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ON LIOUVILLE THEOREMS FOR HARMONIC FUNCTIONS WITH FINITE DIRICHLET INTEGRAL

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ABSTRACT. A criterion for the validity of the D-Liouville theorem is proved. In §1 it is shown that the question of L^∞- and D-Liouville theorems reduces to the study of the so-called massive sets (in other words, the level sets of harmonic functions in the classes L^∞ and $L^\infty \cap D$). In §2 some properties of capacity are presented. In §3 the criterion of D-massiveness is formulated—the central result of this article—and examples are presented. In §4 a criterion for the D-Liouville theorem is formulated, and corollaries are derived. In §§5–9 the main theorems are proved.

Figures: 5. Bibliography: 17 titles.

Introduction

The classical theorem of Liouville states that any bounded harmonic function on $\mathbf{R}^n$ is constant. It is easy to verify that the following assertions are also true:

1) If the harmonic function u on $\mathbf{R}^n$ has finite Dirichlet integral then $u \equiv \text{const}$.

2) If $u \in L^p(\mathbf{R}^n)$ is a harmonic function, $1 \le p < \infty$, then $u \equiv 0$.

The list of theorems of this kind can be extended; they are known in the literature under the general category of Liouville-type theorems.

After Moser's paper [1], which in particular proved Liouville's theorem for entire solutions of the uniformly elliptic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0, \tag{1}$$

it became possible to study the solutions of the Laplace-Beltrami equation(1) on arbitrary Riemannian manifolds. The main efforts here are directed towards finding under what geometric conditions one or another Liouville theorem is true.

Let M be a smooth Riemannian manifold with boundary ∂M (possibly empty). The function $u \in C^\infty(M)$ is called *harmonic* if it satisfies the Laplace-Beltrami equation

$$\Delta u = 0 \tag{2}$$

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(1)Having the form (1) in local coordinates.

(see the definition of the operator Δ in [2]) and the boundary condition

$$\partial u/\partial \nu = 0, \tag{3}$$

where ν is the normal to the boundary ∂M.([2])

If A is some class of functions on M, then by the *A-Liouville theorem* we mean the assertion that any harmonic function in the class A is equal to a constant. The monograph [5] is devoted to the classification of Liouville theorems. In this article we shall primarily take up the D-Liouville theorem, where D is the class of functions on M with finite Dirichlet integral.

It is well known [3] that, for $1 < p < \infty$, the L^p-Liouville theorem is satisfied on any complete manifold. For the P-, L^1-, L^∞-, and D-theorems this is not so (P is the class of positive functions). Existing counterexamples [6] suggest that at least the P-, L^∞-, and D-Liouville theorems fail in the presence of "narrow" places on the manifold. On the other hand, the known sufficient conditions for these theorems to hold (see [4] and [7]–[11]) in some sense or other exclude "narrow" places.

Up to now, for none of the Liouville theorems indicated above was there known a necessary and sufficient condition ensuring its satisfaction. In this article, a criterion for the D-Liouville theorem is proved, which confirms that the sole obstruction for this theorem to be true is the presence of a "narrow" place on the manifold.

By a theorem of Ahlfors (see [5]), if there exists a nontrivial harmonic function with finite Dirichlet integral, then there exists a like function which is bounded as well. Therefore, throughout the following, we shall consider only bounded harmonic functions.

The main theorem is formulated in §4. Let us present two corollaries.

1. If the Riemannian manifolds M_1 and M_2 are quasi-isometric and the D-Liouville theorem holds on M_1, then it also holds on M_2.

It would be interesting to find out whether the analogous assertion is true for other Liouville theorems.

2. Let M be a complete, n-dimensional, spherically symmetric manifold. Then for $n \geq 4$ the D-Liouville theorem holds on M.

For $n = 2$ this is not so, and for $n = 3$ it is not known (for more details see §4).

A few words on the structure of this article. In §1 we prove that the question of L^∞- and D-Liouville theorems reduces to the study of the so-called massive sets (in other words, the level sets of harmonic functions in the classes L^∞ and $L^\infty \cap D$). In §2 some properties of capacity are given. In §3 the criterion of D-massiveness is formulated—the central result of this article—and examples are given. In §4 the criterion for the D-Liouville theorem is formulated and corollaries are derived. In §§5–9 the main theorems are proved.

The numbering of the theorems, lemmas, etc. is sequential through the whole article, while the formulas have their own numbering in each section.

A variant of Green's formula is proved in the Appendix.

Notation and terminology. M is a smooth, connected, noncompact Riemannian manifold; ∂M is the boundary of M; $n = \dim M$; ∇ and Δ are the gradient and the Laplacian on M; $\partial/\partial\nu$ is the derivative in the direction of ν; dV is the volume element of M; dS is the $(n-1)$-volume element on $(n-1)$-submanifolds on M; $D(u,\Omega) = \int_\Omega |\nabla u|^2\, dV$; $D(u) = D(u,\Omega)$, where Ω is the domain of u; $u\,|_A$ is the restriction of the function u to $A \cap \Omega$, where Ω is the domain of u; a smooth hypersurface is a C^∞-submanifold of codimension 1 transversal to the boundary of the manifold; Ω has smooth boundary $\Leftrightarrow$ $\partial\Omega$ is

([2])Condition (3) does not restrict generality, although the first impression may be to the contrary. If we wish to consider solutions of (2) without a boundary condition, then the boundary can be excluded from the manifold.

a smooth hypersurface; and $\{B_k\}$ is an exhaustion of M, i.e. a sequence of precompact open subsets B_k such that $\overline{B}_k \subset B_{k+1}$, $\bigcup_1^\infty B_k = M$, and ∂B_k are smooth.

§1. Massive sets

DEFINITION 1. Let $\Omega \subset M$ be an open subset with smooth boundary. The set Ω is called *massive* if there exists a function $u \in C^\infty(\overline{\Omega} \backslash (\partial\Omega \cap \partial M))$ such that

$$0 < u < 1 \quad \text{in } \Omega, \tag{1}$$

$$\Delta u = 0, \tag{2}$$

$$u \,|_{\partial\Omega} = 0, \tag{3}$$

$$\partial u / \partial \nu \,|_{\partial M} = 0. \tag{4}$$

If in this case $D(u) < \infty$, then Ω is called *D-massive.*

Obviously, a massive set is noncompact. The significance of the concept of massiveness for our purposes is shown by the following proposition.

PROPOSITION 1. *A nontrivial bounded harmonic function (resp., with finite Dirichlet integral) exists on the manifold M if and only if there exists a smooth hypersurface Γ dividing M into two massive (resp., D-massive) subsets.*

The hypersurface Γ is the "narrow" place discussed in the Introduction.

Note that massiveness is an intrinsic property of the set Ω. Massiveness can also be interpreted thus: the massive sets are the sets in which there is no Phragmén-Lindelöf type theorem, i.e. a positive harmonic function with zero Dirichlet boundary condition need not go to infinity.

PROOF OF PROPOSITION 1. If there exists a nontrivial bounded harmonic function u on M, then as Γ one can take the level set $\{u = a\}$, where $a \in (\inf u, \sup u)$ is a regular value of u (i.e. a common regular value of the functions $u \,|_{\dot{M}}$ and $u \,|_{\partial M}$). Clearly, the sets $\{u > a\}$ and $\{u < a\}$ are massive, and D-massive if $D(u) < \infty$.

Now let the smooth hypersurface Γ divide M into two massive subsets Ω and $M \backslash \overline{\Omega}$. We construct on M a bounded harmonic function which is not equal to a constant. First we construct an increasing sequence $\{\Omega_m\}$, $m = 1, 2, \ldots$, of open sets with smooth boundaries, such that $\Omega_m \supset \Omega$ and $\bigcup_1^\infty \Omega_m = M$. Let us show that all the Ω_m are massive. Let u be a function on Ω satisfying (1)–(4). We construct a function u_m in Ω_m satisfying the analogous conditions. Let $\{B_k\}$ be an exhaustion of the manifold M (see the notation list), where ∂B_k is transversal to $\partial\Omega_m$ for all k and m. We solve in $\Omega_m \cap B_k$ the following boundary value problem for the unknown function v_k (see Figure 1):

$$\Delta v_k = 0, \quad v_k \,|_{\partial\Omega_m} = 0, \quad v_k \,|_{\partial B_k \backslash \Omega} = 0, \quad v_k \,|_{\partial B_k \cap \Omega} = u, \quad \partial v_k / \partial v \,|_{\partial M} = 0.$$

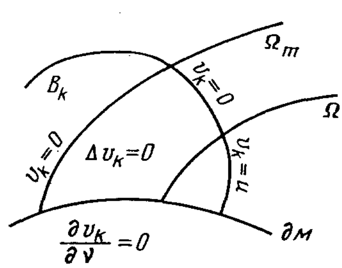

FIGURE 1

It follows from the maximum principle that $0 \leq v_k \leq 1$ in $\Omega_m \cap B_k$, and $v_k \geq u$ in $\Omega \cap B_k$. From these inequalities and the maximum principle again, it follows that $v_{k+1} \geq v_k$ in the common domain of definition. We thus have a bounded increasing sequence of harmonic functions $\{v_k\}$. It has a limit as $k \to \infty$, which we denote u_m, satisfying conditions (1)–(4) in Ω_m.

