\alpha = L^2$.

L. 3.2. Let $f \in L^2$ and $\alpha > 0$.

(a) $f \geq 0 \Rightarrow R_\alpha f \geq 0$

(b) $f \leq 1 \Rightarrow R_\alpha f \leq \alpha^{-1}$.

Proof. It suffices to prove that if $f \le c$ for some $c > 0$ then $R_\alpha f \le c\alpha^{-1}$. Indeed, for $c = 1$ we obtain (b), for $c \to 0$ obtain (a).

Without loss of generality, set $c = \alpha$ and prove that if $f \le \alpha$ then $u := R_\alpha f \le 1$.

Function $u = (-\Delta + \alpha)^{-1} f$ belongs to $\operatorname{dom}(\Delta) = W_0^2$ and satisfies

$$-\Delta u + \alpha u = f. \qquad (1)$$

In order to prove $u \le 1$, it suffices to show that $v := (u-1)_+$ is $\equiv 0$. Since $u \in W_0^1$ then also $v \in W_0^1$ and

$$\nabla v = \begin{cases} \nabla u, & \text{on } \{u > 1\} \\ 0, & \text{on } u \le 1. \end{cases}$$

Multiplying (1) by v and integrating, we obtain

$$(-\Delta u, v) + \alpha (u, v) = (f, v)$$

Green's formula gives

$$(-\Delta u, v) = \int (\nabla u, \nabla v) = \int_{\{u > 1\}} |\nabla u|^2 \ge 0,$$

while

$$(u, v) = \int_{\{v > 0\}} (v+1) v \, d\mu = \|v\|_{L^2}^2 + \int_{\{v > 0\}} v \, d\mu$$

Hence, $\alpha \|v\|_{L^2}^2 + \alpha \int_M v \, d\mu \le (f, v) \le \alpha \int_M v \, d\mu$

$\Rightarrow \|v\|_{L^2} \le 0$, $v = 0$.

Lemma 3.3. For any $t>0$,

$$P_t = \lim_{k\to\infty} \left(\frac{k}{t}\right)^k \left(R_{k/t}\right)^k, \qquad (1)$$

where the limit is understood in the strong operator topology.

Proof. We have

$$e^{-t\lambda} = \lim_{k\to\infty} \left(1+\frac{t\lambda}{k}\right)^{-k}.$$

Since the function in the RHS remains uniformly bdd by 1, it follows that

$$\int_0^\infty e^{-t\lambda}\, dE_\lambda = s\text{-}\lim_{k\to\infty} \int_0^\infty \left(1+\frac{t\lambda}{k}\right)^{-k} dE_\lambda,$$

where E_λ is res. of identity of $-\Delta$.

Observe that

$$\int_0^\infty \left(1+\frac{t\lambda}{k}\right)^{-k} dE_\lambda = \left(\frac{k}{t}\right)^k \int \left(\frac{k}{t}+\lambda\right)^{-k} dE_\lambda$$

$$= \left(\frac{k}{t}\right)^k \cdot \left(\frac{k}{t}-\Delta\right)^{-k}$$

$$= \left(\frac{k}{t}\right)^k R_{k/t}^k,$$

whence (1) follows.

Proof of Th. 3.1

(a) If $f \geq 0$ then also $R_\alpha f \geq 0 \Rightarrow R_\alpha^k f \geq 0$ for any pos. integer k. Hence, $R_{k/t}^k f \geq 0$

$\Rightarrow P_t f \geq 0$ by (1).

(b) If $f \leq 1$ then $R_\alpha f \leq \alpha^{-1} \Rightarrow R_\alpha^k f \leq \alpha^{-k}$

$$\alpha^{+k} R_\alpha^k f \leq 1 \Rightarrow \left(\frac{k}{t}\right)^k R_{k/t}^k f \leq 1$$

$\Rightarrow P_t f \leq 1$.

End of lec 2.

§4. Smoothness properties and existence of heat kernel

Let us define for any non-neg. integer k the following Sobolev space:

$$W^{2k}(M) = \{ f \in L^2 : \Delta f \in L^2, \dots \Delta^k f \in L^2 \}.$$

Δ - distr.

The norm in W^{2k} is given by

$$\| f \|^2_{W^{2k}} = \sum_{j=0}^{k} \| \Delta^j u \|^2_{L^2} ,$$

which makes W^{2k} into a Hilbert space.

In particular, $W_0^2 = \{ f \in W_0^1 : \Delta f \in L^2 \} \subset W^2$.

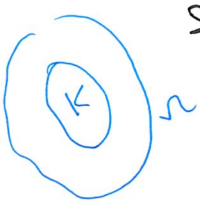
Define also the local version $W^{2k}_{loc}(M)$, when on L^2 are replaced by L^2_{loc}.

Th 4.1 Let $\dim M = n$ and $u \in W^{2k}_{loc}(M)$.

(a) If $k > \frac{n}{4}$ then $u \in C(M)$. Moreover, for any precompact open set $\Omega \subset M$ and any compact set $K \subset \Omega$, $\exists\, C = C(K, \Omega)$ s.t.

$$\sup_K |u| \leq C \| u \|_{W^{2k}(\Omega)} . \qquad (1)$$

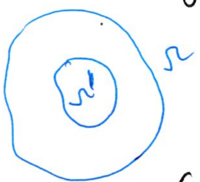
(b) If $k > \frac{n}{4} + \frac{m}{2}$ for some pos. integer m, then $u \in C^m(M)$. Moreover, for any precompact chart $\Omega \subset M$ and any compact set $K \subset \Omega$, $\exists\, C(K,\Omega)$ s.t.



$$\|u\|_{C^m(K)} \leq C \|u\|_{W^{2k}(\Omega)} . \quad (2)$$

<u>Sketch of proof</u>. It suffices to do all in a precompact chart Ω, where Δ is represented by a divergence form elliptic operator L with smooth coeff.

