Lectures on heat kernels on Riemannian manifolds

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0 Introduction: Laplace-Beltrami operator and heat kernel

Let (M, g) be a connected Riemannian manifold. The Laplace-Beltrami operator Δ is given in the local coordinates by

$$\Delta = \Delta_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $n = \dim M$, $g = (g_{ij})$ and $(g^{ij}) = g^{-1}$. This operator is symmetric with respect to the Riemannian measure

$$d\mu = \sqrt{\det g} dx^1 ... dx^n,$$

that is, for all $u, v \in C_0^\infty(M)$,

$$\int_M (\Delta u) v \, d\mu = - \int_M \langle \nabla u, \nabla v \rangle \, d\mu = \int_M u \Delta v \, d\mu$$

Furthermore, the operator Δ with the domain $C_0^\infty(M)$ is admits the Friedrichs extension to a self-adjoint operator in $L^2(M, \mu)$, which will also be denoted by Δ.

The heat semigroup $P_t = \exp(t\Delta)$, $t \geq 0$, is defined by means of spectral theory as a family of bounded self-adjoint operators in $L^2(M, \mu)$. For any $f \in L^2(M, \mu)$, the function

$$u(t, x) = P_t f(x)$$

is a smooth function of $(t,x) \in \mathbb{R}_+ \times M$, satisfies the heat equation $\frac{\partial u}{\partial t} = \Delta u$ and the initial condition

$$u(t,\cdot) \xrightarrow{L^2} f \quad \text{as } t \to 0+.$$

The *heat kernel* $p_t(x,y)$ is a function of $t > 0$ and $x, y \in M$ such that

$$P_t f(x) = \int_M p_t(x,y) f(y)\, d\mu(y),$$

for all $f \in L^2(M,\mu)$. It is known that $p_t(x,y)$ exists on any Riemannian manifold and is unique. Besides, the heat kernel satisfies the following properties.

- Smoothness: $p_t(x,y) \in C^\infty(\mathbb{R}_+ \times M \times M)$
- Positivity: $p_t(x,y) > 0$
- Symmetry: $p_t(x,y) = p_t(y,x)$;
- The semigroup identity:
$$p_{t+s}(x,y) = \int_M p_t(x,z)\, p_s(z,y)\, d\mu(z). \tag{1}$$
- Submarkovian property:
$$\int_M p_t(x,y)\, d\mu(y) \le 1.$$
- For any $y \in M$, the function $u(x,t) = p_t(x,y)$ satisfies the heat equation and the initial condition
$$u(t,x) \to \delta_y(x) \quad \text{as } t \to 0+,$$
that is, $p_t(x,y)$ is a fundamental solution of the heat equation. Moreover, $p_t(x,y)$ is the smallest positive fundamental solution of the heat equation.

Recall that in $\mathbb{R}^n$, Δ is the classical Laplace operator $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$, and its heat kernel is given by the Gauss-Weierstrass formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

Explicit formulas for the heat kernel exist also in hyperbolic spaces $\mathbb{H}^n$. For example in $\mathbb{H}^3$

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right), \tag{2}$$

where $r = d(x,y)$ is the geodesic distance between x, y. For arbitrary $\mathbb{H}^n$ the formula looks complicated, but it implies the following estimate, for all $t > 0$ and $x, y \in \mathbb{H}^n$:

$$p_t(x,y) \simeq \frac{(1+r+t)^{\frac{n-3}{2}}(1+r)}{t^{n/2}} \exp\left(-\lambda t - \frac{r^2}{4t} - \sqrt{\lambda} r\right), \tag{3}$$

where $\lambda = \frac{(n-1)^2}{4}$ is the bottom of the spectrum of the Laplace operator on $\mathbb{H}^n$.

1 Uniqueness in the Cauchy problem

Fix $0 < T \leq \infty$ and consider the following Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } (0,T) \times M, \\ u|_{t=0} = u_0, \end{cases} \tag{4}$$

where u_0 a given initial function, a solution u is sought in the class $u \in C^\infty((0,T)\times M)$, and the initial condition means that $u(t,\cdot) \to u_0$ as $t \to 0$ in certain sense to be specified. As it was already mentioned above if $u_0 \in L^2(M)$ then solution of (4) exists: $u(t,\cdot) = P_t u_0$, and it satisfies the initial condition in L^2 sense.

For a more general u_0, set

$$u(t,x) = P_t u_0(x) := \int_M p_t(x,y)\, u_0(y)\, d\mu,$$

assuming that the integral in the right hand side is finite. For u_0 in certain function classes, it is possible to prove that this function u solves the heat equation and also satisfies the initial condition in an appropriate sense. For example, if $u_0 \in C_b(M)$ then u is finite, solves the heat equation and satisfies the initial condition locally uniformly. If $u_0 \in L^p(M)$ with $1 \leq p < \infty$ then u is again a solution and satisfies the initial condition in L^p sense.

In this section, we investigate uniqueness of solution of (4) in a certain class of function. Clearly, this is equivalent to the uniqueness of solution $u \equiv 0$ of the homogeneous Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } (0,T) \times M, \\ u|_{t=0} = 0, \end{cases} \tag{5}$$

where the initial condition will be understood in L^2_{loc}-sense.

As it is well-known, even in $\mathbb{R}^n$ the Cauchy problem (5) has non-zero solution, and in order to ensure the uniqueness one has to restrict a function class where a solution is sought.

In order to state the main result of this section, we need some notation. Let $d(x,y)$ be the geodesic distance on M. Denote by $B(x,r)$ the geodesic ball of radius r centered at x, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Recall that a manifold M is geodesically complete if and only if all the geodesic balls are precompact (theorem of Hopf–Rinow).

Theorem 1 *Let $(M, \mathbf{g})$ be a geodesically complete connected Riemannian manifold, and let $u(x,t)$ be a solution to the Cauchy problem (5). Assume that, for some $x_0 \in M$ and for all $R > 0$,*

$$\int_0^T \int_{B(x_0,R)} u^2(t,x)\, d\mu(x) dt \leq \exp(f(R)), \tag{6}$$

where $f(r)$ is a positive increasing function on $(0,+\infty)$ such that

$$\int^\infty \frac{r dr}{f(r)} = \infty. \tag{7}$$

Then $u \equiv 0$ in $(0,T) \times M$.

Theorem 1 provides the uniqueness class (6) for the Cauchy problem (4). The condition (7) holds if, for example, $f(r) = Cr^2$, but fails for $f(r) = Cr^{2+\varepsilon}$ when $\varepsilon > 0$.

Proof of Theorem 1. Denote for simplicity $B_r = B(x_0, r)$. Let us extend the solution u to $t = 0$ by setting $u(0, \cdot) = 0$. The main technical part of the proof is the following claim.

Claim. *Under the hypotheses of Theorem 1, for all $R > 0$ and $0 \le b < a$ satisfying the condition*

$$a - b \le \frac{R^2}{8f(4R)}, \tag{8}$$

the following inequality holds:

$$\int_{B_R} u^2(a, \cdot)d\mu \le \int_{B_{4R}} u^2(b, \cdot)d\mu + \frac{4}{R^2}. \tag{9}$$

Let us first show how this Claim allows to prove that $u \equiv 0$. Fix $R > 0$ and $t \in (0, T)$. For any non-negative integer k, set

$$R_k = 4^k R$$

and, for any $k \ge 1$, choose (so far arbitrarily) a number τ_k to satisfy the condition

$$0 < \tau_k \le \frac{R_{k-1}^2}{8f(4R_{k-1})} = \frac{1}{128}\frac{R_k^2}{f(R_k)}. \tag{10}$$

Then define a decreasing sequence of times $\{t_k\}$ inductively by $t_0 = t$ and $t_k = t_{k-1} - \tau_k$ (see Fig. 1).

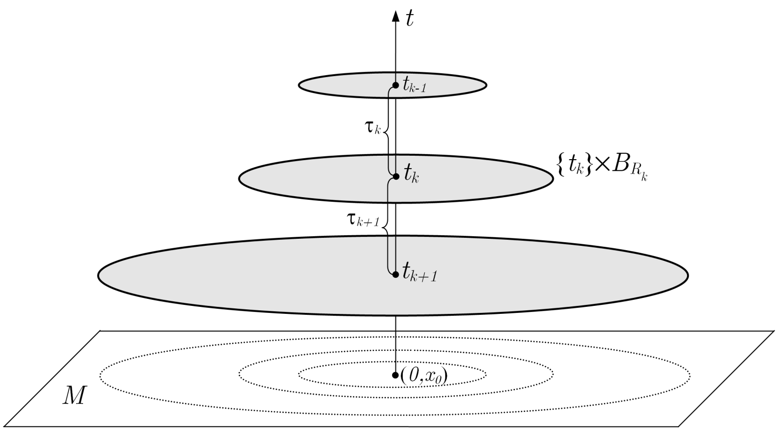

Figure 1: The sequence of the balls B_{R_k} and the time moments t_k.