Now let $m \to \infty$. We redenote the function v_k by v_{km} and observe that, by the maximum principle, we have $v_{k(m+1)} \geq v_{km}$ in $\Omega_m \cap B_k$. As $k \to \infty$ we obtain $u_{m+1} \geq u_m \geq u$. Therefore, there exists a bounded harmonic function $u_\infty = \lim_{m\to\infty} u_m$ defined on M, where in Ω we have $u_\infty \geq u$.

Let us prove that u_∞ is not equal to a constant. Indeed, we use $u_\infty \geq u$ and the massiveness of the set $M\backslash\overline{\Omega}$, which so far has not been applied. Let v be a function on $M\backslash\overline{\Omega}$ satisfying conditions (1), (2), and (4), and, in place of (3), satisfying $v\,|_\Gamma = 1$ (i.e. we subtract from 1 a function satisfying (1)–(4)). Moreover, it can be assumed that $\inf v = 0$. From the maximum principle it follows that in $B_k \cap (\Omega_m\backslash\overline{\Omega})$ we have $v \geq v_{km}$. Therefore, on passing to the limit as $k \to \infty$ and $m \to \infty$, we get $v \geq u_\infty$ in $M\backslash\overline{\Omega}$. But it is easy to see that a constant cannot simultaneously satisfy the conditions $u_\infty \geq u$ and $u_\infty \leq v$. Therefore, u_∞ is the desired nontrivial bounded harmonic function on M.

It remains to prove that if Ω and $M\backslash\overline{\Omega}$ are D-massive, then $D(u_\infty) < \infty$. It suffices for us that the set Ω be D-massive, i.e. $D(u) < \infty$. Indeed, from the definition of the functions v_k and from Green's formula([3]) it follows that

$$\begin{aligned} D(v_k) &= \int_{\Omega_m \cap B_k} |\nabla v_k|^2 dV = \int_{\partial(\Omega_m \cap B_k)} v_k \frac{\partial v_k}{\partial v} dS + \int_{\partial M \cap (\Omega_m \cap B_k)} v_k \frac{\partial v_k}{\partial v} dS \\ &= \int_{\partial B_k \cap \Omega} v_k \frac{\partial v_k}{\partial v} dS = \int_{\partial B_k \cap \Omega} u \frac{\partial v_k}{\partial v} dS. \end{aligned}$$

From $v_k \geq u$ and $v_k\,|_{\partial B_k \cap \Omega} = u$ it follows that $\partial v_k/\partial \nu \leq \partial u/\partial \nu$ on $\partial B_k \cap \Omega$, where ν is the outward normal to ∂B_k. Thus,

$$D(v_k) \leq \int_{\partial B_k \cap \Omega} u \frac{\partial u}{\partial \nu} dS = \int_{B_k \cap \Omega} |\nabla u|^2 dV \leq D(u).$$

Therefore, passing to the limit as $k \to \infty$ and $m \to \infty$, we obtain $D(u_m) \leq D(u)$ and $D(u_\infty) \leq D(u) < \infty$. Proposition 1 is completely proved.

Observe that we have actually proved the following property of massive sets: if Ω and Ω' are subsets of M with smooth boundaries and $\Omega \subset \Omega'$, then the massiveness (D-massiveness) of Ω implies the massiveness (D-massiveness) of Ω'. We present another useful property: if $\Omega' \subset \Omega$ and $\overline{\Omega}\backslash\Omega'$ is compact, then the massiveness (D-massiveness) of Ω implies the massiveness (D-massiveness) of Ω'. We omit the proof, since we shall not be needing this property. From it and from Proposition 1 the following result can also be derived (first proved by N. S. Nadirashvili by a different method). Let the manifolds M_1 and M_2 be such that, if a compact subset is removed from each of them, the the remaining parts M_1' and M_2' are isometric. If the L^∞-Liouville theorem (or the D-Liouville theorem) is satisfied on M_1, then it is also satisfied on M_2.

§2. Capacity and type

DEFINITION 2. A *capacitor* on the manifold M is any triple of sets $(F_1, F_2; \Omega)$, where F_1 and F_2 are closed and Ω is open. The *capacity* of the capacitor $(F_1, F_2; \Omega)$ is the number

$$\operatorname{cap}(F_1, F_2; \Omega) = \inf_\varphi \int_\Omega |\nabla \varphi|^2 dV,$$

([3])See §5 and the Appendix.

where the infimum is taken over all admissible functions φ, i.e. locally Lipschitz functions on $\overline{\Omega}$ such that $\varphi\,|_{F_1} = 1$ and $\varphi\,|_{F_2} = 0$.

Note that a value of ∞ is allowed for capacity.

Let us present some well-known properties of capacity.

1.

$$\begin{aligned}\operatorname{cap}(F_1, F_2; \Omega) &= \operatorname{cap}(F_2, F_1; \Omega) = \operatorname{cap}(\partial F_1, \partial F_2; \Omega)\\ &= \operatorname{cap}(\partial F_1, \partial F_2; \Omega\backslash(F_1 \cup F_2)).\end{aligned}$$

2. If $\Omega \subset \Omega'$, then

$$\operatorname{cap}(F_1, F_2; \Omega) \leq \operatorname{cap}(F_1, F_2; \Omega').$$

Indeed, if φ is admissible for $(F_1, F_2; \Omega')$ and φ "almost" realizes the capacity, then

$$\begin{aligned}\operatorname{cap}(F_1, F_2; \Omega') &= \int_{\Omega'} |\nabla\varphi|^2 dV - \varepsilon \geq \int_{\Omega} |\nabla\varphi|^2 dV - \varepsilon\\ &\geq \operatorname{cap}(F_1, F_2; \Omega) - \varepsilon,\end{aligned}$$

since $\varphi\,|_{\overline{\Omega}}$ is admissible for $(F_1, F_2; \Omega)$. It remains to let $\varepsilon \to 0$.

3. If $F_1 \subset F_1'$, then

$$\operatorname{cap}(F_1, F_2; \Omega) \leq \operatorname{cap}(F_1', F_2; \Omega).$$

Indeed, broadening F_1 restricts the class of admissible functions, and thereby raises the inf in the definition of capacity.

4. Let Ω be a precompact set, and let F_1 and F_2 be the closures of open sets, where the boundaries $\partial\Omega$, ∂F_1, and ∂F_2 are smooth and pairwise transversal. Let $F_1 \cap F_2 = \varnothing$. Let u be a solution in $\Omega_0 = \Omega\backslash(F_1 \cup F_2)$ of the following boundary value problem:

$$\Delta u = 0, \qquad u\,|_{\partial F_1} = 1, \qquad u\,|_{\partial F_2} = 0, \qquad \partial u/\partial\nu\Big|_{\partial\Omega\cup\partial M} = 0.$$

Then

$$\operatorname{cap}(F_1, F_2; \Omega) = \int_{\Omega_0} |\nabla u|^2 dV = \int_{\partial F_1 \cap \overline{\Omega}_0} \frac{\partial u}{\partial\nu} dS, \tag{1}$$

where ν is the normal to ∂F_1 which is outward with respect to Ω_0.

By smoothness of the boundary and the compactness of $\overline{\Omega}$, the classical solution of the above-indicated boundary value problem exists and is unique. The function u is called the *capacity potential* of the capacitor $(F_1, F_2; \Omega)$. The proof of (1) is standard and will be omitted (see, for example, [15]).