The regularity theory of elliptic operators given for $\Omega' \Subset \Omega$



$$\|u\|_{W^{k+2}(\Omega')} \leq C\left(\|u\|_{L^2(\Omega)} + \|Lu\|_{W^k(\Omega)}\right)$$

(first $k=0$, then induction). Here W^k are Euclidean Sob. spaces.

Applying the same inequality to $\|Lu\|_{W^k(\Omega)}$ and using induction, we obtain:

$$\|u\|_{W^{2k}(\Omega')} \leq C \sum_{j=0}^{k} \|L^j u\|_{L^2(\Omega)}$$

If $2k > \frac{n}{2} + m$ then Sobolev embedding theorem says $W^{2k}(\Omega') \hookrightarrow C(\Omega')$ and for any compact $K \subset \Omega$

$$\|u\|_{C^m(K)} \le C \|u\|_{W^{2k}(\Omega')}. \quad (3)$$

Hence, we obtain

$$\|u\|_{C^m(K)} \le C \sum_{j=0}^{k} \|L^j u\|_{L^2(\Omega)}, \quad (4)$$

Note: $L^2_{Euc}(\Omega) \simeq L^2_{Rien.}$

which implies both (a) and (b).

Recall. $P_t = e^{t\Delta}$ – the heat semigroup.

Th 4.2 For any $f \in L^2(M)$, and any $t>0$, $P_t f \in C^\infty(M)$. Moreover, for any $K \Subset M$ we have

$$\boxed{\sup_K |P_t f| \le F_K(t) \|f\|_{L^2},}$$

where $F_K(t) = C(1+t^{-\sigma})$, $C = C(K)$, ~~[illegible]~~ σ is the smallest integer $> \frac{n}{4}$.

Moreover, if K lies in a chart then for any m

$$\boxed{\|P_t f\|_{C^m(K)} \le F_K(t) \|f\|_{L^2},}$$

where σ is the smallest integer $> \frac{n}{4} + \frac{m}{2}$.

Proof. Fix $t>0$ and set $u = e^{t\Delta} f$. We have

$$\Delta^j e^{t\Delta} = \int_0^\infty (-\lambda)^j e^{-t\lambda} dE_\lambda.$$

Since the function $\lambda \mapsto (-\lambda)^j e^{-t\lambda}$

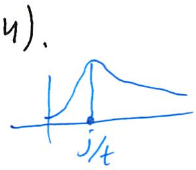
is bounded on $[0,+\infty)$, the operator $\Delta^j e^{t\Delta}$ is bounded, whence

$$\Delta^j e^{t\Delta} f \in L^2$$

$$\Delta^j u \in L^2.$$

Therefore, $u \in W^{2k}(M)$ for any k, which implies by Th 4.1 that $u \in C^\infty(M)$.

Let us prove the estimates (3) and (4).



The function $\lambda \mapsto \lambda^j e^{-t\lambda}$ takes its maximal value at $\lambda = j/t$, which implies that

$$\sup_{\lambda \in [0,+\infty)} |\lambda^j e^{-t\lambda}| = \left(\frac{j}{t}\right)^j e^{-j},$$

and

$$\| \Delta^j e^{t\Delta} \| \leq \left(\frac{j}{t}\right)^j e^{-j}$$

$$\| \Delta^j \underbrace{e^{t\Delta} f}_{u} \|_{L^2} \leq \left(\frac{j}{t}\right)^j e^{-j} \|f\|_{L^2}$$

It follows that

$$\|u\|_{W^{2\sigma}} \leq \sum_{j=0}^{\sigma} \|\Delta^j u\|_{L^2}$$

$$\leq C\left(1 + \sum_{j=1}^{\sigma} \left(\frac{j}{t}\right)^j e^{-j}\right) \|f\|_{L^2}$$

$$\leq C\left(1 + t^{-\sigma}\right) \|f\|_{L^2}.$$

By Th 4.1 (b) we obtain that if $\sigma > \frac{n}{4} + \frac{m}{2}$, then

~~that is, $\sigma >$~~

$$\|u\|_{C^m(K)} \le C \|u\|_{W^{2\sigma}} \le C(1+t^{-\sigma})\|f\|_{L^2},$$

which finishes the proof.

Pre-heat kernel

Th 4.3 For any $x \in M$ $\forall t>0$ $\exists$ a unique function $p_{t,x} \in L^2(M)$ s.t. $\forall f \in L^2(M)$

$$P_t f(x) = \int_M p_{t,x}(y) f(y)\, d\mu(y). \qquad (5)$$

Moreover, for any compact set $K \subset M$

$$\sup_{x \in K} \|p_{t,x}\|_{L^2(M)} \le C(1+t^{-\sigma}) \quad (6)$$

$\sigma > n/4$.