If $t_k \ge 0$ then, applying the Claim with $a = t_{k-1}$ and $b = t_k$, we obtain from (9)

$$\int_{B_{R_{k-1}}} u^2(t_{k-1}, \cdot)d\mu \le \int_{B_{R_k}} u^2(t_k, \cdot)d\mu + \frac{4}{R_{k-1}^2}, \tag{11}$$

which implies by induction that

$$\int_{B_R} u^2(t,\cdot)d\mu \le \int_{B_{R_k}} u^2(t_k,\cdot)d\mu + \sum_{i=1}^{k} \frac{4}{R_{i-1}^2}. \tag{12}$$

If it happens that $t_k = 0$ for some k then, by the initial condition in (5),

$$\int_{B_{R_k}} u^2(t_k,\cdot)d\mu = 0.$$

In this case, it follows from (12) that

$$\int_{B_R} u^2(t,\cdot)d\mu \le \sum_{i=1}^{\infty} \frac{4}{R_{i-1}^2} = \frac{C}{R^2}.$$

If we have this inequality for any $R > 0$, then, by letting $R \to \infty$, we obtain $u(t,\cdot) \equiv 0$.

Hence, to finish the proof of $u \equiv 0$, it suffices to construct, for any $R > 0$ and $t \in (0,T)$, a sequence $\{t_k\}$ as above that vanishes at a finite k. The condition $t_k = 0$ is equivalent to

$$t = \tau_1 + \tau_2 + ... + \tau_k\,. \tag{13}$$

The only restriction on τ_k is the inequality (10). The hypothesis that $f(r)$ is an increasing function implies that

$$\int_R^\infty \frac{rdr}{f(r)} = \sum_{k=0}^\infty \int_{R_k}^{R_{k+1}} \frac{rdr}{f(r)} \le \sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)},$$

which together with (7) yields

$$\sum_{k=1}^\infty \frac{R_k^2}{f(R_k)} = \infty.$$

Therefore, the sequence $\{\tau_k\}_{k=1}^\infty$ can be chosen to satisfy simultaneously (10) and

$$\sum_{k=1}^\infty \tau_k = \infty.$$

By diminishing some of τ_k, we can achieve (13) for any finite t, which finishes the proof of $u \equiv 0$.

Now we prove the above Claim. Since the integral in

$$\int_{B_{4R}} u^2(b,\cdot)d\mu$$

is continuous in $b \in [0,T)$ up to $b = 0$, it suffices to prove (9) for $b > 0$. In particular, we can assume that $u(t,x)$ is C^∞-smooth in $[b,a] \times M$.

Let $\rho(x)$ be a Lipschitz function on M (to be specified below) with the Lipschitz constant 1. Fix a real $s \notin [b,a]$ (also to be specified below) and consider the following the function

$$\xi(t,x) := \frac{\rho^2(x)}{4(t-s)},$$

which is defined on $\mathbb{R} \times M$ except for $t = s$, in particular, on $[b, a] \times M$. The distributional gradient $\nabla\rho$ is in $L^\infty(M)$ and satisfies the inequality $|\nabla\rho| \leq 1$, which implies, for any $t \neq s$,

$$|\nabla\xi(t,x)| \leq \frac{\rho(x)}{2(t-s)}.$$

Since

$$\frac{\partial\xi}{\partial t} = -\frac{\rho^2(x)}{4(t-s)^2},$$

we obtain

$$\frac{\partial\xi}{\partial t} + |\nabla\xi|^2 \leq 0. \tag{14}$$

For a given $R > 0$, let φ be the cutoff function of B_{2R} in B_{3R} (see Fig. 2), that is,

$$\varphi(x) = \min\left(\left(3 - \tfrac{d(x,x_0)}{R}\right)_+, 1\right)$$

Obviously, we have $0 \leq \varphi \leq 1$ on M, $\varphi \equiv 1$ in B_{2R}, and $\varphi \equiv 0$ outside B_{3R}. Since the function $d(\cdot, x_0)$ is Lipschitz with the Lipschitz constant 1, we obtain that φ is Lipschitz with the Lipschitz constant $1/R$. It follows that $|\nabla\varphi| \leq 1/R$. By the completeness of M, all the balls in M are relatively compact sets, which implies that φ has a compact support.

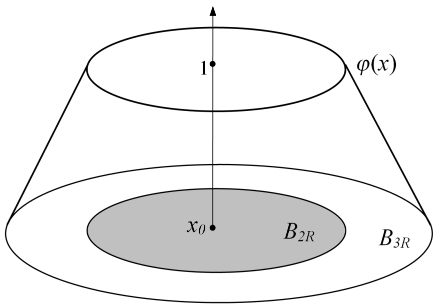

Figure 2: Function $\varphi(x)$

Consider the function $u\varphi^2 e^\xi$ as a function of x for any fixed $t \in [b, a]$. Clearly, this is a Lipschitz function with compact support. Multiplying the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

by $u\varphi^2 e^\xi$ and integrating it over $[b, a] \times M$, we obtain

$$\int_b^a \int_M \frac{\partial u}{\partial t} u\varphi^2 e^\xi d\mu dt = \int_b^a \int_M (\Delta u) u\varphi^2 e^\xi d\mu dt. \tag{15}$$

Since both functions u and ξ are smooth in $t \in [b, a]$ and φ does not depend on t, the time integral on the left hand side can be computed as follows:

$$\int_b^a \frac{\partial u}{\partial t} u\varphi^2 e^\xi dt = \frac{1}{2}\int_b^a \frac{\partial(u^2)}{\partial t}\varphi^2 e^\xi dt = \frac{1}{2}\left[u^2\varphi^2 e^\xi\right]_b^a - \frac{1}{2}\int_b^a \frac{\partial\xi}{\partial t} u^2\varphi^2 e^\xi dt. \tag{16}$$

Integrating this identity over M, using (15) and (14), we obtain

$$\begin{aligned}\left[\int_M u^2\varphi^2 e^{\xi} d\mu\right]_b^a &= \int_b^a\int_M \frac{\partial \xi}{\partial t} u^2\varphi^2 e^{\xi} d\mu dt + 2\int_b^a\int_M \frac{\partial u}{\partial t} u\varphi^2 e^{\xi} d\mu dt \\ &\leq -\int_b^a\int_M |\nabla \xi|^2 u^2\varphi^2 e^{\xi} d\mu dt + 2\int_b^a\int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu dt \quad (17)\end{aligned}$$

Using the Green formula to evaluate the spatial integral on the right hand side of (17), we obtain

$$\int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu = -\int_M \langle \nabla u, \nabla (u\varphi^2 e^{\xi})\rangle d\mu.$$

Applying the product rule and the chain rule to compute $\nabla(u\varphi^2 e^{\xi})$, we obtain

$$\begin{aligned}-\langle \nabla u, \nabla (u\varphi^2 e^{\xi})\rangle &= -|\nabla u|^2 \varphi^2 e^{\xi} - \langle \nabla u, \nabla \xi\rangle u\varphi^2 e^{\xi} - 2\langle \nabla u, \nabla \varphi\rangle u\varphi e^{\xi} \\ &\leq -|\nabla u|^2 \varphi^2 e^{\xi} - \langle \nabla u, u\nabla \xi\rangle \varphi^2 e^{\xi} + \left(\frac{1}{2}|\nabla u|^2\varphi^2 + 2|\nabla\varphi|^2 u^2\right) e^{\xi} \\ &= -\left(\frac{1}{2}|\nabla u|^2 + \langle \nabla u, u\nabla\xi\rangle\right)\varphi^2 e^{\xi} + 2|\nabla\varphi|^2 u^2 e^{\xi},\end{aligned}$$

whence

$$2\int_M (\Delta u)\, u\varphi^2 e^{\xi} d\mu \leq -\int_M -\left(|\nabla u|^2 + 2\langle \nabla u, u\nabla\xi\rangle\right)\varphi^2 e^{\xi} d\mu + 4\int_M |\nabla\varphi|^2 u^2 e^{\xi}.$$

Substituting into (17), we obtain

$$\begin{aligned}\left[\int_M u^2\varphi^2 e^{\xi} d\mu\right]_b^a &\leq -\int_b^a\int_M \left(|\nabla\xi|^2 u^2 + |\nabla u|^2 + 2\langle \nabla u, u\nabla\xi\rangle\right)\varphi^2 e^{\xi} d\mu dt \\ &\quad +4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^{\xi} d\mu dt \\ &= -\int_b^a\int_M (u\nabla\xi + \nabla u)^2 \varphi^2 e^{\xi} d\mu dt \\ &\quad +4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^{\xi} d\mu dt\end{aligned}$$

whence

$$\left[\int_M u^2\varphi^2 e^{\xi} d\mu\right]_b^a \leq 4\int_b^a\int_M |\nabla\varphi|^2 u^2 e^{\xi} d\mu dt. \quad (18)$$

So far we have used only that φ is compactly supported. Using the specific shape of φ, in particular, $|\nabla\varphi| \leq 1/R$, we obtain from (18)

$$\int_{B_R} u^2(a,\cdot) e^{\xi(a,\cdot)} d\mu \leq \int_{B_{4R}} u^2(b,\cdot) e^{\xi(b,\cdot)} d\mu + \frac{4}{R^2}\int_b^a\int_{B_{4R}\setminus B_{2R}} u^2 e^{\xi} d\mu dt. \quad (19)$$

Let us now specify ξ. For that we choose $\rho(x)$ and s as follows. Set $\rho(x)$ to be the distance function from the ball B_R, that is,

$$\rho(x) = d\left(x, B_R\right) = \left(d(x, x_0) - R\right)_+$$

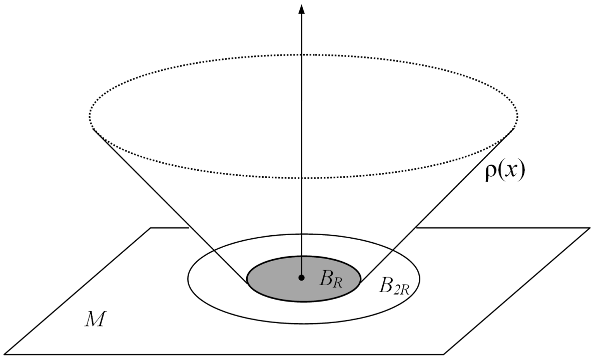

Figure 3: Function $\rho(x)$.