With the help of capacity, the notion of type of an open set (parabolic or hyperbolic) is defined. Let Ω be an open subset of M with smooth boundary, and let F be a compact set lying in $\overline{\Omega}$. Let $\{B_k\}$ be an exhaustion of M. We define

$$\operatorname{cap}(F, \infty; \Omega) = \lim_{k\to\infty} \operatorname{cap}(F, \overline{\Omega}\backslash B_k; \Omega).$$

From property 3 of capacity it follows that the sequence of capacities is monotonically decreasing, so that the limit exists. Hence the limit does not depend on the choice of exhaustion sequence $\{B_k\}$. Finally, by the compactness of F and $\overline{B}_k$, the successive capacities are finite, so that $\operatorname{cap}(F, \infty; \Omega) < \infty$.

DEFINITION 3. We say that Ω has *parabolic type* if, for any compact set $F \subset \overline{\Omega}$, $\operatorname{cap}(F, \infty; \Omega) = 0$. Otherwise, Ω has *hyperbolic type*.

REMARK. This terminology is analogous to that used in the theory of Riemann surfaces. As we know, a simply connected noncompact Riemann surface is conformally equivalent to the plane or the disk. In the first case we say it has parabolic type, and in the second, hyperbolic. It is easy to prove that the capacity of any compact set in the plane is zero, but not for one in the disk, so that the definition of type in the theory of Riemann surfaces is compatible with Definition 3. The problem of determining the type of a set reduces to obtaining estimates of its capacity. Some of these estimates, as well as sufficient conditions for parabolic or hyperbolic type are given in [12]. In particular, the cone([4]) in $\mathbf{R}^n$, $n \geq 3$, has hyperbolic type.

We shall need the following properties of sets of hyperbolic type.

1. If Ω and Ω' are open sets with smooth boundaries, $\Omega \subset \Omega'$, and Ω is of hyperbolic type, then Ω' is also of hyperbolic type.

The proof follows from properties 2 and 3 of capacity.

2. If Ω has hyperbolic type and the compact set $F \subset \overline{\Omega}$ has nonempty interior, then $\operatorname{cap}(F, \infty; \Omega) > 0$.

PROOF. Let G be a nonempty open set with smooth boundary lying in $F \cap \Omega$. It suffices to prove that $\operatorname{cap}(\overline{G}, \infty; \Omega) > 0$. By the hyperbolicity of Ω there exists in Ω a compact set of positive capacity. We extend it to a precompact open set $G' \supset G$. It can be assumed that the boundary $\partial G'$ is smooth and transversal to $\partial\Omega$. Let u_k and u'_k be the capacity potentials for the capacitors $(\partial G, \partial B_k; B_k \backslash \overline{G})$ and $(\partial G', \partial B_k; B_k \backslash \overline{G}')$. From the maximum principle it follows that the sequences $\{u_k\}$ and $\{u'_k\}$ increase monotonically, and therefore have limits u and u', where $u_k \leq u'_k$ and $u \leq u'$. Observe that

$$\int_{\partial G} \frac{\partial u}{\partial \nu} dS = \operatorname{cap}(\overline{G}, \infty; \Omega), \qquad \int_{\partial G'} \frac{\partial u'}{\partial \nu} dS = \operatorname{cap}(\overline{G}', \infty; \Omega).$$

Indeed, by properties of capacity,

$$\operatorname{cap}(\overline{G}, \overline{\Omega} \backslash B_k; \Omega) = \operatorname{cap}(\partial G, \partial B_k; B_k \backslash \overline{G}) = \int_{\partial G} \frac{\partial u_k}{\partial \nu} dS \xrightarrow[k \to \infty]{} \int_{\partial G} \frac{\partial u}{\partial \nu} dS.$$

Passage to the limit is possible thanks to a Schauder estimate of the solution up to the boundary see ([17], Appendix IV, §5). The second relation is proved the same way. Since $\operatorname{cap}(\overline{G}', \infty; \Omega) > 0$, it follows that $u' \not\equiv \text{const}$ and $u' < 1$ outside $\overline{G}'$. From $u \leq u'$ it follows that $u \not\equiv \text{const}$, and by the lemma on the normal derivative

$$\int_{\partial G} \frac{\partial u}{\partial \nu} dS > 0, \qquad \operatorname{cap}(\overline{G}, \infty; \Omega) > 0.$$

§3. A criterion for D-massiveness

THEOREM 1. *Let $\Omega \subset M$ be an open set with smooth boundary. Then Ω is D-massive if and only if Ω contains a subset Ω_1 of hyperbolic type whose closure $\overline{\Omega}_1$ is noncompact, lies in Ω, and satisfies* $\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega \backslash \overline{\Omega}_1) < \infty$.

This theorem will be proved in §§5–8. Right now, let us present some examples of D-massive sets.

EXAMPLE 1. Let $\Omega \subset \mathbf{R}^n$ be the exterior of the domain of revolution

$$F = \{x \in \mathbf{R}^n \mid x_n \geq 0,\ r \leq f(x_n)\}, \tag{1}$$

where $r = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$, and $f\colon [0, +\infty) \to [0, +\infty)$ is monotonically decreasing for $x_n \geq 1$ (the differential properties of f are such that $\partial\Omega$ is a smooth hypersurface in $\mathbf{R}^n$).

([4])Smoothed at the vertex.

Then Ω is D-massive if and only if

$$\int^{\infty} f(x)^{n-3} dx < \infty, \qquad n > 3; \tag{2}$$

$$\int^{\infty} \frac{dx}{\ln(1 + x/f(x))} < \infty, \qquad n = 3.\text{([5])} \tag{3}$$

PROOF. Let (2) or (3) be satisfied. The D-massiveness of Ω will follow from

$$\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) = \operatorname{cap}(F, \overline{\Omega}_1; \mathbf{R}^n) < \infty, \tag{4}$$

where Ω_1 is the exterior of a sufficiently large cone containing F.([6]) For the proof of (4) we use the semiadditivity of capacity:

$$\operatorname{cap}(F, \overline{\Omega}_1; \mathbf{R}^n) \leq \sum_{m=0}^{\infty} \operatorname{cap}(F_m, \overline{\Omega}_1; \mathbf{R}^n),$$

where

$$F_m = \{x \in \mathbf{R}^n \mid 2^m \leq x_n \leq 2^{m+1},\ r \leq f(2^m)\}, \quad m \geq 1, \qquad F_0 = F \cap \{x_n \leq 2\}.$$

Further,

$$\operatorname{cap}(F_m, \overline{\Omega}_1; \mathbf{R}^n) \leq \operatorname{cap}(F_m, \partial G_m; G_m),$$

where G_m is a 2^{m-1}-neighborhood of F_m.

If σ_t is the $(n-1)$-measure of the set of points in $\mathbf{R}^n$ whose distance from F_m is t, then by a capacity estimate of [12] we have

$$\operatorname{cap}(F_m, \partial G_m; G_m) \leq \left(\int_0^{2^{m-1}} \frac{dt}{\sigma_t}\right)^{-1}.$$

Clearly, $\sigma_t \leq \operatorname{const}(t + f(2^m))^{n-2} 2^m$ $(m \geq 1)$, so that

$$\operatorname{cap}(F_m, \partial G_m; G_m) \leq \operatorname{const} \begin{cases} f(2^m)^{n-3} 2^m, & n > 3, \\ \dfrac{2m}{\ln(1 + 2^{m-1}/f(2^m))}, & n = 3. \end{cases}$$

By (2) or (3), we get $\sum_1^{\infty} \operatorname{cap}(F_m, \partial G_m; G_m) < \infty$, whence follows (4).

Now let integral (2) diverge, and let Ω_1 be an arbitrary subset of Ω of hyperbolic type, $\overline{\Omega}_1$ noncompact. We prove that $\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) = \infty$.

Let S_r be the sphere in $\mathbf{R}^n$ of radius r with center at the point O, $A_r = S_r\backslash\Omega$, $B_r = S_r \cap \overline{\Omega}_1$, $a_r = \operatorname{meas}_{n-1} A_r$, $b_r = \operatorname{meas}_{n-1} B_r$, and ? $_r = \operatorname{meas}_{n-1} S_r$. From the definition of capacity it follows that

$$\operatorname{cap}(\partial\Omega; \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) \geq \int_1^{\infty} \operatorname{cap}(A_r, B_r; S_r)\, dr. \tag{5}$$

From a capacity estimate of [12] it follows that

$$\operatorname{cap}(A_r, B_r; S_r) \geq \left(\int_{a_r}^{2\sigma_r - b_r} \frac{dv}{g(v)^2}\right)^{-1}, \tag{6}$$

([5]) In $\mathbf{R}^2$ there are no massive subsets, due to the parabolic type of $\mathbf{R}^2$.