Proof. By Th 4.2 we have $\forall t>0$ $\forall x \in K$

$$|P_t f(x)| \le C(1+t^{-\sigma})\|f\|_{L^2} \quad \forall f \in L^2. \qquad (*)$$

Hence, the mapping $\underset{L^2}{f} \longmapsto \underset{\mathbb{R}}{P_t f(x)}$

is a bounded linear functional on $L^2(M)$.

By the Riesz representation theorem,

$\exists\, P_{t,x} \in L^2(M)$, s.t. $P_t f(x) = (f, P_{t,x})$,

which proves the first claim (5).

Setting in ~~(7)~~ (*) $f = P_{t,x}$, we obtain

$$\| P_{t,x} \|_{L^2}^2 \leq C(1+t^{-\sigma}) \| P_{t,x} \|,$$

whence (6) follows

Example. In $\mathbb{R}^n$ we have

$$P_{t,x}(y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

whence $\| P_{t,x} \|_2^2 = \frac{1}{(8\pi t)^{n/4}}$.

Hence, the estimate (6) with $\sigma > n/4$ is close to optimal one for $t < 1$.

$\boxed{\text{Smoothness of semigroup solution}}$

Thm 4.4. For any $f \in L^2(M)$, the function $(t,x) \mapsto P_t f(x)$ is C^∞ smooth on $(0,+\infty) \times M$ and satisfies the classical heat equation.

<u>Proof</u> (Sketch) $\quad u(t) = P_t f = \int_0^\infty e^{-\lambda t} \, dE_\lambda f$

As we already know,

$$\frac{du}{dt} = -\int_0^\infty \lambda e^{-\lambda t} \, dE_\lambda f = \Delta u$$

L^2-der

Similarly, $\quad \frac{d^2u}{dt^2} = \int_0^\infty \lambda^2 e^{-\lambda t} \, dE_\lambda f$

~~Since~~ $\Rightarrow \quad \frac{d^2u}{dt^2} + \Delta u = \int_0^\infty (\lambda^2 - \lambda) \, e^{-\lambda t} \, dE_\lambda f.$

Since $(\lambda^2 - \lambda) e^{-\lambda t}$ is bounded on any interval $t \in [\varepsilon, +\infty)$, $\varepsilon > 0$, the function $\frac{d^2u}{dt^2} + \Delta u$ belongs to $L^2(M)$ with L^2 norm bounded uniformly for $t \geq \varepsilon$. $\Rightarrow$

$$\frac{d^2u}{dt^2} + \Delta u \in L^2_{loc}(N), \quad N = M \times (0, +\infty).$$

$L = \frac{\partial^2}{\partial t^2} + \Delta$ coincides with Laplacian on N,

so that $\quad Lu \in L^2_{loc}(N)$. $\quad$ Note that also $u \in L^2_{loc}(N)$.

Similarly, $\quad L^k u = \int_0^\infty (\lambda^2 - \lambda)^k e^{-\lambda t} \, dE_\lambda f \in L^2_{loc}(N).$

By theorem 4.1, $u \in C^\infty(N)$.

$\boxed{\text{Back to heat kernel}}$

The function $p_{t,x}(y)$ is a candidate for the heat kernel. It is L^2 functioning, but we want to have a smooth $p_t(x,y)$.

Set:
$$\boxed{p_t(x,y) = (p_{t/2,x},\ p_{t/2,y})_{L^2}}. \qquad (7)$$

This function is called <u>the heat kernel on M</u>. ⇐ defined for $t>0$, $\forall x,y \in M$.

<u>Th 4.5</u>. The heat kernel satisfies the following.

(a) $p_t(x,y) = p_t(y,x)$ - symmetry.

(b) $\forall f \in L^2$ $\forall x \in M$ $\forall t>0$
$$P_t f(x) = \int_M p_t(x,y) f(y)\, d\mu(y).$$

(c) $p_t(x,y) \geq 0$ and $\int_M p_t(x,y)\, d\mu(y) \leq 1$.

(d) $p_{t+s}(x,y) = \int_M p_t(x,z) p_s(z,y)\, d\mu(z)$

(e) $\forall y \in M$, the function $u(t,x) = p_t(x,y)$ is C^∞ smooth in $M \times \mathbb{R}_+$ and satisfies $\frac{\partial u}{\partial t} = \Delta u$.

~~(f) $\forall f \in C_0^\infty(M)$, $\int_M p_t(x,y) f(y)\, d\mu(y) \xrightarrow[t\to 0]{} f(x)$, convergence in $C^\infty(M)$.~~

(f) $p_t(x,y) \in C^\infty$ in (t,x,y).

Proof. (a) is obvious from Def.

Let us show that $P_t(x,y) \geq 0$. For that, it suffices to verify that $P_{t,x} \geq 0$ a.e.

Set $f = (P_{t,x})_- \geq 0$ Then, by Th 3.1, $P_t f \overset{(*)}{\geq} 0$.

$$\Rightarrow \quad 0 \leq P_t f(x) = (P_{t,x}, f) = ((P_{t,x})_+, f) - ((P_{t,x})_-, f) = -(f,f)$$

$\Rightarrow$ $f=0$ a.e. and $P_{t,x} \geq 0$ a.e.

Before we prove other parts, two claims.