(see Fig. 3).

Set $s = 2a - b$ so that, for all $t \in [b, a]$,

$$a - b \le s - t \le 2(a - b),$$

whence

$$\xi(t, x) = -\frac{\rho^2(x)}{4(s-t)} \le -\frac{\rho^2(x)}{8(a-b)} \le 0. \tag{20}$$

Consequently, we can drop the factor e^ξ on the left hand side of (19) because $\xi = 0$ in B_R, and drop the factor e^ξ in the first integral on the right hand side of (19) because $\xi \le 0$. Clearly, if $x \in B_{4R} \backslash B_{2R}$ then $\rho(x) \ge R$, which together with (20) implies that

$$\xi(t, x) \le -\frac{R^2}{8(a-b)} \quad \text{in } [b, a] \times B_{4R} \backslash B_{2R}.$$

Hence, we obtain from (19)

$$\int_{B_R} u^2(a, \cdot) d\mu \le \int_{B_{4R}} u^2(b, \cdot) d\mu + \frac{4}{R^2} \exp\left(-\frac{R^2}{8(a-b)}\right) \int_b^a \int_{B_{4R}} u^2 d\mu dt.$$

By (6) we have

$$\int_b^a \int_{B_{4R}} u^2 d\mu dt \le \exp(f(4R))$$

whence

$$\int_{B_R} u^2(a, \cdot) d\mu \le \int_{B_{4R}} u^2(b, \cdot) d\mu + \frac{4}{R^2} \exp\left(-\frac{R^2}{8(a-b)} + f(4R)\right).$$

Finally, applying the hypothesis (8), we obtain (9). ■

Corollary 2 *If $M = \mathbb{R}^n$ and $u(t, x)$ is a solution to (5) satisfying the condition*

$$|u(t, x)| \le C \exp\left(C |x|^2\right) \quad \text{for all } t \in (0, T), \ x \in \mathbb{R}^n, \tag{21}$$

then $u \equiv 0$. Moreover, the same is true if u satisfies instead of (21) the condition

$$|u(t,x)| \leq C \exp\left(f\left(|x|\right)\right) \quad \textit{for all}\, t \in (0,T)\,,\ x \in \mathbb{R}^n, \tag{22}$$

where $f(r)$ is a convex increasing function on $(0,+\infty)$ satisfying (7).

Proof. Since (21) is a particular case of (22) for the function $f(r) = Cr^2$, it suffices to treat the condition (22). In $\mathbb{R}^n$ we have

$$\mu\left(B\left(x,R\right)\right) = c_n R^n.$$

Therefore, (22) implies that

$$\int_0^T \int_{B(0,R)} u^2(t,x)\, d\mu(x) dt \leq CTR^n \exp\left(2f\left(R\right)\right) = \exp(\widetilde{f}\left(R\right)),$$

where

$$\widetilde{f}\left(r\right) := 2f\left(r\right) + n\log r + \mathrm{const}\,.$$

The convexity of f implies that $f(r) \geq cr$ for large enough r. Hence, $\widetilde{f}(r) \leq Cf(r)$ for large r, and the function $\widetilde{f}$ also satisfies the condition (7). By Theorem 1, we conclude $u \equiv 0$. ■

The class of functions u satisfying (21) is called the *Tikhonov class*, and the conditions (22) and (7) define the *Täcklind class*. The uniqueness of the Cauchy problem in $\mathbb{R}^n$ in each of these classes is a classical result.

2 Stochastic completeness

Consider the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } (0,\infty) \times M, \\ u|_{t=0} = 1. \end{cases}$$

Clearly, it always has a solution $u_1 \equiv 1$. On the other hand, it also has a solution

$$u_2\left(t,x\right) = P_t 1\left(x\right) = \int_M p_t\left(x,y\right) d\mu\left(y\right).$$

Moreover, by the properties of the heat kernel, we have

$$0 \leq u_2 \leq 1.$$

Hence, both u_1 and u_2 are bounded solutions of the same Cauchy problem. If $u_1 \not\equiv u_2$ then we obtain non-uniqueness of the Cauchy problem in the class of bounded functions. Of course, the condition $u_1 \not\equiv u_2$ is equivalent to $P_t 1 \not\equiv 1$, that is, to $P_t 1\left(x\right) < 1$ for some t and x.

Definition. If $P_t 1 \equiv 1$ then the manifold M is called *stochastically complete*, and if $P_t 1 \not\equiv 1$ then M is called *stochastically incomplete*.

Since $p_t\left(x,y\right)$ is the transition density of Brownian motion $\{X_t\}_{t\geq 0}$ on M, the quantity $P_t 1\left(x\right)$ is equal to the probability that the Brownian particle stays on M at the time t. It may be smaller than 1 if the particle escapes to ∞ in finite time. This phenomenon is called *explosion* of Brownian motion. Hence, the stochastic incompleteness is equivalent to the explosion.

Let us summarize this discussion in the following statement.

Proposition 3 *If M is stochastically incomplete then the Cauchy problem has non-uniqueness in the class of bounded functions.*

In fact, it is possible to prove that M is stochastically complete *if and only if* the Cauchy problem has uniqueness in the class of bounded functions.

Let us introduce the notation

$$V(x,r) = \mu(B(x,r)).$$

Theorem 4 *If M is geodesically complete and, for some $x_0 \in M$,*

$$\int^{\infty} \frac{rdr}{\log V(x_0,r)} = \infty, \tag{23}$$

then M is stochastically complete.

Proof. To prove that M is stochastically complete, it suffices to verify that the Cauchy problem has a unique solution in the class of bounded functions. Indeed, if u is a bounded solution of (5) with zero initial function then setting

$$C := \sup |u| < \infty$$

we obtain, for any $T > 0$,

$$\int_0^T \int_{B(x_0,R)} u^2(t,x)d\mu(x) \leq C^2 T V(x_0,R) = \exp(f(R)),$$

where

$$f(r) := \log\left(C^2 T V(x_0,r)\right) = \log V(x_0,r) + \mathrm{const}.$$

It follows from the hypothesis (23) that the function f satisfies (7), that is,

$$\int^{\infty} \frac{rdr}{f(r)} = \infty.$$

Hence, by Theorem 1, we obtain $u \equiv 0$. ■

For example, the condition (23) is satisfied if, for large r,

$$V(x_0,r) = \exp\left(Cr^2\right)$$

and is not satisfied if

$$V(x_0,r) = \exp\left(Cr^{2+\varepsilon}\right)$$

for some $\varepsilon > 0$.

Our next purpose is to construct examples of stochastically incomplete but geodesically complete manifolds. For that, we need the following tool.

Lemma 5 *Let there exist a non-negative function $u \in L^1(M)$ such that*

$$-\Delta u \geq f,$$

where f is a non-negative continuous function on M such that $f \not\equiv 0$. Then $P_t 1 \not\equiv 1$, that is, M is stochastically incomplete.

Proof. For any $\alpha > 0$, denote by R_α the resolvent operator:

$$R_\alpha = (-\Delta + \alpha)^{-1} = \int_0^\infty \frac{1}{\lambda + \alpha} dE_\lambda,$$

where $\{E_\lambda\}$ is the spectral resolution of $-\Delta$. Clearly, we have

$$-\Delta u + \alpha u \geq f. \tag{24}$$

It is known that the minimal non-negative solution of the inequality (24) is the function $R_\alpha f$, so that we obtain

$$u \geq R_\alpha f.$$

Since

$$\frac{1}{\lambda + \alpha} = \int_0^\infty e^{-\alpha t} e^{-\lambda t} dt,$$

it follows that

$$R_\alpha = \int_0^\infty e^{-\alpha t} e^{t\Delta} dt = \int_0^\infty e^{-\alpha t} P_t dt,$$

so that

$$u \geq \int_0^\infty e^{-\alpha t} P_t f \, dt.$$

Letting $\alpha \to 0+$, we obtain

$$u(x) \geq \int_0^\infty P_t f(x) \, dt = \int_0^\infty \int_M p_t(x, y) f(y) \, d\mu(y) \, dt.$$

Since $u \in L^1(M)$, we have

$$\int_M u(x) \, d\mu(x) < \infty,$$

whence

$$\int_M \int_0^\infty \int_M p_t(x, y) f(y) \, d\mu(y) \; dt \, d\mu(x) < \infty,$$

that is

$$\int_0^\infty \int_M P_t 1(y) f(y) \, d\mu(y) \, dt < \infty. \tag{25}$$

However, if $P_t 1 \equiv 1$ then we have

$$\int_0^\infty \int_M P_t 1(y) f(y) \, d\mu(y) \, dt = \int_0^\infty \int_M f(y) \, d\mu(y) \, dt = \int_0^\infty \|f\|_{L^1} \, dt = \infty,$$

which contradicts (25). Hence, $P_t 1 \not\equiv 1$, which was to be proved. ■

Let (r, θ) be the polar coordinates in $\mathbb{R}^n$, that is, for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$r = |x| \in \mathbb{R}_+ \text{ and } \theta = \frac{x}{|x|} \in \mathbb{S}^{n-1}.$$

The canonical Euclidean metric

$$\mathbf{g}_{\mathbb{R}^n} = dx_1^2 + ... + dx_n^2$$

has in the polar coordinates the form

$$\mathbf{g}_{\mathbb{R}^n} = dr^2 + r^2 \mathbf{g}_{\mathbb{S}^{n-1}},$$

where $\mathbf{g}_{\mathbb{S}^{n-1}}$ is the canonical metric of the sphere $\mathbb{S}^{n-1}$.