([6]) Ω_1 has hyperbolic type, since it contains a cone.

where $g(v)$ is the isoperimetric function on S_r, i.e. $g(v) = \text{const} \cdot v^{(n-2)/(n-1)}$ for $v \le \sigma_r$ and $g(v) = g(2\sigma_r - v)$ for $v \ge \sigma_r$. Evaluating the integral (6) and substituting into (5), we obtain

$$\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) \ge \text{const} \int_1^\infty \frac{dr}{a_r^{-(n-3)/(n-1)} + b_r^{-(n-3)/(n-1)}}. \tag{7}$$

Note that the divergence of (2) implies

$$\int^\infty \frac{dr}{a_r^{-(n-3)/(n-1)}} = \infty. \tag{8}$$

Since Ω_1 has hyperbolic type, it follows that (see [12])

$$\int^\infty \frac{dr}{b_r} < \infty. \tag{9}$$

By Lemma 1, which we shall prove in the next section, it follows from (8), (9), and the boundedness of a_r that the integral (7) diverges. By Theorem 1, Ω is D-massive.

The case $n = 3$ is treated analogously.

EXAMPLE 2. We construct in $\mathbf{R}^n$, $n \ge 3$, a subset Ω diffeomorphic to half-space, in which there exists a nontrivial bounded harmonic function with finite Dirichlet integral.

Let the function f satisfy (2) or (3), let F be the set (1), and let $\Gamma = \partial F \cap \{x_1 = 0\}$. Let the region Ω be such that $\partial\Omega$ contains Γ, and the sets $\Omega_+ = \Omega \cap \{x_1 > 0\}$ and $\Omega_- = \Omega \cap \{x_1 < 0\}$ contain the cones K_+ and K_- respectively (see Figure 2). Then Ω is the desired region.

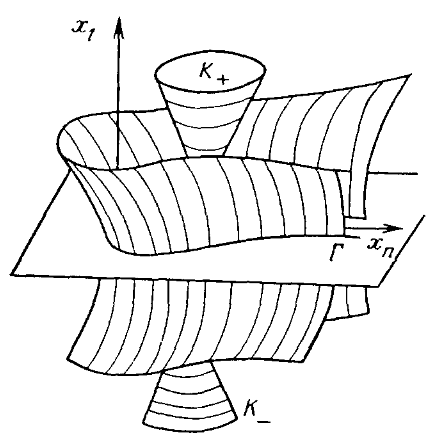

FIGURE 2

Indeed, by Proposition 1, it suffices to prove that Ω_+ and Ω_- are D-massive. By Theorem 1 this will follow from $\operatorname{cap}(F \cap \{x_1 = 0\}, K_+; \Omega_+) < \infty$ and the analogous inequality for K_- and Ω_-.

By the properties of capacity in §2, we have

$$\operatorname{cap}(F \cap \{x_1 = 0\}, K_+; \Omega_+) \le \operatorname{cap}(F, K_+; \mathbf{R}^n) \le \operatorname{cap}(F, \overline{\Omega}_1; \mathbf{R}^n) < \infty,$$

where Ω_1 is the exterior of a cone containing F but not containing K_+ or K_- (see Example 1).

EXAMPLE 3. In the Lobachevsky plane, any angle(7) is a D-massive set.

(7) Smoothed at the vertex.

Note, first, that an angle has hyperbolic type. This follows, for example, from the isoperimetric inequality [13] and the hyperbolicity condition [12]. Consider the two angles Ω and Ω_1 ($\Omega_1 \subset \Omega$) in the Lobachevsky plane, whose sides meet at the point O. Let these angles have a common symmetry axis, and let the angular measure of Ω be greater than the angular measure of Ω_1. We prove that the D-massiveness condition is satisfied: $\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) < \infty$.

Let (r, θ) be polar coordinates with pole at O. For suitable c_1 and c_2 the function $\varphi(r,\theta) = c_1\theta + c_2$ is admissible (more precisely, φ is defined thus for $r \geq r_0 > 0$, and is extended in an admissible fashion to $r < r_0$). It suffices to prove that $\int_{r\geq r_0} |\nabla\theta|^2 dV < \infty$. In polar coordinates, the metric of the Lobachevsky plane has the form $ds^2 = dr^2 + f(r)^2 d\theta^2$. From this it follows that, first, the length of the circle of radius r is $\sigma_r = 2\pi f(r)$, and second, $|\nabla\theta| = 1/f(r)$. Therefore,

$$\int_{r\geq r_0} |\nabla\theta|^2 dV = \int_{r_0}^{\infty} \left(\int_{S_r} f(r)^{-2} dS\right) dr = 4\pi^2 \int_{r_0}^{\infty} \frac{dr}{\sigma_r} < \infty,$$

since σ_r grows exponentially.

§4. Criterion for the D-Liouville theorem

THEOREM 2. *The D-Liouville theorem holds on the manifold M if and only if, for any two open sets Ω_1 and Ω_2 of hyperbolic type having noncompact closures $\overline{\Omega}_1$ and $\overline{\Omega}_2$,*

$$\operatorname{cap}(\overline{\Omega}_1, \overline{\Omega}_2; M) = \infty.$$

The proof, which relies on Theorem 1, is given in §9. Here we consider some corollaries.

COROLLARY 1. *If the manifolds M_1 and M_2 are quasi-isometric and the D-Liouville theorem holds on M_1, then the D-Liouville theorem holds on M_2 also.*

Indeed, quasi-isometry means that, under an identification of M_1 and M_2 by means of a diffeomorphism, the first quadratic forms on M_1 and M_2 are finitely proportional. Therefore, the capacities are finitely proportional, whence follows the desired result.

COROLLARY 2. *Let M be a complete manifold, and let $\rho \in C^\infty(M)$ be a Lipschitz exhaustion function* (*i.e.* $|\nabla\rho| \leq \text{const}$ *and all the level sets* $\{\rho \leq t\}$ *are compact*). *On the hypersurface $S_t = \{\rho = t\}$ for almost all t,* ([8]) *let the isoperimetric inequality* $\operatorname{meas}_{n-2}\Gamma \geq f(v)$ *be satisfied, where Γ is any smooth $(n-2)$-dimensional surface dividing S_t into two sets having $(n-1)$-dimensional volume at least v, and f is a positive monotonically increasing function on* $(0, +\infty)$.

a) *If*

$$\int^{\infty} \frac{dv}{f(v)^2} < \infty, \tag{1}$$

then the D-Liouville theorem holds on M.

b) *Let the integral* (1) *diverge. Put* $I(\sigma) = \int_\varepsilon^\sigma (dv/f(v)^2)$ (*where* $\varepsilon > 0$ *is fixed*) *and* $2\sigma_t = \operatorname{meas}_{n-1} S_t$. *If*

$$\int^{\infty} \frac{dt}{I(\sigma_t + 2\varepsilon)} = \infty, \tag{2}$$

then the D-Liouville theorem holds on M.

PROOF. We shall need the following fact.

([8])By Sard's theorem, for almost all t, S_t is a smooth hypersurface.

LEMMA 1. *Let α and β be measurable functions on $[t_0, +\infty)$, $\alpha(t) \geq \alpha_0 > 0$, and $\alpha+\beta > 0$ (t_0 and α_0 are constants). Let $\alpha\colon \mathbf{R} \to [0, +\infty]$ be a continuous, monotonically increasing function, where $\varphi(\alpha_0) > 0$. Finally, let*

$$\int^{\infty} \frac{d}{\alpha(t)} = \infty, \qquad \int^{\infty} \varphi(\beta(t))dt < \infty.$$

Then

$$\int^{\infty} \frac{dt}{\alpha(t) + \beta(t)} = \infty.$$

PROOF. Put

$$E_+ = \{t \geq t_1 \mid \alpha(t) > \beta(t)\}, \qquad E_- = \{t \geq t_1 \mid \alpha(t) \leq \beta(t)\},$$

where t_1 is sufficiently large. Then

$$\infty > \int_{E_-} \varphi(\beta(t))dt \geq \int_{E_-} \varphi(\alpha(t))dt \geq \varphi(\alpha_0)\alpha_0 \int_{E_-} \frac{dt}{\alpha(t)}.$$

Therefore,

$$\int_{E_+} \frac{dt}{\alpha(t)} = \infty, \qquad \int_{E_+} \frac{dt}{\alpha(t) + \beta(t)} = \infty,$$

whence follows the desired result.