Claim 1. $\forall f \in L^2$, $x \in M$, $t, s > 0$

$$P_{t+s} f(x) = \int_M (P_{t,z}, P_{s,x}) f(z) \, d\mu(z). \qquad (8)$$

Indeed, $P_{t+s} f(x) = P_s (P_t f)(x)$

$$= (P_{s,x}, P_t f) = (P_t P_{s,x}, f)$$

$$= \int_M P_t P_{s,x}(z) f(z) \, d\mu(z)$$

$$= \int_M (P_{t,z}, P_{s,x}) f(z) \, d\mu(z). \qquad \boxtimes$$

Claim 2 $\forall x, y$ and $t > 0$,

$$(P_{s,x}, P_{t-s,y})$$

does not depend on s.

Indeed, for $0 < r < s < t$, we have

$$(p_{s,x}, p_{t-s,y}) = P_s\, p_{t-s,y}(x) = (P_r \circ P_{s-r}\, p_{t-s,y})(x)$$

$$= \int p_{r,x}(z)\, P_{s-r}\, p_{t-s,y}(z)\, d\mu(z)$$

$$= \int p_{r,x}(z)\, \underline{(p_{s-r,z}, p_{t-s,y})}\, d\mu(z)$$

$$\overset{(8)}{=} p_{\underbrace{s-r+t-s}_{t-r}}\, p_{r,x}(y)$$

$$= (p_{t-r,y}, p_{r,x}) = (p_{r,x}, p_{t-r,y}). \quad \blacksquare$$

<u>Proof of (b)</u>. By (8) we have

$$P_t f(x) = \int_M (p_{t/2,y}, p_{t/2,x})\, f(y)\, d\mu(y)$$

$$= \int_M p_t(x,y)\, f(z)\, d\mu(y).$$

<u>Proof of (c)</u> By T 3.1, $f \le 1 \Rightarrow P_t f \le 1$.

Taking $f = 1_K$, K - compact, we obtain:

$$P_t f(x) = \int_K p_t(x,y)\, d\mu(y) \le 1.$$

$K \to M$ implies $\int_M p_t(x,y)\, d\mu(y) \le 1$.

Proof of (d). We have

$$P_t f(x) = \int_M P_{t,x}(y) f(y) d\mu(y) = \int_M P_t(x,y) f(y) d\mu(y)$$

$\Rightarrow \quad \forall x, \quad P_{t,x} = P_t(x,\cdot)$ a.e.

Therefore,

$$P_{t+s}(x,y) = (P_{t,x}, P_{s,y}) = \int P_{t,x}(z) P_{s,y}(z) d\mu(z)$$

$$= \int_M P_t(x,z) P_s(y,z) d\mu(z).$$

Proof of (e). Fix $s>0$, $y \in M$ and consider function

$$u(x,t) = P_{t+s}(x,y).$$

We have $\quad u(x,t) = (P_{t,x}, P_{s,y}) = P_t P_{s,y}(x) = P_t f(x)$
$f = P_{s,y}$.

By Theorem 4.4, the function $u \in C^\infty(M \times \mathbb{R}_+)$ and satisfies the heat equation $\Rightarrow$ which implies the same for $u(x,t) = P_t(x,y)$.

end of lect 3.

Proof of (f). Let us first prove that $P_t(x,y)$ is jointly cont in t, x, y. It suffices to prove that $P_t(x,y)$ is cont. in x loc. uniformly in (t,y) (as it is cont in (t,y)).

By Theorem T.4.2, in any compact $K \subset$ chart,

$F(t) = c(1+t^{-\sigma})$

$\sigma = \sigma(m)$

$$\| P_t f \|_{C^m(K)} \leq F(t) . \| f \|_{L^2}$$

Use it for $m=1$ to obtain, that all partial derivatives $\frac{\partial}{\partial x_j} P_t f$ are uniformly bounded on K ~~and~~.

$$\Rightarrow \quad | P_t f(x) - P_t f(x') | \leq F(t) \| f \|_{L^2} |x - x'|$$

$y \in M$. Since $p_{t+s}(x, y) - p_{t+s}(x', y) = P_t\, p_{s,y}(x) - P_t\, p_{s,y}(x')$

$$\Rightarrow \quad | p_{t+s}(x, y) - p_{t+s}(x', y) | \leq F(t) \| p_{s,y} \|_{L^2} |x - x'|$$

Restricting y to another compact set $\widetilde{K} \subset M$ and using (6):

$$\sup_{y \in \widetilde{K}} \| p_{s,y} \|_{L^2} \leq \widetilde{F}(s),$$

we obtain

$$| p_{t+s}(x, y) - p_{t+s}(x', y) | \leq F(t) \widetilde{F}(s) |x - x'|$$

$\forall x, x' \in K$, $\forall y \in \widetilde{K}$. Since $F(t)$ and $\widetilde{F}(s)$ are uniformly bounded for $t, s \geq \varepsilon > 0$, we see that $p_t(x, y)$ is cont in x loc. uniformly in (t, y).

We consider $P_t(x,y)$ as function on $M \times M \times \mathbb{R}_+$.

As distribution on this manifold, it satisfies

$$\frac{\partial}{\partial t} P_t(x,y) = \Delta_x P_t \equiv \Delta_y P_t$$

$$\Rightarrow \quad \frac{\partial}{\partial t} P_t(x,y) = \frac{1}{2} \underbrace{(\Delta_x + \Delta_y)}_{\parallel} P_t$$

L - Laplacian on $\underbrace{M \times M}_{N}$.