Let $(M, \mathbf{g})$ be a *model manifold* based on $\mathbb{R}^n$, that is, $M = \mathbb{R}^n$ as a smooth manifold, and the metric $\mathbf{g}$ has in the polar coordinates (r, θ) in $\mathbb{R}^n$ the form

$$\mathbf{g} = dr^2 + \psi(r)^2 \mathbf{g}_{\mathbb{S}^{n-1}}, \tag{26}$$

where $\psi(r)$ is a smooth positive function on $(0, +\infty)$, such that $\psi(r) = r$ for small r.

The geodesic ball $B(0, R)$ of the metric $\mathbf{g}$ coincides with the Euclidean ball, and its volume in $(M, \mathbf{g})$ is equal to

$$V(R) := \mu(B(0, R)) = \int_0^R \omega_n \psi^{n-1}(r)\, dr = \int_0^R S(r)\, dr,$$

where

$$S(r) := \omega_n \psi^{n-1}(r).$$

The function $V(R)$ is called the *volume function* of M and $S(r)$ is called the *area function* of M. The Laplace-Beltrami operator of $(M, \mathbf{g})$ has in the polar coordinates the following form:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial}{\partial r} + \frac{1}{\psi^2(r)} \Delta_{\mathbb{S}^{n-1}}. \tag{27}$$

Proposition 6 *If on the model manifold M*

$$\int^\infty \frac{V(r)}{S(r)} dr < \infty, \tag{28}$$

then M is stochastically incomplete. Consequently, the Cauchy problem on M features non-uniqueness in the class of bounded solutions.

Proof. By Lemma 5, it suffices to construct on M a non-negative function $u \in L^1(M)$ such that

$$-\Delta u = f, \tag{29}$$

where $f \in C_0^\infty(M)$, $f \geq 0$ and $f \not\equiv 0$. Both functions u and f will depend only on the polar radius r so that (29) in the domain of the polar coordinates becomes

$$u'' + \frac{S'}{S} u' = -f. \tag{30}$$

Choose $f(r)$ to be any non-negative non-zero function from $C_0^\infty(1, 2)$. Then (30) has on $(0, \infty)$ a solution

$$u(R) = \int_R^\infty \frac{dr}{S(r)} \int_0^r S(t) f(t)\, dt. \tag{31}$$

Indeed, since f is bounded, the condition (28) implies that u is finite. It is easy to see that u satisfies the equation

$$(Su')' = -Sf,$$

which is equivalent to (30).

The function $u(R)$ is clearly positive and monotone decreasing. It is constant on the interval $0 < R < 1$ because $f(t) \equiv 0$ for $0 < t < 1$. Hence, u extends by continuity to the origin and satisfies (29) on the whole manifold.

We are left to verify that $u \in L^1(M)$. Since $f(t) \equiv 0$ for $t > 2$, we have for $R > 2$

$$u(R) = C \int_R^\infty \frac{dr}{S(r)}$$

where $C = \int_0^2 S(t) f(t) \, dt$. Therefore,

$$\begin{aligned} \int_{\{R>2\}} u d\mu &= \int_2^\infty u(R) S(R) \, dR \\ &= C \int_2^\infty \left(\int_R^\infty \frac{dr}{S(r)} \right) S(R) \, dR \\ &= C \int_2^\infty \left(\int_2^r S(R) \, dR \right) \frac{dr}{S(r)} \\ &\leq C \int_2^\infty \frac{V(r)}{S(r)} dr < \infty, \end{aligned}$$

which gives $u \in L^1(M)$. ■

For example, (28) is satisfied if, for large r,

$$V(r) = \exp\left(r^{2+\varepsilon}\right),$$

where $\varepsilon > 0$, since in this case

$$\frac{S(r)}{V(r)} = (\log V(r))' = (2+\varepsilon) r^{1+\varepsilon}.$$

Hence, a model manifold with such a volume function $V(r)$ is stochastically incomplete, and the Cauchy problem on this manifold has non-unique solution even in the class of bounded functions.

In fact, the condition (28) is not only sufficient but also necessary for a model manifold to be stochastically incomplete.

3 Takeda's inequality and consequences

Recall the following theorem from lectures of 2017.

Theorem 7 (Davies-Gaffney inequality) *Let S be a measurable subset of a manifold M. Then, for any function $f \in L^2(S)$ and for all positive R, t,*

$$\int_{(S_R)^c} (P_t f)^2 \, d\mu \leq \|f\|_2^2 \exp\left(-\frac{R^2}{2t}\right), \tag{32}$$

where S_R is the R-neighborhood of A.

Similarly to Theorem 7, the next theorem provides a certain L^2-estimate for a solution to the heat equation. However, the setting and the estimate are essentially different.

Theorem 8 *Let B be a precompact open subset of a Riemannian manifold M and let $u(t,x)$ be a solution of the heat equation in $(0,T)\times B$ such that*

$$u(t,\cdot)\overset{L^2_{loc}(B)}{\longrightarrow}0 \quad as \quad t\to 0. \tag{33}$$

Then, for any open set $A\Subset B$ and any $t\in(0,T)$,

$$\int_A u^2(t,\cdot)\,d\mu\le\mu(B\setminus A)\|u\|_\infty^2\max\left(\frac{R^2}{2t},\frac{2t}{R^2}\right)\exp\left(-\frac{R^2}{2t}+1\right), \tag{34}$$

where $R=d(A,B^c)$ (see Fig. 4).

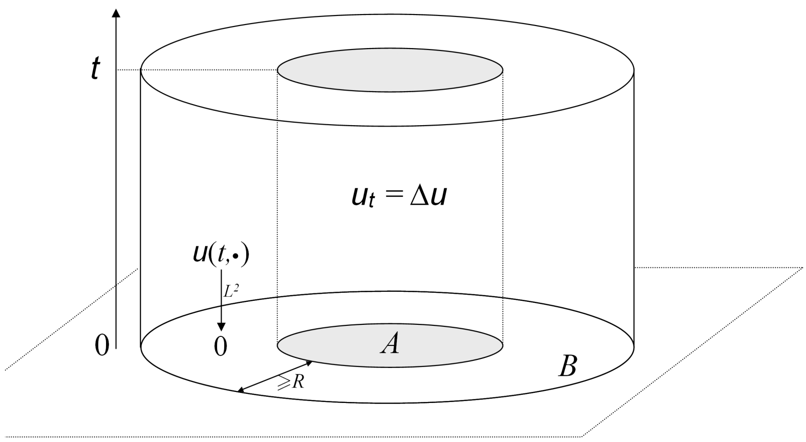

Figure 4: The function $u(t,x)$ in $(0,T)\times B$.

Remark. In fact, the following inequality is true:

$$\int_A u(t,\cdot)\,d\mu\le 16\mu(B)\|u\|_{L^\infty}\int_R^\infty\frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{r^2}{4t}\right)dr. \tag{35}$$

A version of this inequality for local regular Dirichlet forms was proved by Masayoshi Takeda in 1989 using probabilistic methods. In the form (35) it was proved by Terry Lyons in 1990. Estimating in a certain way the integral on the right hand side, one obtains

$$\int_A u(t,\cdot)\,d\mu\le\frac{16}{\sqrt{\pi}}\mu(B)\|u\|_{L^\infty}\frac{\sqrt{t}}{R}\exp\left(-\frac{R^2}{4t}\right).$$

The inequality (34) can be considered as an L^2 version of Takeda's inequality (35).

Remark. The hypotheses of Theorem 8 are in particular satisfied if $u(t,\cdot)=P_tf$ where f is a non-negative function from $L^\infty(B^c)$. If in addition $f\in L^2(B^c)$ then Theorem 7 with $S=B^c$ yields in this case the following estimate

$$\int_A u^2(t,\cdot)\,d\mu\le\|f\|_2^2\exp\left(-\frac{R^2}{2t}\right),$$

because $A \subset (S_R)^c$. The advantage of (34) is that it can be applied to functions like $f = 1_{B^c}$ that are in L^∞ but are not necessarily in L^2.