Let us return to the proof of Corollary 2. Let Ω_1 and Ω_2 be subsets of M of hyperbolic type with $\overline{\Omega}_1$ and $\overline{\Omega}_2$ noncompact and $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \varnothing$. We prove that, under condition (1) or (2), $\operatorname{cap}(\overline{\Omega}_1, \overline{\Omega}_2; M) = \infty$.

Denote

$$A_t = \overline{\Omega}_1 \cap S_t, \quad B_t = \overline{\Omega}_2 \cap S_t, \quad a_t = \operatorname{meas}_{n-1} A_t, \quad b_t = \operatorname{meas}_{n-1} B_t.$$

Then

$$\operatorname{cap}(\overline{\Omega}_1, \overline{\Omega}_2; M) \geq \int_{-\infty}^{\infty} \operatorname{cap}(A_t, B_t; S_t)dt.$$

If G is an open subset of S_t with smooth boundary Γ, and $\operatorname{meas}_{n-1} G = v$, then, by the condition of Corollary 2, $\operatorname{meas}_{n-2}\Gamma \geq f_t(v)$, where

$$f_t(v) = \begin{cases} f(v), & v \leq \sigma_t, \\ f(2\sigma_t - v), & v \geq \sigma_t. \end{cases}$$

By an estimate of capacity [12] we have

$$\begin{aligned} \operatorname{cap}(A_t, B_t; S_t) &\geq \left(\int_{a_t}^{2\sigma_t - b_t} f_t(v)^{-2} dv \right)^{-1} = \left(\left(\int_{a_t}^{\sigma_t} + \int_{b_t}^{\sigma_t} \right) \frac{dv}{f_t(v)^2} \right)^{-1} \\ &\geq \left(\left(\int_{a_t}^{\sigma_t} + \int_{b_t}^{\sigma_t} \right) \frac{dv}{f(v)^2} \right)^{-1} = \frac{1}{2I(\sigma_t) - I(a_t) - I(b_t)}. \end{aligned}$$

It remains to prove that

$$\int^{\infty} \frac{dt}{2I(\sigma_t) - I(a_t) - I(b_t)} = \infty. \tag{3}$$

In this case we shall use (1) or (2), as well as the following consequence of the fact that Ω_1 and Ω_2 are hyperbolic (see [12]):

$$\int^{\infty} \frac{dt}{a_t} < \infty, \qquad \int^{\infty} \frac{dt}{b_t} < \infty.$$

a) Let (1) be satisfied. Put $I_1(\sigma) = \int_\sigma^\infty (dv/f(v)^2)$. Then in place of (3) it suffices to prove that

$$\int^\infty \frac{dt}{I_1(a_t) + I_1(b_t) - 2I_1(\sigma_t)} = \infty,$$

or, more crudely,

$$\int^\infty \frac{dt}{1 + I_1(a_t) + I_1(b_t)} = \infty.$$

Using Lemma 1 for $\alpha(t) \equiv 1$, $\beta(t) = I_1(a_t)$, and $\varphi(\beta) = 1/I_1^{-1}(\beta)$, we get

$$\int^\infty (dt/(1 + I_1(a_t))) = \infty.$$

Applying Lemma 1 again for $\alpha(t) = 1 + I_1(a_t)$ and $\beta(t) = I_1(b_t)$, we obtain the desired result.

b) Let (2) be satisfied. Clearly, in place of (3) it suffices to prove that

$$\int^\infty \frac{dt}{2I(\sigma_t + 2\varepsilon) - I(a_t) - I(b_t)} = \infty,$$

and this is obtained by the same twofold application of Lemma 1 as in a).

COROLLARY 3. *Let M be a complete, spherically symmetric manifold, i.e. on M there acts a group of isometries* $\mathrm{SO}(n)$ *having fixed point $O \in M$. Let σ_r be the $(n-1)$-dimensional volume of the geodesic sphere of radius r with center at O. Then, under any of conditions* a), b), *or* c) *the D-Liouville theorem holds on the manifold M, where*

a) $n \geq 4$;

b)

$$n = 3, \qquad \int^\infty \frac{dr}{\ln(\sigma_r + 2)} = \infty; \tag{4}$$

c)

$$n = 2, \qquad \int^\infty \frac{dr}{\sigma_r} = \infty. \tag{5}$$

PROOF. Clearly, the geodesic sphere on M is isometric to a sphere in $\mathbf{R}^n$, so that the isoperimetric inequality with function $f(v) = \text{const} \cdot v^{(n-2)/(n-1)}$ is satisfied on it. For $n \geq 4$ we have $\int^\infty (dv/f(v)^2) < \infty$, and the D-Liouville theorem holds by assertion a) of Corollary 2. For $n = 3$, in the notation of Corollary 2, we have $I(\sigma) = \text{const} \cdot \ln(\sigma/\varepsilon)$, and for $\varepsilon = 1$ condition (2) turns into (4). The case $n = 2$ can be analyzed analogously but, in fact, condition (5) without spherical symmetry, and for any n, already implies that M has parabolic type (see [12]). Also, for $n = 2$ condition (5) is necessary for Corollary 3 to hold, which was actually proved by us in Example 3 of §3. How essential (4) is for $n = 3$ remains unknown.

From assertions a) and b) of Corollary 3, it follows that the D-Liouville theorem is satisfied in Lobachevsky spaces of dimension $n \geq 3$. For $n = 2$ this is not the case, as follows from Example 3 and Proposition 1.

It follows from Corollaries 2 and 3, as well, that the D-Liouville theorem holds on any one-sided surface of revolution in Euclidean space and on any one-sided domain of revolution (like manifolds with boundary), independently of dimension.

§5. Proof of Theorem 1. Necessity

Here we shall prove that if Ω is D-massive then Ω contains a subset Ω_1 of hyperbolic type whose closure $\overline{\Omega}_1$ is noncompact and such that $\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) < \infty$. The D-massiveness of Ω entails the existence of a harmonic function on Ω with properties

(1)–(4) of §1. Let $\varepsilon > 0$ be a regular value of the function u such that the set $\{u > \varepsilon\}$ is nonempty. Put $\Omega_1 = \{x \in \Omega \mid u(x) > \varepsilon\}$. Then Ω_1 has smooth boundary, the closure $\overline{\Omega}_1$ is noncompact, and $\operatorname{cap}(\partial\Omega_1, \partial\Omega; \Omega\backslash\overline{\Omega}_1) < \infty$ since the admissible function $u/\varepsilon\,|_{\Omega\backslash\overline{\Omega}_1}$ has finite Dirichlet integral. It remains to prove that Ω_1 has hyperbolic type.

We redenote $u - \varepsilon$ by u, and then the function u satisfies conditions (1)–(4) of §1 in Ω_1. We use the following variant of Green's formula.

PROPOSITION 2. *Let N be a smooth manifold with boundary having parabolic type. Let $u \in C^\infty(N)$ and $\sup_N |u| < \infty$. Then Green's formula*

$$\int_N \Delta u\, dV = \int_{\partial N} \frac{\partial u}{\partial \nu} dS$$

holds, where both integrals are improper (*the values $\pm\infty$ are allowed*).

The proof is in the Appendix.

Returning to the proof of Theorem 1, assume that Ω_1 has parabolic type. Then the manifold $N = \overline{\Omega}_1 \backslash (\partial\Omega_1 \cap \partial M)$ with boundary also has parabolic type. Applying Proposition 2 to the function u^2, we get

$$\int_N |\nabla u|^2 dV = \int_{\partial N} u \frac{\partial u}{\partial \nu} dS = 0,$$

whence $u \equiv 0$, which contradicts the definition of the function u.

Note that we have actually proved that any massive set has hyperbolic type.

A few words concerning Green's formula. We shall often apply it, and it has already been used once in the following situation. Let G be an open precompact set in M whose boundary consists of several smooth hypersurfaces which intersect transversally. Let the following boundary value problem be solved in G: $\Delta u = 0$ in G, and on ∂G sufficiently smooth Dirichlet or Neumann data is given. We shall write

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, \tag{1}$$

$$\int_G |\nabla u|^2 dV = \int_{\partial G} u \frac{\partial u}{\partial \nu} dS. \tag{2}$$

Meanwhile, to apply the usual Green's formula we need $u \in C^1(\overline{G})$, but we cannot guarantee this, due to breaks in the boundary. The validity of (1) and (2) follows from Proposition 2. Indeed, let N be the manifold with boundary which is obtained if the singularities in the boundary ∂G are removed from $\overline{G}$. Then N has parabolic type, which follows from the fact that the measure of an ε-neighborhood of the singularities of ∂G is $O(\varepsilon^2)$, and from the parabolicity condition [12]. Since the function u is bounded and infinitely smooth up to ∂G, excluding the breaks in the boundary, then, applying Proposition 2 to the functions u and u^2 on the manifold N, we obtain (1) and (2). The existence of the integrals in (1) and (2) in each concrete case is easy to check.