Hence, $P_t(x,y)$ satisfies the distributional heat equation on $N \times \mathbb{R}_+$ $\Rightarrow$ this function is C^∞.

(using parabolic regularity theory).

§5. Gaussian upper bound for $P_t f$ in integrated form

In this section we always assume that a Riemannian manifold M is connected, so that the geodesic distance $d(x,y)$ is a metric on M.

Th 5.1 Let A be a measurable subset of M. Then, for any $f \in L^2(A)$, we have

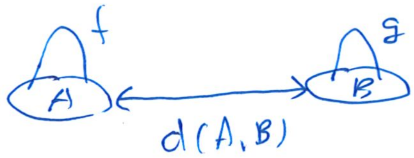
$$\int_{A_r^c} (P_t f)^2 d\mu \le \|f\|_{L^2}^2 \exp\left(-\frac{r^2}{2t}\right), \qquad (1)$$

for all $t, r > 0$, where $A_r = \{x \in M : d(x, A) < r\}$, and $A_r^c = M \setminus A_r$.

Note that $f \in L^2(A)$ extends to $f \in L^2(M)$ by setting $f = 0$ in A^c, so that $P_t f$ is defined as element of $L^2(M)$. Recall also that $\|P_t f\|_{L^2}^2 \le \|f\|_{L^2}^2$.

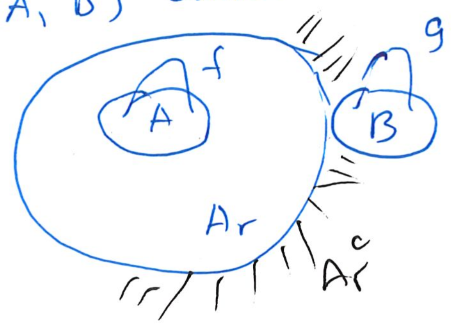
Cor 5.2. (Davies-Gaffney). If A, B - two disjoint meas. subsets of M then, for all $f \in L^2(A)$, $g \in L^2(B)$

$$(2) \qquad |(P_t f, g)_{L^2}| \le \|f\|_{L^2} \|g\|_{L^2} \exp\left(-\frac{d^2(A,B)}{4t}\right)$$



Consequently,

$$(3) \qquad \boxed{\int_A \int_B p_t(x,y)\,d\mu(x)\,d\mu(y) \le \sqrt{\mu(A)\mu(B)}\,\exp\left(-\frac{d^2(A,B)}{4t}\right)}.$$



Proof of Cor 5.2 Set $r = d(A,B)$ and observe that $B \subset A_r^c$.

Therefore, we have, using Th 5.1:

$$|(P_t f, g)| = \left|\int_M P_t f \cdot g\,d\mu\right| = \left|\int_B P_t f\,g\,d\mu\right|$$

$$\le \|P_t f\|_{L^2(B)} \|g\|_{L^2} \le \|P_t f\|_{L^2(A_r^c)} \|g\|_{L^2}$$

$$\overset{(1)}{\le} \|f\|_{L^2} \|g\|_{L^2} \exp\left(-\frac{r^2}{4t}\right),$$

which proves (2).

To prove (3), observe that

$$(P_t f, g) = \int_M P_t f(x)\,g(x)\,d\mu(x) = \int_M\int_M p_t(x,y)\,f(y)\,g(x)\,d\mu(x)\,d\mu(y)$$

Setting $f = 1_A$ and $g = 1_B$, we obtain:

$$\left|\int_A\int_B p_t(x,y)\,d\mu(x)\,d\mu(y)\right| \le \|f\|_{L^2}\|g\|_{L^2}\exp\left(-\frac{r^2}{4t}\right)$$

$$= \sqrt{\mu(A)}\,\sqrt{\mu(B)}\,\exp\left(-\frac{r^2}{4t}\right).$$

Recall that is $\mathbb{R}^n$

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d^2(x,y)}{4t}\right)$$

which implies

$$\int_A \int_B p_t(x,y)\, d\mu(x)\, d\mu(y) \le \frac{\mu(A)\mu(B)}{(4\pi t)^{n/2}} \exp\left(-\frac{r^2}{4t}\right).$$

We see that the term $\exp\left(-\frac{r^2}{4t}\right)$ in (3)
is sharp even in $\mathbb{R}^n$. On the contrary,
the term $\sqrt{\mu(A)\mu(B)}$ is not sharp: in $\mathbb{R}^n$ for
large t one can get much smaller multiple.
However, the possibility to replace $\sqrt{\mu(A)\mu(B)}$ by
a decreasing in $t \to \infty$ term depends on specific
geometry of M, whereas the Gaussian multiple
$\exp\left(-\frac{r^2}{4t}\right)$ has a universal nature.

Proof of Th 5.1 is based on the following lemma.

Lemma 5.3 (Integrated maximum principle)

Let $\xi(x,t)$ be a function on $M\times[0,S)$, $S>0$, with the following properties:

- ~~[illegible]~~ $\frac{\partial \xi}{\partial t}$ exists and is continuous in $M\times[0,S)$
- $x\longmapsto \xi(x,t)$ is locally Lipschitz on M, $\forall t\in[0,S)$.
- $$\frac{\partial \xi}{\partial t}+\frac{1}{2}|\nabla\xi|^2\leq 0 \qquad (4)$$

where $\nabla\xi$ is the weak gradient of ξ. (In particular ξ is cont in $M\times[0,S)$)

Then $\forall f\in L^2(M)$ the function

$$J(t):=\int_M (P_t f)^2 e^{\xi(x,t)}d\mu(x)$$

is monotone decreasing in $t\in[0,S)$.