Proof of Theorem 8. Without loss of generality, we can assume throughout that $|u| \leq 1$. Let $\xi(s,x)$ be a continuous function on $[0,T] \times \overline{B}$ such that $\xi(s,x)$ is Lipschitz in x, continuously differentiable in s, and the following inequality holds almost everywhere on $[0,T] \times B$:

$$\partial_s \xi + \frac{\alpha}{2} |\nabla \xi|^2 \leq 0, \tag{36}$$

for some $\alpha > 1$. We claim that the following inequality is true for any $t \in (0,T)$ and any $\varphi \in Lip_0(B)$:

$$\int_B u(t,\cdot)^2 \varphi^2 e^{\xi(t,\cdot)} d\mu \leq \frac{2\alpha}{\alpha - 1} \int_0^t \int_B |\nabla \varphi|^2 e^{\xi(s,\cdot)} u^2 d\mu ds. \tag{37}$$

A similar inequality (18 was used in the proof of Theorem 1 in the case $\alpha = 2$. Clearly, it suffices to prove (37) for $\varphi \in C_0^\infty(B)$, which will be assumed in the sequel.

Multiplying the heat equation

$$\partial_s u = \Delta u$$

by $u\varphi^2 e^\xi$ and integrating over $[0,t] \times B$, we obtain

$$\int_0^t \int_B \partial_s u \, u\varphi^2 e^\xi d\mu ds = \int_0^t \int_B (\Delta u) \, u\varphi^2 e^\xi d\mu ds. \tag{38}$$

On the left hand side we have

$$\int_0^t \partial_s u \, u\varphi^2 e^\xi ds = \frac{1}{2} \int_0^t \partial_s \left(u^2\right) \varphi^2 e^\xi ds = \frac{1}{2} \left[u^2 \varphi^2 e^\xi\right]_0^t - \frac{1}{2} \int_0^t \partial_s \xi \, u^2 \varphi^2 e^\xi ds.$$

Integrating this identity over B, using (36), (38) and the initial condition (33), we obtain

$$\begin{aligned} \int_B u(t,\cdot)^2 \varphi^2 e^{\xi(t,\cdot)} d\mu &= \int_0^t \int_B \partial_s \xi \, u^2 \varphi^2 e^\xi d\mu ds + 2 \int_0^t \int_B \partial_s u \, u\varphi^2 e^\xi d\mu ds \\ &\leq -\frac{\alpha}{2} \int_0^t \int_B |\nabla \xi|^2 u^2 \varphi^2 e^\xi d\mu dt + 2 \int_0^t \int_B (\Delta u) \, u\varphi^2 e^\xi d\mu ds. \end{aligned} \tag{39}$$

By the Green formula, we have

$$\int_B (\Delta u) \, u\varphi^2 e^\xi d\mu = -\int_B \langle \nabla u, \nabla (u\varphi^2 e^\xi) \rangle d\mu.$$

By the product and chain rules, we obtain, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} -\langle \nabla u, \nabla (u\varphi^2 e^\xi) \rangle &= -|\nabla u|^2 \varphi^2 e^\xi - \langle \nabla u, \nabla \xi \rangle u\varphi^2 e^\xi - 2\langle \nabla u, \nabla \varphi \rangle u\varphi e^\xi \\ &\leq -|\nabla u|^2 \varphi^2 e^\xi - \langle \nabla u, u\nabla \xi \rangle \varphi^2 e^\xi + \left(\varepsilon |\nabla u|^2 \varphi^2 + \frac{1}{\varepsilon} |\nabla \varphi|^2 u^2\right) e^\xi \\ &= -\left[(1-\varepsilon) |\nabla u|^2 + \langle \nabla u, u\nabla \xi \rangle\right] \varphi^2 e^\xi + \frac{1}{\varepsilon} |\nabla \varphi|^2 u^2 e^\xi. \end{aligned}$$

Substituting into (39), we obtain

$$\begin{aligned}\int_B u(t,\cdot)^2\varphi^2 e^{\xi(t,\cdot)}\,d\mu &\leq -\int_0^t\int_B\left[\frac{\alpha}{2}|\nabla\xi|^2u^2+2(1-\varepsilon)|\nabla u|^2+2\langle\nabla u,u\nabla\xi\rangle\right]\varphi^2e^{\xi}d\mu ds\\ &\quad +\frac{2}{\varepsilon}\int_0^t\int_B|\nabla\varphi|^2u^2e^{\xi}d\mu ds\end{aligned}$$

Choose ε to satisfy

$$1-\varepsilon=\frac{1}{\alpha}.$$

Then the expression in the square brackets is a complete square:

$$\begin{aligned}\frac{\alpha}{2}|\nabla\xi|^2u^2+2(1-\varepsilon)|\nabla u|^2+2\langle\nabla u,u\nabla\xi\rangle &= \frac{\alpha}{2}|\nabla\xi|^2u^2+\frac{2}{\alpha}|\nabla u|^2+2\langle\nabla u,u\nabla\xi\rangle\\ &= \left(\sqrt{\frac{\alpha}{2}}u\nabla\xi+\sqrt{\frac{2}{\alpha}}\nabla u\right)^2.\end{aligned}$$

Since $\varepsilon=1-\frac{1}{\alpha}=\frac{\alpha-1}{\alpha}$, it follows that

$$\int_B u(t,\cdot)^2\varphi^2e^{\xi(t,\cdot)}\,d\mu\leq\frac{2\alpha}{\alpha-1}\int_0^t\int_B|\nabla\varphi|^2u^2e^{\xi}d\mu ds,$$

which proves (37).

Now let us specify the functions φ and ξ in (37). In all cases, we will have $\varphi\equiv1$ on A, whence also $|\nabla\varphi|=0$ on A, so that (37) implies

$$\int_A u(t,\cdot)^2e^{\xi(t,\cdot)}d\mu\leq\frac{2\alpha}{\alpha-1}\int_{B\setminus A}|\nabla\varphi|^2\left(\int_0^t e^{\xi(s,\cdot)}\,ds\right)d\mu. \tag{40}$$

In order to prove (34) for $R=d(A,B^c)$, it suffices to prove (34) for any $R<d(A,B^c)$. Fix $R<d(A,B^c)$, $t\in(0,T)$, set

$$\rho(x)=d(x,A),$$

and consider the function

$$\varphi(x)=\psi(\rho(x))$$

where $\psi(r)$ is a Lipschitz function on $[0,+\infty)$ such that

$$\psi(0)=1\text{ and }\psi(r)=0\text{ if }r\geq R$$

(see Fig. 5).

This ensures that $\varphi\in Lip_0(B)$ and $\varphi\equiv1$ on A. We have

$$\nabla\varphi=\psi'(\rho)\nabla\rho,$$

and since $\|\nabla\rho\|_{L^\infty}\leq1$, it follows that

$$|\nabla\varphi(x)|\leq|\psi'(\rho(x))|\text{ for almost all }x\in B\setminus A. \tag{41}$$

To specify further ψ and ξ, consider two cases.

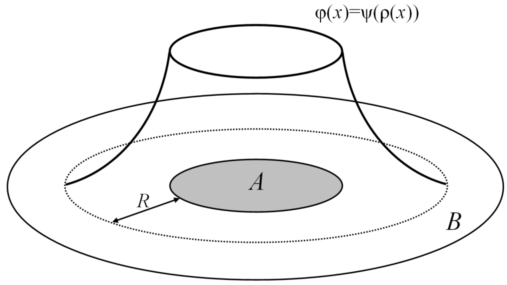

Figure 5: Function $\varphi(x)$

Case 1. In the trivial case
$$\frac{R^2}{2t} \leq 1,$$
we set $\xi \equiv 0$ and
$$\psi(r) = \frac{(R-r)_+}{R}.$$
By (41) we have $|\nabla \varphi| \leq \frac{1}{R}$, and it follows from (40) that
$$\int_A u^2(t,\cdot)\, d\mu \leq \frac{2\alpha}{\alpha-1} \frac{t}{R^2} \mu(B \setminus A).$$
Letting $\alpha \to \infty$, we obtain
$$\int_A u^2(t,\cdot)\, d\mu \leq \frac{2t}{R^2} \mu(B \setminus A) \leq \frac{2t}{R^2} e^{-\frac{R^2}{2t}+1} \mu(B \setminus A). \tag{42}$$

Case 2. In the main case
$$\frac{R^2}{2t} > 1,$$
we set
$$\xi(s,x) = -2a\rho(x) - bs,$$
for some positive constants a and b. Clearly, ξ satisfies (36) provided
$$b = 2a^2\alpha.$$
Below we will specify α and a, while b will always be determined by this identity. Note also that
$$\int_0^t e^{\xi(s,x)}\, ds = \frac{1-e^{-bt}}{b} e^{-2a\rho(x)}. \tag{43}$$
Next, define ψ as follows:
$$\psi(r) = \frac{\left(e^{aR} - e^{ar}\right)_+}{e^{aR}-1}.$$
Then we have
$$\psi'(r) = -ce^{ar} \text{ for } r \in (0,R),$$

where

$$c := \frac{a}{e^{aR}-1},$$

and it follows that

$$\left|\nabla\varphi\left(x\right)\right|^2 \leq c^2 e^{2a\rho(x)} \tag{44}$$

for almost all $x \in B \setminus A$. Substituting (43) and (44) into (40) and observing that $\xi|_A = -bt$, we obtain

$$\begin{aligned}
\int_A u\left(t,\cdot\right)^2 \, d\mu &= e^{bt}\int_A u\left(t,\cdot\right)^2 e^{\xi(t,\cdot)} d\mu \\
&\leq \frac{2\alpha}{\alpha-1} e^{bt} \int_{B\setminus A} \left|\nabla\varphi\right|^2 d\mu \int_0^t e^{\xi(s,\cdot)}\, ds \\
&\leq \frac{2\alpha}{\alpha-1} e^{bt} \mu\left(B\setminus A\right) c^2 e^{2a\rho(x)} \frac{1-e^{-bt}}{b} e^{-2a\rho(x)} \\
&= \frac{2\alpha}{\alpha-1}\frac{e^{bt}-1}{b}\left(\frac{a}{e^{aR}-1}\right)^2 \mu\left(B\setminus A\right) \\
&= \frac{2\alpha}{\alpha-1}\frac{e^{2a^2\alpha t}-1}{2a^2\alpha}\frac{a^2}{\left(e^{aR}-1\right)^2}\mu\left(B\setminus A\right) \\
&= \frac{1}{\alpha-1}\frac{e^{2a^2\alpha t}-1}{\left(e^{aR}-1\right)^2}\mu\left(B\setminus A\right).
\end{aligned}$$