§6. Idea behind the sufficiency proof

In the set Ω from the condition of Theorem 1, let there exist a subset Ω_1 of hyperbolic type, where $\overline{\Omega}_1$ is noncompact and $\operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1) < \infty$. We wish to prove that Ω

is D-massive, i.e. in Ω there exists a function $w \in C^\infty(\overline{\Omega}\backslash(\partial\Omega \cap \partial M))$ such that

$$0 < w < 1 \quad \text{in } \Omega, \tag{1}$$

$$\Delta w = 0, \tag{2}$$

$$w \,|\, _{\partial\Omega} = 1, \tag{3}$$

$$\partial w/\partial \nu \,|\, _{\partial M} = 0, \tag{4}$$

$$D(w) < \infty. \tag{5}$$

Note that in place of the condition $u \,|\, _{\partial\Omega} = 0$ from §1, we have written $w \,|\, _{\partial\Omega} = 1$, i.e. we have passed from the function u to the function $w = 1 - u$.

The function w will be constructed by passages to the limit. First, we construct the capacity potential u for the capacitor $(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1)$ as the limit of the capacity potentials v_k for the compact capacitors $(\partial\Omega, \partial\Omega_1; G_k)$, where $G_k = (\Omega\backslash\overline{\Omega}_1) \cap B_k$, and $\{B_k\}$ is an exhaustion of M such that ∂B_k is transversal to $\partial\Omega$ and $\partial\Omega_1$. For the potentials v_k we have (see §2)

$$D(v_k) = \operatorname{cap}(\partial\Omega, \partial\Omega_1; G_k). \tag{6}$$

By the properties of capacity we have

$$\operatorname{cap}(\partial\Omega, \partial\Omega_1; G_k) \le \operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1). \tag{7}$$

The sequence of harmonic functions $\{v_k\}$ is bounded; hence it has a limit function which is harmonic in $\Omega\backslash\overline{\Omega}_1$ and satisfies the boundary conditions $u \,|\, _{\partial\Omega} = 1$, $u \,|\, _{\partial\Omega_1} = 0$, and $\partial u/\partial\nu \,|\, _{\partial M} = 0$. Moreover, from (6) and (7) it follows that

$$D(u) \le \operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1).$$

Since the function u is admissible for the capacitor $(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1)$, the reverse inequality also holds, so that

$$D(u) = \operatorname{cap}(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1).\text{([9])}$$

After this, we take instead of Ω_1 the smaller set Ω_2, and construct the capacity potential u_2 for the capacitor $(\partial\Omega, \partial\Omega_2; \Omega\backslash\overline{\Omega}_2)$, and so forth. It is easy to see that we can construct an increasing sequence $u_1 = u, u_2, u_3, \dots$ of harmonic functions, where $u_m \,|\, _{\partial\Omega} = 1$, $u_m \,|\, _{\partial\Omega_m} = 0$, and $\partial u_m/\partial\nu \,|\, _{\partial M} = 0$, and $D(u_m) = \operatorname{cap}(\partial\Omega, \partial\Omega_m; \Omega\backslash\overline{\Omega}_m)$. This sequence has a limit, which we denote by w. Clearly, properties (2)–(5) are satisfied. It is only unclear why $w \not\equiv 1$. It can be proved that if $\overline{\Omega}_{m+1}$ differs from $\overline{\Omega}_m$ by a compact set, then the hyperbolicity of Ω_1 guarantees

$$\lim_{m\to\infty} D(u_m) = \lim_{m\to\infty} \operatorname{cap}(\partial\Omega, \partial\Omega_m; \Omega\backslash\overline{\Omega}_m) > 0. \tag{8}$$

But it still does not follow that $D(w) > 0$, since $D(w)$ need not equal the limit of (8). To prove $w \not\equiv 1$ we shall use the flows $\int_{\partial\Omega}(\partial w/\partial\nu)dS$ and $\int_{\partial\Omega}(\partial u_m/\partial_\nu)dS$, where ν is the outward normal to $\partial\Omega$. Since $u_{m+1} \ge u_m$ and $u_{m+1} = u_m = 1$ on $\partial\Omega$, it follows that $\partial u_{m+1}/\partial\nu \le \partial u_m/\partial\nu$, so that on $\partial\Omega$ we have the monotonically decreasing sequence of functions $\{\partial u_m/\partial\nu\}$, converging to $\partial w/\partial\nu$. Below we shall prove that $\int_{\partial\Omega}(\partial u_1/\partial\nu)dS < \infty$, so that by Lebesgue's theorem

$$\int_{\partial\Omega} \frac{\partial w}{\partial\nu} dS = \lim_{m\to\infty} \int_{\partial\Omega} \frac{\partial u_m}{\partial\nu} dS.$$

If we knew that the flow $\int_{\partial\Omega}(\partial u_m/\partial\nu)dS$ were equal to the corresponding Dirichlet integral $D(u_m)$, then it would follow from (8) that $\int_{\partial\Omega}(\partial w/\partial\nu)dS > 0$, and so $w \not\equiv \text{const}$.

([9]) This does not exclude the case of a nonunique capacity potential u.

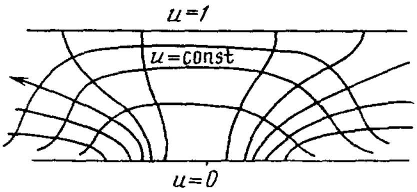

FIGURE 3

But the whole difficulty is that we are considering noncompact capacitors for which the flow is not necessarily equal to the capacity (in contrast to the compact case considered in §2), and may be less. In Figure 3 the gradient curves and level curves for such a capacity potential are shown.

Fortunately, it turns out that a diminished flow in comparison with the Dirichlet integral is a fortuitous circumstance related to an unpropitious arrangement of the hypersurface $\partial\Omega$. It turns out that the flow of the vector field ∇u across almost all level sets $\{u = t\}$ nevertheless equals the capacity. Therefore, instead of $\partial\Omega$, we can take one of these level sets.

§7. The capacity potential of a noncompact capacitor

LEMMA 2. *Let u be the capacity potential for the capacitor $(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1)$ constructed in the previous section. Let $\Gamma_t = \{x \mid u(x) = t\}$, where $0 \leq t \leq 1$. Then for almost all t*

$$\int_{\Gamma_t} \frac{\partial u}{\partial \nu} dS = D(u),$$

where the normal ν is in the direction of increasing u.

PROOF. Denote $p_t(u) = \int_{\{u=t\}} (\partial u/\partial \nu) dS$. By a well-known formula of Federer [14] we have

$$D(u) = \int_{\Omega\backslash\overline{\Omega}_1} |\nabla u|^2 dV = \int_0^1 \left(\int_{\Gamma_t} |\nabla u| dS \right) dt = \int_0^1 p_t(u) dt. \tag{1}$$

If we show that $p_t(u) \leq D(u)$ for almost all t, then this, together with (1), will give the desired result. To prove that $p_t(u) \leq D(u)$, recall that the function u is a limit point of the sequence $\{v_k\}$ of §6. It can be assumed that $v_k \to u$ as $k \to \infty$. For each of the functions v_k we have, by property 4 of capacity in §2 and Green's formula, $D(v_k) = p_t(v_k)$ for almost all t.