Examples of ξ. $\xi(x,t)=\frac{d^2(x)}{2t}$, where $d(x)$ is a Lipschitz function on M with $\mathrm{Lip}=1$, for example, $d(x)=d(x,A)$, $A\subset M$.

Then $\frac{\partial\xi}{\partial t}=-\frac{d^2(x)}{2t^2}$, $\nabla\xi=\frac{2d\nabla d}{2t}$, $\frac{1}{2}|\nabla\xi|^2\leq\frac{d^2}{2t}$

$\Rightarrow \frac{\partial\xi}{\partial t}+\frac{1}{2}|\nabla\xi|^2\leq 0$.

The same applies to $\boxed{\xi(x,t)=\frac{d^2(x)}{2(t+b)}}$, $b\in\mathbb{R}$.

~~(infact, a must be outside $[0,S)$), for example, $a=S$).~~

Another example: $\boxed{\xi(x,t)=\alpha d(x)-\frac{\alpha^2}{2}t,}$ $\alpha\in\mathbb{R}$.

If $\xi \equiv 0$ then $J(t) = \| P_t f \|_{L^2}^2$,

and we proved in Th 2.2 that $\frac{dJ}{dt} \le 0$.

Hence, L.5.3 can be regarded as generalization of Th 2.2

Note also that J may be equal to $+\infty$.

Consider in $\mathbb{R}^n$ the following example:

$$f = \frac{1}{(4\pi a)^{n/2}} \exp\left(-\frac{|x|^2}{4a}\right) = p_a(x,0)$$

Then
$$P_t f = P_t p_a = p_{t+a} = \frac{1}{(4\pi(t+a))^{n/2}} e^{-\frac{|x|^2}{4(t+a)}}$$

Set also $\xi(x,t) = \frac{|x|^2}{2(t+b)}$, $b>0$, for all $t \ge 0$.

Then
$$J(t) = \frac{1}{[4\pi(t+a)]^n} \int_{\mathbb{R}^n} \exp\left(\frac{|x|^2}{2(t+b)} - \frac{|x|^2}{4(t+a)}\right) dx$$

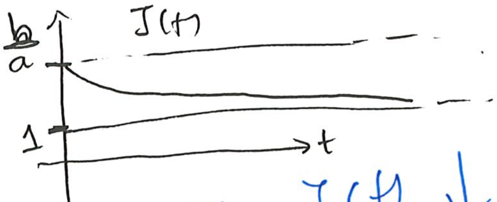
$$= \frac{1}{[4\pi(t+a)]^n} \int_{\mathbb{R}^n} \exp\left(-\frac{(b-a)\,|x|^2}{2(t+a)(t+b)}\right) dx.$$

Clearly, if $b \le a$ then $J(t) \equiv +\infty$.

If $b > a$, then using $\int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4\tau}\right) dx = (4\pi\tau)^{n/2}$,

$4\tau = \frac{2(t+a)(t+b)}{b-a}$,

we obtain
$$J(t) = \frac{1}{[4\pi(t+a)]^n}\left(\frac{2\pi(t+a)(t+b)}{b-a}\right)^{n/2} = c\cdot\left(\frac{t+b}{t+a}\right)^{n/2}$$
$$= c\left(1+\frac{b-a}{t+a}\right)^{n/2}$$



$\Rightarrow J(t) \downarrow$. We see that even in this case L.5.3 is quite delicate.

L.5.3 Idea of proof, so far without justification:

Set $u(t,x) = P_t f(x)$, so that $J(t) = \int_M u^2 e^{\xi} d\mu$.

(!) Justification of ~~term by te~~ $\frac{\partial}{\partial t}\int = \int \frac{\partial}{\partial t}$

Then

$$\frac{dJ}{dt} = \int_M \left(\frac{\partial}{\partial t} u^2\right) e^{\xi} d\mu + \int_M u^2 \frac{\partial}{\partial t} e^{\xi} d\mu$$

$$= \int_M 2u \frac{\partial u}{\partial t} e^{\xi} d\mu + \int_M u^2 \frac{\partial \xi}{\partial t} e^{\xi} d\mu.$$

Use: $\frac{\partial u}{\partial t} = \Delta u$ and $\frac{\partial \xi}{\partial t} \le -\frac{1}{2}|\nabla \xi|^2$.

Hence,

$$\frac{dJ}{dt} \le 2\int_M u\, \Delta u\, e^{\xi} d\mu - \frac{1}{2}\int_M u^2 |\nabla \xi|^2 e^{\xi} d\mu$$

(!) Justification of Green's fla.

$$= -2\int_M (\nabla u, \nabla(u e^{\xi}))\, d\mu - \frac{1}{2}\int_M u^2 |\nabla \xi|^2 e^{\xi} d\mu$$

$$= -2\int_M \left(|\nabla u|^2 + u(\nabla u, \nabla \xi) + \frac{1}{4} u^2 |\nabla \xi|^2\right) e^{\xi} d\mu$$

$$= -2\int_M \left(\nabla u + \frac{1}{2} u \nabla \xi\right)^2 e^{\xi} d\mu \le 0.$$

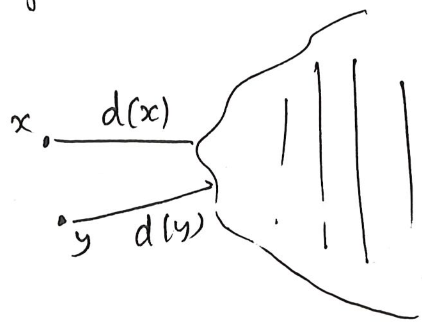
So, we need to justify: $\frac{\partial}{\partial t}\int_M = \int_M \frac{\partial}{\partial t}$ and the use of the Green formula, convergence of integrals?