Let us require further that

$$2a^2\alpha t = aR,$$

that is,

$$a = \frac{R}{2\alpha t},$$

where $\alpha > 1$ is still to be specified. Then we have

$$aR = \frac{R^2}{2\alpha t}$$

and, hence,

$$\int_A u\left(t,\cdot\right)^2 \, d\mu \leq \frac{1}{\alpha-1}\frac{1}{e^{\frac{R^2}{2\alpha t}}-1}\mu\left(B\setminus A\right).$$

Finally, we choose α by

$$\alpha = \frac{\frac{R^2}{2t}}{\frac{R^2}{2t}-1}$$

so that

$$\frac{R^2}{2\alpha t} = \frac{R^2}{2t} - 1 =: \delta \tag{45}$$

and

$$\alpha - 1 = \frac{1}{\delta}.$$

Hence, we obtain

$$\int_A u\left(t,\cdot\right)^2 \, d\mu \leq \frac{\delta}{e^{\delta}-1}\mu\left(B\setminus A\right). \tag{46}$$

Since $e^{\delta} \geq 1+\delta$, we have

$$1-e^{-\delta} \geq 1-\frac{1}{1+\delta}=\frac{\delta}{1+\delta}$$

whence

$$e^{\delta}-1 \geq \frac{\delta}{1+\delta} e^{\delta}$$

and

$$\frac{\delta}{e^{\delta}-1} \leq (1+\delta)\, e^{-\delta}.$$

Substituting into (46) and using (45), we obtain

$$\int_A u\,(t,\cdot)^2\, d\mu \leq \frac{R^2}{2t} e^{-\frac{R^2}{2t}+1} \mu\,(B \setminus A)\,. \tag{47}$$

Combining (42) and (47), we obtain (34). ■

Remark. As one can see from the proof, the same estimate holds if u is a non-negative subsolution of the heat equation in $(0,T)\times B$.

Corollary 9 *Under the conditions of Theorem 8, the following inequalities are satisfied:*

$$\int_A u^2\,(t,\cdot)\, d\mu \leq \mu\,(B)\, \|u\|_{L^\infty}^2 \max\left(\frac{R^2}{2t},1\right) \exp\left(-\frac{R^2}{2t}+1\right) \tag{48}$$

and

$$\int_A u\,(t,\cdot)\, d\mu \leq \sqrt{\mu\,(A)\,\mu\,(B)}\|u\|_{L^\infty} \max\left(\frac{R}{\sqrt{2t}},1\right) \exp\left(-\frac{R^2}{4t}+\frac{1}{2}\right). \tag{49}$$

Proof. If $R^2/2t \geq 1$ then (48) trivially follows from (34). If $R^2/2t \leq 1$ then

$$\int_A u^2\,(t,\cdot)\, d\mu \leq \mu\,(A)\, \|u\|_{L^\infty}^2 \leq \mu\,(B)\, \|u\|_{L^\infty}^2 \exp\left(-\frac{R^2}{2t}+1\right),$$

which again implies (48).

Inequality (49) follows from (48) and the Cauchy-Schwarz inequality. ■

Corollary 10 *Let M be a geodesically complete manifold. Assume that there exists a sequence $\{R_k\}$ with $R_k \to \infty$ such that, for some $x \in M$,*

$$V\,(x,R_k) \leq \exp\left(CR_k^2\right). \tag{50}$$

Then M is stochastically complete.

Proof. One can show that (50) implies that

$$\int^{\infty} \frac{rdr}{\log V\,(x,r)}=\infty$$

so that the stochastic completeness follows from Theorem 4. Let us give a different proof based on (48). It is suffices to prove that any bounded solution u of the heat equation in $(0,T)\times M$ with the zero initial condition must be identical zero. For any $k\in\mathbb{N}$, set $A_k=B(x,R_k/2)$ and $B_k=B(x,R_k)$. Then by (48) we have, for any $t>0$ and large enough k,

$$\int_{A_k} u^2(t,\cdot)\,d\mu \le \|u\|^2_{L^\infty} V(x,R_k)\frac{R_k^2}{8t}\exp\left(-\frac{R_k^2}{8t}\right)\le \|u\|^2_{L^\infty}\exp\left(CR_k^2-\frac{R_k^2}{16t}\right).$$

Choosing $t<t_0:=1/(16C)$, we obtain

$$\int_{A_k} u^2(t,\cdot)\,d\mu \le C'\exp\left(-cR_k^2\right)$$

with $c=c(t)>0$, which implies as $k\to\infty$ that $u(t,\cdot)\equiv 0$, provided $t<t_0$. By iterating this argument, we obtain that $u(t,\cdot)\equiv 0$ for all $t<T$. ■

Next we will obtain some pointwise estimates of the solution $u(t,x)$ from the above statements. For that we meed the following lemma.

Lemma 11 (Mean value inequality) *Let $B(x,r)$ be a precompact ball on M. Assume that $u(t,y)$ is a solution to the heat equation in the cylinder $(0,r^2)\times B(x,r)$. Then, for any $0<t\le r^2$,*

$$u^2(t,x)\le\frac{\lambda(x,r)}{tV\left(x,\sqrt{t}\right)}\int_0^t\int_{B(x,\sqrt{t})}u^2(s,y)d\mu(x)ds, \tag{51}$$

where the constant $\lambda(x,r)$ depends only on the geometry inside the ball $B(x,r)$.

The inequality (51) is also true if $u(t,y)$ is a non-negative subsolution of the heat equation.

This lemma is illustrated on Fig. 6.

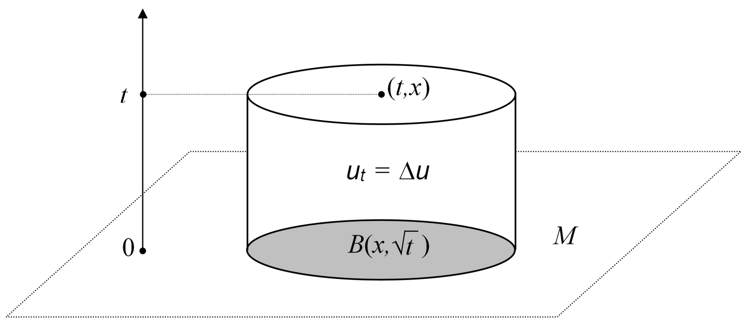

Figure 6: Illustration to the mean value inequality

In fact, we will need Lemma 11 only for sufficiently small balls $B(x,r)$. In this case, we can work entirely in a chart and write the heat equation in the form

$$\partial_t u = Lu,$$

where L is a uniformly elliptic operator in the divergence form. For solutions of such equations, the estimate (51) was proved by J.Moser in 1964. The coefficient $\lambda(x,r)$

depends in this case on r, $\dim M$ and on the ellipticity constant of L. One can give a more geometric proof of (51) that works for any precompact ball $B(x,r)$. Then the coefficient $\lambda(x,r)$ depends on the constant in a certain Sobolev inequality inside the ball $B(x,r)$.

Under certain additional assumption about the manifold, the constants $\lambda(x,r)$ may be independent of x and r. For example, this is the case when M is a geodesically complete manifold with non-negative Ricci curvature. In this case $\lambda(x,r)$ can be replaced by a constant.

Theorem 12 *Let ball $B(x,R)$ be precompact. Let u be a solution to the heat equation in a cylinder $(0,\infty)\times B(x,R)$ such that $0\le u\le 1$ and*

$$u(t,\cdot)\to 0 \quad \text{as } t\to 0 \text{ in } L^2_{loc}(B(x,R)).$$

Fix $r\in(0,R)$. Then, for any $t\ge r^2$, we have

$$u(t,x)\le\sqrt{\lambda(x,r)\frac{V(x,R)}{V(x,r)}}\max\Big(\frac{R}{\sqrt{t}},1\Big)\exp\left(-\frac{(R-r)^2}{4t}\right), \tag{52}$$

and, for any $t\le r^2$,

$$u(t,x)\le\sqrt{\lambda(x,r)\frac{V(x,R)}{V\left(x,\sqrt{t}\right)}\frac{R}{\sqrt{t}}}\exp\left(-\frac{(R-r)^2}{4t}\right), \tag{53}$$

where $\lambda(x,r)$ depends only on the geometry inside $B(x,r)$.