Let F be an arbitrary compact set with smooth boundary on the submanifold Γ_t, where t is a regular value of the functions u and v_k. We prove that

$$\int_F \frac{\partial u}{\partial \nu} dS \leq D(u).$$

For that, let us see how the hypersurface $\Gamma_t^k = \{v_k = t\}$ is arranged. Through each point of F, draw the gradient curve of the function u (note that $\partial u/\partial\nu\,|_{\Gamma_t} = |\nabla u|\,|_{\Gamma_t} > 0$) of length 2ε, with length ε on each side of the point. Since $v_k \to u$ together with all the derivatives uniformly on each compact set, then for sufficiently large k, on each gradient curve constructed above, there will be exactly one point at which $v_k = t$. We denote that part of Γ_t^k which intersects the gradient curves by F^k (see Figure 4). Taking a sufficiently small ε (and large k), we obtain

$$\left| \int_F \frac{\partial u}{\partial \nu} dS - \int_{F^k} \frac{\partial u}{\partial \nu} dS \right| < \delta, \tag{2}$$

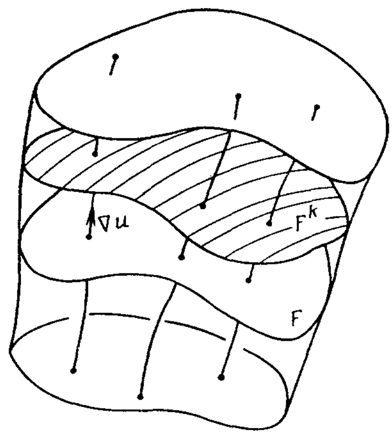

FIGURE 4

where $\delta > 0$ is a number given in advance. Indeed, the above-indicated difference does not exceed the integral of $|\nabla u|$ over the lateral surface of the figure Φ swept out by the gradient lines of length 2ε constructed above, which is clearly equal to $O(\varepsilon)$. Further, for sufficiently large k the derivatives of the functions u and v_k are uniformly close on $\overline{\Phi}$, so that

$$\left|\int_{F^k} \frac{\partial u}{\partial \nu} dS - \int_{F^k} \frac{\partial v_k}{\partial \nu} dS\right| < \delta. \tag{3}$$

Therefore, from (2) and (3) it follows that

$$\int_F \frac{\partial u}{\partial \nu} dS \le \int_{F^k} \frac{\partial v_k}{\partial \nu} dS + 2\delta \le \int_{\Gamma_t^k} \frac{\partial v_k}{\partial \nu} dS + 2\delta = D(v_k) + 2\delta.$$

But, as was shown in §6, $D(v_k) \le D(u)$, so that

$$\int_F \frac{\partial u}{\partial \nu} dS \le D(u) + 2\delta.$$

Letting $\delta \to 0$ and $F \to \Gamma_t$, we obtain, finally, $p_t(u) \le D(u)$.

§8. Conclusion of the proof of Theorem 1

So, in §6 we have constructed the capacity potential u for the capacitor

$$(\partial\Omega, \partial\Omega_1; \Omega\backslash\overline{\Omega}_1),$$

and in §7 we proved that there exists a smooth level hypersurface $\Gamma_t = \{u = t\}$ for which $\int_{\Gamma_t} (\partial u/\partial \nu) dS = D(u)$.

Fix $t > 0$, and put $\Omega_0 = \Omega\backslash\{u \ge t\}$ and $u_1 = u/t\,|_{\Omega_0}$. We shall prove that Ω_0 is D-massive; then from the results of §1 it will follow that Ω is also D-massive. The desired function w satisfying conditions (1)–(5) of §6 in Ω_0 will be the limit of the sequence $\{u_m\}$, which we now construct.

Let $\{B_k\}$, as usual, be an exhaustion sequence, where the ∂B_k are transversal to $\partial\Omega_0$ and $\partial\Omega_1$. Denote $G_1 = \Omega_0\backslash\overline{\Omega}_1$, $G_m = \Omega_0\backslash(\overline{\Omega}_1\backslash B_m)$, $m \ge 2$, and $G_m^k = G_m \cap B_k$, where $k > m$. In the region G_m^k, $m \ge 2$, we solve the following boundary value problem:

$$\Delta v_m^k = 0, \qquad v_m^k\,|_{\partial\Omega_0} = 1, \qquad v_m^k\,|_{\partial B_k} = u_1\,|_{\partial B_k}, \qquad v_m^k\,|_{\partial\Omega_1\backslash B_m} = 0,$$

$$v_m^k\,|_{\partial B_m \cap \Omega_1} = 0, \qquad \partial v_m^k/\partial\nu\Big|_{\partial M} = 0$$

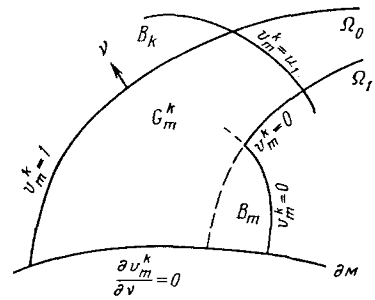

FIGURE 5

(see Figure 5). From the maximum principle it follows that

$$v_{m+1}^k \geq v_m^k \geq u_1. \tag{1}$$

The sequence v_m^k is bounded, and therefore, as $k \to \infty$, it has the limit function $u_m \geq u_1$. Using the diagonal process, choose a subsequence of k which serves for all m. Then, by (1), $u_{m+1} \geq u_m \geq u_1$. Put $w = \lim_{m\to\infty} u_m$. This will be the desired function if we prove that $D(w) < \infty, w \not\equiv 1$.

To prove $D(w) < \infty$ it suffices to prove that $D(v_m^k) \leq D(u_1)$, and the rest follows by passing to the limit. By Green's formula we have

$$\begin{aligned} D(v_m^k) &= \int_{G_m^k} |\nabla v_m^k|^2 dV = \int_{\partial\Omega_0 \cap \overline{G}_m^k} v_m^k \frac{\partial v_m^k}{\partial \nu} dS + \int_{\partial B_k \cap \overline{G}_m^k} v_m^k \frac{\partial v_m^k}{\partial \nu} dS \\ &= \int_{\partial\Omega_0 \cap \overline{G}_m^k} \frac{\partial v_m^k}{\partial \nu} dS + \int_{\partial B_k \cap \overline{G}_m^k} u_1 \frac{\partial v_m^k}{\partial \nu} dS, \end{aligned}$$

where ν is the outward normal to ∂G_m^k. Further, we use that $v_m^k \geq u_1$, and on $\partial\Omega_0$ and ∂B_k we have $v_m^k = u_1$. Consequently, on $\partial\Omega_1$ and ∂B_k the inequality $\partial v_m^k/\partial\nu \leq \partial u_1/\partial\nu$ is satisfied. Therefore,

$$\begin{aligned} D(v_m^k) &\leq \int_{\partial\Omega_0 \cap \overline{G}_m^k} \frac{\partial u_1}{\partial \nu} dS + \int_{\partial B_k \cap \overline{G}_m^k} u_1 \frac{\partial u_1}{\partial \nu} dS \\ &= \int_{\partial\Omega_0 \cap \overline{G}_1^k} \frac{\partial u_1}{\partial \nu} dS + \int_{\partial B_k \cap \overline{G}_1^k} u_1 \frac{\partial u_1}{\partial \nu} dS \\ &= \int_{G_1^k} |\nabla u_1|^2 dS \leq D(u_1). \end{aligned}$$

We now prove that $\int_{\partial\Omega_0} (\partial w/\partial\nu) dS > 0$, from which it follows that $w \not\equiv 1$ (here ν is the outward normal to $\partial\Omega_0$). From the fact that $u_{m+1} \geq u_m$ and $u_m \mid_{\partial\Omega_0} = u_{m+1} \mid_{\partial\Omega_0} = 1$, it follows that $0 \leq \partial u_{m+1}/\partial\nu \leq \partial u_m/\partial\nu$. Since

$$\int_{\partial\Omega_0} (\partial u_1/\partial\nu) dS = (1/t) \int_{\Gamma_t} (\partial u/\partial\nu) dS < \infty,$$

by Lebesgue's Theorem we may assert that

$$\int_{\partial\Omega_0} \frac{\partial w}{\partial \nu} dS = \lim_{m\to\infty} \int_{\partial\Omega_0} \frac{\partial u_m}{\partial \nu} dS.$$

Next, we shall prove the following two facts.

1. $\int_{\partial\Omega_0} (\partial u_m/\partial\nu) dS \geq D(u_m) \geq \operatorname{cap}(\partial\Omega_0, \partial\Omega_m; G_m)$, where $m \geq 1$ and $\Omega_m = \Omega_0 \backslash \overline{G}_m$.

2. $\operatorname{cap}(\partial\Omega_0, \partial\Omega_m; G_m) \geq \text{const} > 0$, where const is independent of m.

Proof of 1. The second inequality follows from the definition of capacity. To prove the first, we use the fact that for $m = 1$

$$\int_{\partial\Omega_0} \frac{\partial u_1}{\partial \nu} dS = D(u_1). \tag{2}$$

Indeed, by the choice of $\partial\Omega_0$, for almost all $\tau \in [0,1]$ we have $p_\tau(u_1) = p_{\tau t}(u) = p_t(u) = p_1(u_1)$ and $p_\tau(u_1) = \int_{\partial\Omega_0}(\partial u_1/\partial\nu)dS$, so that (2) follows from Federer's formula (see §7).