Proof of Th 5.1 using Lemma 5.3.

Consider function

$$\xi(x,t) = -\alpha d(x) - \frac{\alpha^2}{2}t$$

where $\alpha > 0$, $d(x)$ = the distance function to some set in M.



By the triangle inequality,

$|d(x) - d(y)| \le d(x,y)$

$\Rightarrow$ d is Lipschitz with Lip $= 1$.

Then $|\nabla d| \le 1$.

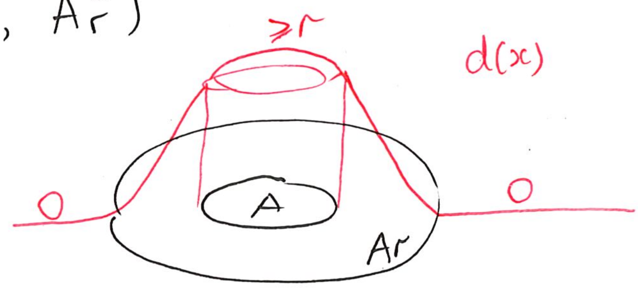
Clearly, ξ is also Lip in x, cont. diff. in t,

and $\frac{\partial \xi}{\partial t} = -\frac{\alpha^2}{2}$, $|\nabla \xi| \le \alpha \Rightarrow \frac{\partial \xi}{\partial t} + \frac{1}{2}|\nabla \xi|^2 \le 0$.

Now specify $d(x) = d(x, A_r^c)$

Clearly, $d(x) = 0$ in A_r^c

and $d(x) \ge r$ in A.



By L 5.3, the function

$$J(t) = \int_M (P_t f)^2 e^{-\alpha d(x) - \frac{\alpha^2}{2}t} d\mu(x)$$

is monotone decreasing in $t \ge 0$.

Hence, $J(t) \le J(0)$.

We have $\quad J(0) = \int_M f(x)^2 e^{-\alpha d(x)} d\mu(x)$

(Since $f \in L^2(A)$) $\quad = \int_A f(x)^2 e^{-\alpha d(x)} d\mu(x)$

(Since $d \geq r$ on A) $\quad \leq \int_A f(x)^2 e^{-\alpha r} d\mu(x)$

$$= e^{-\alpha r} \|f\|_{L^2}^2 .$$

$$J(t) \geq \int_{A_r^c} (P_t f)^2 e^{-\alpha d(x) - \frac{\alpha^2}{2} t} d\mu(x)$$

($d = 0$ on A_r^c) $\quad = \int_{A_r^c} (P_t f)^2 e^{-\frac{\alpha^2}{2} t} d\mu(x)$

$$\Rightarrow \int_{A_r^c} (P_t f)^2 d\mu \leq e^{-\alpha r + \frac{\alpha^2}{2} t} \|f\|_{L^2}^2 .$$

Since this is true for any $\alpha > 0$, we can now choose α to minimize $-\alpha r + \frac{\alpha^2}{2} t$. This is

$$\alpha = \frac{r}{t} \Rightarrow -\alpha r + \frac{\alpha^2}{2} t = -\frac{r^2}{t} + \frac{r^2}{2t} = -\frac{r^2}{2t}$$

$$\Rightarrow \int_{A_r^c} (P_t f)^2 d\mu \leq e^{-\frac{r^2}{2t}} \|f\|_{L^2}^2 ,\ \text{q.e.d.}$$

<u>Proof of L. 5.3</u> We prove this lemma under additional assumptions about $\xi(x,t)$:
(proof without assumptions require more tools):

(*) Functions $\xi(x,t)$, $\frac{\partial \xi}{\partial t}(x,t)$, $\frac{\partial^2 \xi}{\partial t^2}(x,t)$ are continuous and <u>bounded</u> on $M \times [0, S)$.

The function $\quad \xi(x,t) = -\alpha d(x) - \frac{\alpha^2}{2} t$

Satisfies (*). Indeed, we can always restrict t to a bounded interval $[0,S)$, and can assume without loss of generality that A is bounded $\Rightarrow$
$d(x) \le r + \operatorname{diam}(A) \Rightarrow d$ is bounded.
Also, $\frac{\partial \xi}{\partial t} = -\frac{\alpha^2}{2}$, $\frac{\partial^2 \xi}{\partial t^2} \equiv 0 \leftarrow$ no problem.

Note that if ξ satisfies (*) then e^{ξ} also satisfies (*), as $\frac{\partial e^{\xi}}{\partial t} = \frac{\partial \xi}{\partial t} e^{\xi}$, $\frac{\partial^2}{\partial t^2} e^{\xi} = \frac{\partial^2 \xi}{\partial t^2} e^{\xi} + \left(\frac{\partial \xi}{\partial t}\right)^2 e^{\xi}$.