Proof. Let $t\ge r^2$. By the mean value inequality of Lemma 11 in the cylinder $(t-r^2,t)\times B(x,r)$, we have

$$u^2(t,x)\le\frac{\lambda(x,r)}{r^2V(x,r)}\int_{t-r^2}^{t}\int_{B(x,r)}u^2(s,y)d\mu(y)ds. \tag{54}$$

By Corollary 9 with $A=B(x,r)$ and $B=B(x,R)$, we obtain, for any $s\in(0,t)$,

$$\begin{aligned}\int_{B(x,r)}u^2(s,y)d\mu(y) &\le V(x,R)\max\Big(\frac{(R-r)^2}{2s},1\Big)\exp\left(-\frac{(R-r)^2}{2s}+1\right)\\ &\le V(x,R)\max\Big(\frac{(R-r)^2}{2t},1\Big)\exp\left(-\frac{(R-r)^2}{2t}+1\right),\end{aligned}$$

where we have used in the last line the fact that the function $\max(\xi,1)\exp(-\xi)$ is monotone decreasing in $\xi\in(0,\infty)$. Substituting this inequality into (54) and renaming $e\lambda(x,r)$ into $\lambda(x,r)$, we obtain (52).

Let $t\le r^2$. By the mean value inequality of Lemma 11 in the cylinder $(0,t)\times B\left(x,\sqrt{t}\right)$, we have

$$u^2(t,x)\le\frac{\lambda(x,r)}{tV\left(x,\sqrt{t}\right)}\int_0^t\int_{B(x,\sqrt{t})}u^2(s,y)\,d\mu(y)\,ds. \tag{55}$$

By Corollary 9 with $A = B\left(x, \sqrt{t}\right)$ and $B = B(x, R)$, we obtain, for any $s \in (0, t)$,

$$\begin{aligned}\int_{B\left(x,\sqrt{t}\right)} u^2(s,y)\,d\mu(y) &\leq V(x,R)\max\Big(\frac{\left(R-\sqrt{t}\right)^2}{2s},1\Big)\exp\left(-\frac{\left(R-\sqrt{t}\right)^2}{2s}+1\right)\\ &\leq V(x,R)\max\Big(\frac{\left(R-\sqrt{t}\right)^2}{2t},1\Big)\exp\left(-\frac{\left(R-\sqrt{t}\right)^2}{2t}+1\right).\end{aligned}$$

Substituting this inequality into (55), we obtain

$$u^2(t,x) \leq \frac{\lambda(x,r)}{V\left(x,\sqrt{t}\right)} V(x,R)\max\Big(\frac{R^2}{t},1\Big)\exp\left(-\frac{\left(R-\sqrt{t}\right)^2}{2t}+1\right).$$

Since $R^2/t \geq 1$ and $R - \sqrt{t} \geq R - r$, we obtain (53). ■

Corollary 13 *For any $R > 0$, we have*

$$\int_{B(x,R)^c} p_t(x,y)\,d\mu(y) = o(t) \quad \textit{as } t \to 0. \tag{56}$$

Proof. It suffices to prove (56) for small enough R. In particular, we can assume that $B(x,R)$ is precompact. Consider the function

$$u = P_t \mathbf{1}_{B(x,R)^c} = \int_{B(x,R)^c} p_t(\cdot, y)\,d\mu(y).$$

that solves the heat equation in $(0,\infty) \times M$ and satisfies the initial condition $u(t,\cdot) \xrightarrow{L^2} 0$ in $B(x,R)$ as $t \to 0$. By Theorem 12, we have, for a fixed $0 < r < R$ and for all $t < r^2$ that

$$u(t,x) \leq \sqrt{\lambda(x,r)\frac{V(x,R)}{V\left(x,\sqrt{t}\right)}}\frac{R}{\sqrt{t}}\exp\left(-\frac{(R-r)^2}{4t}\right).$$

Since $V\left(x,\sqrt{t}\right) \simeq t^{n/2}$ as $t \to 0$ where $n = \dim M$, it follows that

$$u(t,x) \leq \frac{C(x,r,R)}{t^N}\exp\left(-\frac{(R-r)^2}{4t}\right),$$

where $N = n/4 + 1/2$. It follows that

$$u(t,x) = o(t) \quad \text{as } t \to 0,$$

which is equivalent to (56). ■

4 Brownian motion and escape rate

Using the heat kernel, one can construct on an arbitrary Riemannian manifold M a stochastic process $\{X_t\}_{t\geq 0}$ whose transition density is $p_t(x,y)$. The latter means that, for any Borel set $A \subset M$ and for all $x \in M$, $t > 0$,

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x,y)\,d\mu(y).$$

Moreover, the process $\{X_t\}$ is Markov, that is, for any finite sequence $\{A_i\}_{i=1}^k$ of Borel sets $A_i \subset M$ and for any sequence $0 < t_1 < ... < t_k$,

$$\mathbb{P}_x\left(X_{t_1} \in A_1, ..., X_{t_k} \in A_k\right) = \int\limits_{A_k} ... \int\limits_{A_2} \int\limits_{A_1} p_{t_1}(x, x_1) p_{t_2-t_1}(x_1, x_2) ... p_{t_k-t_{k-1}}(x_{k-1}, x_k) \times d\mu(x_1) d\mu(x_2) ... d\mu(x_k). \tag{57}$$

Let us discuss construction of such a process. Given a point $x \in M$, we need to construct a probability space $\{\Omega_x, \mathbb{P}_x\}$ and a family $\{X_t\}_{t\geq 0}$ of random variables with values in M such that their joint distributions are given by (57). Assume for simplicity that M is stochastically complete. Kolmogorov's extension theorem says that a family of random variables with predefined joint distributions exists if and only if the distributions satisfy the compatibility condition:

$$\mathbb{P}_x\left(X_{t_1} \in A_1, ..., X_{t_i} \in M, ..., X_{t_k} \in A_{t_k}\right) = \mathbb{P}_x(X_{t_1} \in A_1, ..., \overset{i}{\checkmark}, ..., X_{t_k} \in A_k),$$

where in the right hand side the i-th condition is omitted. The validity of this condition follows from the stochastic completeness and the semigroup identity of the heat kernel. For example, in the case $k = 2$, we have

$$\begin{aligned} \mathbb{P}_x\left(X_{t_1} \in A_1, X_{t_2} \in M\right) &= \int\limits_M \int\limits_{A_1} p_{t_1}\left(x, x_1\right) p_{t_2-t_1}\left(x_1, x_2\right) d\mu\left(x_1\right) d\mu\left(x_2\right) \\ &= \int\limits_{A_1} p_{t_1}\left(x, x_1\right) \left(\int_M p_{t_2-t_1}\left(x_1, x_2\right) d\mu\left(x_2\right)\right) d\mu\left(x_1\right) \\ &= \int\limits_{A_1} p_{t_1}\left(x, x_1\right) d\mu\left(x_1\right) = \mathbb{P}_x\left(X_{t_1} \in A_1\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_x\left(X_{t_1} \in M, X_{t_2} \in A_2\right) &= \int\limits_{A_2} \int\limits_M p_{t_1}\left(x, x_1\right) p_{t_2-t_1}\left(x_1, x_2\right) d\mu\left(x_1\right) d\mu\left(x_2\right) \\ &= \int\limits_{A_2} p_{t_2}\left(x, x_2\right) d\mu\left(x_2\right) = \mathbb{P}_x\left(X_{t_2} \in A_2\right). \end{aligned}$$

An important property of the process $\{X_t\}$ is that it is a *diffusion,* that is, all the paths X_t are continuous with probability 1. By a general theory of symmetric Markov processes, the continuity of sample paths follows from the following property of the transition probabilities: for any point $x \in M$ and for any open set U containing x,

$$\mathbb{P}_x\left(X_t \in U^c\right) = o\left(t\right) \text{ as } t \to 0. \tag{58}$$

In term of the heat kernel, (58) amounts to the following: for any $x \in M$ and any $r > 0$,

$$\int_{B(x,r)^c} p_t\left(x, y\right) d\mu\left(y\right) = o\left(t\right) \text{ as } t \to 0, \tag{59}$$

which is indeed the case by Corollary 13.

Definition. The diffusion process $\{X_t\}$ constructed as above is called *Brownian motion* on M.

In the case $M = \mathbb{R}^n$, it coincides with the classical Brownian motion with the transition density

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

It is easy to see that, for this function,

$$\int_{(B_r)^c} p_t(0,y)\,dy \le \text{const} \exp\left(-\frac{r^2}{5t}\right) = o(t) \quad \text{as } t \to 0.$$

For comparison, let us consider the function

$$p_t(x,y) = \frac{C_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}},$$

that is the heat kernel the operator $(-\Delta)^{1/2}$ in $\mathbb{R}^n$, or, equivalently, the transition density of a symmetric stable Levy process of index 1. For this heat kernel we have

$$\int_{(B_r)^c} p_t(0,y)\,dy \simeq \frac{t}{r} \quad \text{as } t \to 0$$

so that (59) fails. Indeed, it is known that a symmetric stable Levy process of index $\in (0,2)$ is a jump process, and its trajectories are discontinuous.