Important consequence of (*): the derivative $\frac{\partial \xi}{\partial t}$ exists in the space $C_b(M)$, that is,

$$\sup_{x \in M} \left| \frac{\xi(x,t+h) - \xi(x,t)}{h} - \frac{\partial \xi}{\partial t} \right| \xrightarrow[h \to 0]{} 0,$$

which follows from boundedness of $\frac{\partial^2 \xi}{\partial t^2}$.

Hence, we regard below $\frac{\partial}{\partial t}\xi$ and $\frac{\partial}{\partial t} e^{\xi}$ as $C_b(M)$-derivatives.

Now we prove L. 5.3 under above additional conditions(*) on ξ.

Set $u(t) = P_t f$, s.t. $J(t) = (u(t), e^{\xi} u(t))$.

Since e^{ξ} is bdd, $\Rightarrow e^{\xi} u \in L^2$.

$$\frac{dJ}{dt} = \left(\frac{du}{dt}, e^{\xi} u\right) + \left(u, \frac{d}{dt}(e^{\xi} u)\right),$$

where $\frac{d}{dt}$ - L^2-derivative.

We know that $\frac{du}{dt} = \Delta u$.

Why $\frac{d}{dt}(e^{\xi} u)$ exists? It is easy to verify, that if $\frac{\partial e^{\xi}}{\partial t}$ exists in C_b then $\frac{d}{dt}(e^{\xi} u)$ exists in L^2 and

$$\frac{d}{dt}(e^{\xi} u) = \frac{\partial e^{\xi}}{\partial t} u + e^{\xi} \frac{du}{dt}.$$

Hence,

$$\frac{dJ}{dt} = \left(\frac{du}{dt}, e^{\xi} u\right) + \left(u, \frac{\partial e^{\xi}}{\partial t} u\right) + \left(u, e^{\xi} \frac{du}{dt}\right)$$

$$= 2 \int_M \frac{du}{dt} u e^{\xi} d\mu + \int u^2 \frac{\partial \xi}{\partial t} e^{\xi} d\mu$$

$$\leq 2 \int_M \Delta u \cdot u e^{\xi} d\mu - \frac{1}{2} \int_M u^2 |\nabla \xi|^2 e^{\xi} d\mu.$$

Since $u \in W_0^1(M) \Rightarrow e^{\xi} u \in W_0^1(M)$

and, hence, we can use Green's f-la

$$\int_M \Delta u \, (u e^{\xi}) \, d\mu = -\int_M (\nabla u, \nabla(u e^{\xi})) \, d\mu$$

$$= -\int_M \left(|\nabla u|^2 e^{\xi} + (\nabla u, \nabla \xi) u \xi e^{\xi} \right) d\mu .$$

$$\frac{dJ}{dt} \leq -2 \int_M e^{\xi} \left[|\nabla u|^2 + (\nabla u, \nabla \xi) u \xi + \frac{1}{4} u^2 |\nabla \xi|^2 \right] d\mu .$$

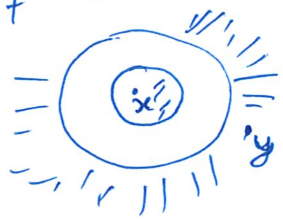
$$= -2 \int_M e^{\xi} \left(\nabla u + \frac{1}{2} u \nabla \xi \right)^2 d\mu \leq 0 .$$

Other related results.

Th 5.4 (Takeda's inequality). Assume A_r is precompact.

Then
$$\int_A \int_{A_r^c} P_t(x,y)\, d\mu(x)\, d\mu(y) \le C\sqrt{\mu(A)\,\mu(A_r)}\,\max\left(\frac{r}{\sqrt{t}}, 1\right) e^{-\frac{r^2}{4t}}.$$

$C = \sqrt{e/2}$



Th 5.5 (Uniqueness class for bounded Cauchy problem). Let M be geod. complete.

Let $u(x,t)$ be the classical solution to the Cauchy problem

$$(5) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{in } M \times \cancel{\mathbb{R}}(0,T) \\ u(\cdot,t) \to 0 & t \to 0, \text{ loc. uniformly. (or } \underline{L^2_{loc}}\text{)} \end{cases}$$

Then $u \equiv 0$ provided u satisfies the following condition: for some $x_0 \in M$ and all $R \gg 1$,

$$(6) \quad \int_0^T \int_{B(x_0,R)} u^2(x,t)\, d\mu(x)\, dt \le e^{f(R)}$$

where f is some function $\uparrow$, s.t.

$$(7) \quad \int^\infty \frac{r\,dr}{f(r)} = \infty. \quad \text{(Täcklind cond.)}.$$

For ex., $f(r) = cr^2$, and $e^{f(R)} = e^{cR^2}$.

Recall that in $\mathbb{R}^n$ there are the following <u>uniqueness classes</u> for solution of (5):

<u>Tikhonov class</u>:

$$\sup_{\substack{x \in B(x_0,R)\\ t\in[0,T]}} |u(x,t)| \le e^{CR^2}$$

<u>Täcklind class</u>

$$\sup_{\substack{x \in B(x_0,R)\\ t\in[0,T]}} |u(x,t)| \le e^{f(R)} \qquad (8)$$

where f satisfies (7).

Given the fact that $\mu(B(x_0,R)) = cR^n$,

we see that (8) $\Rightarrow$ (6) so that Täcklind's theorem in $\mathbb{R}^n$ follows from T. 5.5.