Let us know discuss how fast X_t goes away from the origin as $t \to \infty$. If X_t is Brownian motion in $\mathbb{R}^n$ then it is known that $\mathbb{E}_0 X_t = 0$ and

$$\mathbb{E}_0 |X_t|^2 = \int_{\mathbb{R}^n} |x|^2 p_t(x,0)\,dx = c_n t.$$

Hence, $|X_t|$ is on average of the order $\sqrt{t}$. A more precise *law of iterated logarithm* says that

$$\limsup_{t\to\infty} \frac{|X_t|}{\sqrt{4t \log\log t}} = 1 \qquad \mathbb{P}_0\text{-a.s.} \tag{60}$$

Fix $\varepsilon > 0$ and set

$$R(t) = \sqrt{(4+\varepsilon)\, t \log\log t}.$$

Then it follows from (60) that

$$\mathbb{P}_0\left(|X_t| < R(t) \ \text{ for all large enough } t\right) = 1.$$

One says that this function $R(t)$ is an upper rate function for Brownian motion in $\mathbb{R}^n$.

Let us return to an arbitrary Riemannian manifold M.

Definition. A monotone increasing function $R(t)$ on $\mathbb{R}_+$ is called an *upper rate function* for Brownian motion X_t on M started at $z \in M$ if

$$\mathbb{P}_z\left(d(X_t, z) < R(t) \ \text{ for all large enough } t\right) = 1$$

(see Fig. 7).

The next theorem provides an upper rate function based on the volume growth.

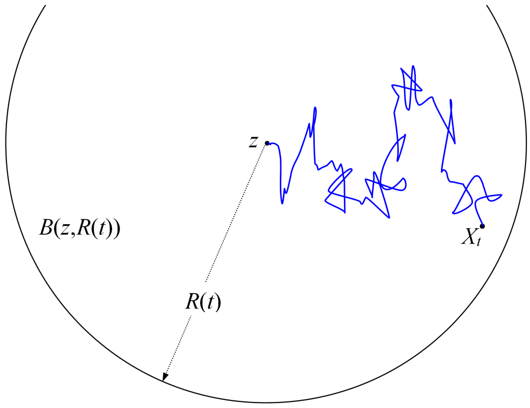

Figure 7: An upper radius function $R(t)$: the process X_t stays in the ball $B(x, R(t))$ for all large enough t with $\mathbb{P}_x$-probability 1.

Theorem 14 *Let M be a geodesically complete manifold. Assume that, for some point $z \in M$ and all large enough R,*

$$V(z, R) \le CR^N, \tag{61}$$

where $C, N > 0$. Then the function $R(t) = \sqrt{\eta t \log t}$ is an upper rate function for the process X_t started at z, for any $\eta > N$. Consequently,

$$\limsup_{t\to\infty} \frac{d(X_t, z)}{\sqrt{Nt\log t}} \le 1 \quad \mathbb{P}_z\text{-}a.s.$$

Proof. The hypothesis (61) implies by Theorem 4 that M is stochastically complete. Assuming that the process X_t starts at z, let us set

$$\mathcal{M}(t) := \sup_{0\le s\le t} d(z, X_s).$$

Given an increasing function $R(t)$, let us introduce a sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ of events as follows:

$$\mathcal{A}_k := \{\mathcal{M}(t) \ge R(t) \quad \text{for some } t \in (t_k, t_{k+1}]\},$$

where so far $\{t_k\}$ is any increasing sequence such that $t_k \to +\infty$ as $k \to \infty$ (see Fig. 8).

Clearly, the function $R(t)$ is an upper rate function if the $\mathbb{P}_z$-probability that only finitely many of the events $\mathcal{A}_k$ occur is equal to 1. By the lemma of Borel–Cantelli, the latter will follow from

$$\sum_k \mathbb{P}_z(\mathcal{A}_k) < \infty. \tag{62}$$

Set

$$R(t) = \sqrt{\eta t \log t}, \tag{63}$$

and

$$t_k := q^k, \quad k = 1, 2, ..., \tag{64}$$

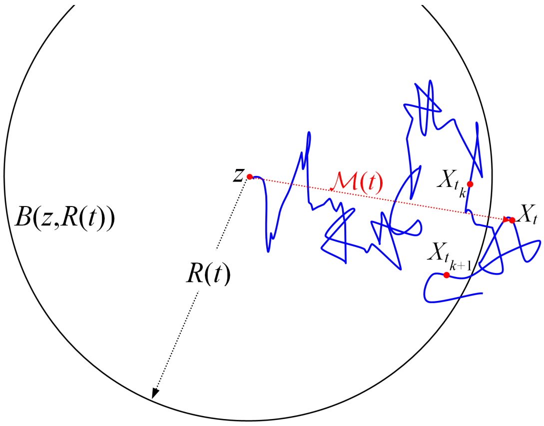

Figure 8: Event $\mathcal{A}_k$

where $q > 1$ and $\eta > 0$ will be chosen later. We will show that, under the hypothesis (61), the condition (62) can be satisfied for an appropriate choice of the parameters η, q.

For any $R > 0$, define the following function for all $t > 0$ and $x \in B(z, R)$:

$$\psi_R(t, x) := \mathbb{P}_x \{X_s \notin B(z, R) \text{ for some } s \in [0, t]\}.$$

It is possible to prove that the function $\psi_R(t, x)$ solves the heat equation in $\mathbb{R}_+ \times B(z, R)$ and tends to 0 in $B(z, R)$ as $t \to 0$.

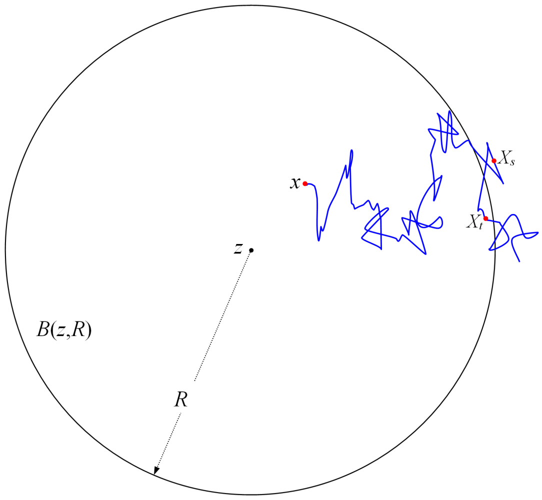

Figure 9: Event $\{X_s \notin B(z, R) \text{ for some } s \in (0, t]\}$ determines the function $\psi_R(t, x)$

Since both $\mathcal{M}(t)$ and $R(t)$ are increasing in t, we have:

$$\mathbb{P}_z(\mathcal{A}_k) \leq \mathbb{P}_z\{\mathcal{M}(t_{k+1}) \geq R(t_k)\} = \psi_{R_k}(z, t_{k+1}), \tag{65}$$

where

$$R_k := R(t_k) = \sqrt{\eta t_k \log t_k}. \tag{66}$$

Let r be any fixed small positive number. Applying Theorem 12 for $u = \psi_R$, we obtain, for all $R > r$ and $t \geq r^2$,

$$\psi_R(t, z) \leq C\sqrt{V(z, R)} \max(\frac{R}{\sqrt{t}}, 1) \exp\left(-\frac{(R-r)^2}{4t}\right), \tag{67}$$

where C depends on z and r. Applying (67) with $R = R_{k+1}$, $t = t_k$ and using (65) and (61), we obtain, for large k,

$$\mathbb{P}_z(\mathcal{A}_k) \leq C R_k^{N/2} \max(\frac{R_k}{\sqrt{t_{k+1}}}, 1) \exp\left(-\frac{(R_k - r)^2}{4t_{k+1}}\right). \tag{68}$$

Assuming further that k is so large that

$$R_k - r \geq \frac{R_k}{q},$$

we obtain

$$\exp\left(-\frac{(R_k - r)^2}{4t_{k+1}}\right) = \exp\left(-\frac{(R_k - r)^2}{4qt_k}\right) \leq \exp\left(-\frac{R_k^2}{4q^3 t_k}\right).$$

By (66), we have

$$\frac{R_k^2}{t_{k+1}} \leq \frac{R_k^2}{t_k} = \eta \log t_k = \eta k \log q.$$

Substituting this into (68) and using (66), we obtain, for all large enough k,

$$\mathbb{P}_z(\mathcal{A}_k) \leq C(t_k \log t_k)^{N/4} \sqrt{k} \exp\left(-\frac{\eta k \log q}{4q^3}\right) = C q^{kN/4} k^{N/4+1/2} q^{-\frac{\eta k}{4q^3}}.$$

Clearly, if

$$\frac{\eta}{4q^3} > \frac{N}{4},$$

then $\sum_k \mathbb{P}(\mathcal{A}_k)$ is dominated by a convergent series $\sum_k k^{N'} q^{-\varepsilon k}$ and, hence, converges. If $\eta > N$ then there exists $q > 1$ such that this condition is satisfied. Hence, for such η, the function (63) is an upper rate function. $\blacksquare$

There is an example showing that in the class of manifolds satisfying (61), the upper rate function $\sqrt{ct \log t}$ is sharp (up to the value of c) and cannot be improved to $\sqrt{ct \log \log t}$.

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with mean zero and variance 1. Set

$$X_n = Y_1 + ... + Y_n.$$

The strong law of large numbers says that

$$\frac{X_n}{n} \xrightarrow{\text{a.s.}} 0 \text{ as } n \to \infty.$$

Hardy and Littlewood proved in 1914 a more precise statement:

$$|X_n| < \sqrt{Cn \log n} \text{ for all large } n \text{ a.s.}$$

In 1924 Khinchin obtained a final result in this direction:

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2n\log\log n}} = 1 \quad \text{a.s.}$$

Hence, our Theorem 14 is an analogue of the Hardy-Littlewood theorem for manifolds with a polynomial volume growth.