Now let $m \geq 2$. Analogously to the way in which we proved the inequality $D(v_m^k) \leq D(u_1)$, we have

$$D(v_m^k) \leq \int_{\partial\Omega_0 \cap \overline{G}_m^k} \frac{\partial v_m^k}{\partial \nu} dS + \int_{\partial B_k \cap \overline{G}_m^k} u_1 \frac{\partial u_1}{\partial \nu} dS.$$

But

$$\int_{\partial B_k \cap \overline{G}_m^k} u_1 \frac{\partial u_1}{\partial \nu} dS = \int_{G_1^k} |\nabla u_1|^2 dV - \int_{\partial\Omega_0 \cap \overline{G}_1^k} \frac{\partial u_1}{\partial \nu} dS,$$

whence

$$\int_{\partial\Omega_0 \cap \overline{G}_m^k} \frac{\partial v_m^k}{\partial \nu} dS - D(v_m^k) \geq \int_{\partial\Omega_0 \cap \overline{G}_1^k} \frac{\partial u_1}{\partial \nu} dS - D(u_1). \tag{3}$$

As $k \to \infty$ the right-hand side of (3) vanishes, by (2). Further, by a well-known property of the Lebesgue integral,

$$\varliminf_{k \to \infty} D(v_m^k) \geq D(u_m). \tag{4}$$

Finally, we prove that

$$\int_{\partial\Omega_0 \cap \overline{G}_m^k} \frac{\partial v_m^k}{\partial \nu} dS \xrightarrow[k \to \infty]{} \int_{\partial\Omega_0} \frac{\partial u_m}{\partial \nu} dS. \tag{5}$$

Indeed, let us extend the function $\partial v_m^k/\partial\nu$ by zero outside $\overline{G}_m^k$ to the whole boundary $\partial\Omega_0$. Then

$$0 \leq \partial v_m^k/\partial\nu|_{\partial\Omega_0} \leq \partial u_1/\partial\nu,$$

and since $\int_{\partial\Omega_0}(\partial u_1/\partial\nu)dS < \infty$ we can apply Lebesgue's theorem and, from the pointwise convergence $\partial v_m^k/\partial\nu \to \partial u_m/\partial\nu$ as $k \to \infty$, obtain the convergence of the integrals. Thus, as $k \to \infty$, from (3)–(5), we obtain the desired inequality.

Proof of 2. Here we use, for the first time, the hyperbolic type of Ω_1. By the connectedness of the manifold M, the set Ω_1 can be extended to an open set Ω_1' with smooth boundary such that $\overline{\Omega}_1' \setminus \Omega_1$ is compact and $\Omega_1' \setminus \overline{\Omega}_0$ is nonempty. Then Ω_1' also has hyperbolic type. Let $F = \overline{\Omega}_1' \setminus \Omega_0$. Since the compact set F has nonempty interior, $\operatorname{cap}(F, \infty; \Omega_1') > 0$ (see §2). But, by properties of capacity,

$$\begin{aligned} \operatorname{cap}(\partial\Omega_0, \partial\Omega_m; G_m) &= \operatorname{cap}(M \setminus \Omega_0, \overline{\Omega}_m; M) \\ &\geq \operatorname{cap}(F, \overline{\Omega}_1' \setminus B_m; \Omega_1') \geq \operatorname{cap}(F, \infty; \Omega_1'), \end{aligned}$$

so that $\operatorname{cap}(F, \infty; \Omega_1')$ is the desired positive constant independent of m.

Thus, in accordance with facts 1 and 2, we can state that

$$\int_{\partial\Omega_0} \frac{\partial w}{\partial \nu} dS \geq \operatorname{cap}(F, \infty; \Omega_1') > 0,$$

so that $w \not\equiv 1$. Theorem 1 is proved.

§9. Proof of Theorem 2

If the D-Liouville theorem does not hold on the manifold M, then there exists a nontrivial, bounded, harmonic function u with $D(u) < \infty$ (see the Introduction). Let a and b be two regular values of u such that $a < b$ and the sets $\{u < a\}$ and $\{u > b\}$ are nonempty. Then we put $\Omega_1 = \{u < a\}$ and $\Omega_2 = \{u > b\}$. By the maximum principle, $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are noncompact. Each of these sets is massive; therefore, by §5, they both have hyperbolic type. Finally,

$$\operatorname{cap}(\overline{\Omega}_1, \overline{\Omega}_2; M) = \operatorname{cap}(\partial\Omega_1, \partial\Omega_2; M \backslash (\overline{\Omega_1 \cup \Omega_2})) < \infty,$$

since the function $(b-u)/(b-a)$ is admissible and has finite Dirichlet integral.

Now let Ω_1 and Ω_2 be open sets satisfying the condition of Theorem 2. Construct the capacity potential u for the capacitor

$$(\partial\Omega_1, \partial\Omega_2; M \backslash (\overline{\Omega_1 \cup \Omega_2}))$$

(see §6), and let $a \in (0,1)$ be a regular value of it. Then each of the sets $\Omega_2 \cup \{u < a\}$ and $\Omega_1 \cup \{u > a\}$ is D-massive, by Theorem 1. Consequently, by Proposition 1, the D-Liouville theorem holds.

Appendix. On Green's formula

Here, we shall prove Proposition 2 from §5. In the proof the following variant of the mean value theorem [16] will be used.

PROPOSITION 3. *Let B_1 and B_2 be precompact open sets with smooth boundaries in the Riemannian manifold (with boundary) N. Let $B_1 \subset B_2$, $U \in C^\infty(\overline{B}_2 \backslash B_1)$, and* $\operatorname{osc} U < K$. *Then there exists a smooth hypersurface Γ separating ∂B_1 and ∂B_2 such that*

$$\int_\Gamma \frac{\partial U}{\partial \nu} dS \leq K \operatorname{cap}(\partial B_1, \partial B_2; G)$$

(the normal ν is directed towards ∂B_2).

PROOF OF PROPOSITION 2. We find an exhaustion of N by open precompact sets G_k with smooth boundaries, such that

$$\lim_{k\to\infty} \int_{G_k} \Delta u \, dV = \lim_{k\to\infty} \int_{\partial N \cap G_k} \frac{\partial u}{\partial \nu} dS, \tag{1}$$

whence follows the desired result. First, let us take any exhaustion $\{B_k\}$ having the properties indicated above, except (1). We apply Proposition 3 to the function u in $B_m \backslash \overline{B}_k$, $m > k$, and find a smooth hypersurface Γ separating ∂B_k and ∂B_m such that

$$\int_\Gamma \frac{\partial u}{\partial \nu} dS \leq K \operatorname{cap}(\partial B_k, \partial B_m; B_m \backslash \overline{B}_k),$$

where $K > \operatorname{osc} u$. For sufficiently large m, the capacity on the right-hand side tends to zero, since N is of parabolic type. For each k, fix m sufficiently large, and choose as G_k an open set containing B_k and having boundary Γ. Applying the usual Green's formula in G_k, we have

$$\int_{G_k} \Delta u \, dV = \int_{\partial N \cap G_k} \frac{\partial u}{\partial \nu} dS + \int_{\partial G_k} \frac{\partial u}{\partial \nu} dS.$$

Passing to the limit as $k \to \infty$, and taking into account that

$$\varlimsup_{k\to\infty} \int_{\partial G_k} \frac{\partial u}{\partial \nu} dS \leq 0,$$

we get

$$\int_N \Delta u\, dV \le \int_{\partial N} \frac{\partial u}{\partial \nu} dS.$$

Applying this inequality to the function $-u$, we obtain the desired result.

Let us now prove Proposition 3. Put $G = B_2 \backslash \overline{B}_1$. It can be assumed that $\inf_G U = 0$, so that $\sup_G U = \operatorname{osc}_G U < K$. Let v be the solution in G of the boundary value problem $\Delta v = 0$, $v\,|_{\partial B_1} = 0$, $v\,|_{\partial B_2} = 1$, $\partial v/\partial \nu\,|_{\partial N} = 0$. Put $w = v - K^{-1}U$. Then $w\,|_{\partial B_2} > 0$ and $w\,|_{\partial B_1} \le 0$. Therefore, for some regular value $\varepsilon > 0$ of the function w, the hypersurface $\Gamma = \{w = \varepsilon\}$ separates ∂B_1 and ∂B_2, where

$$\int_\Gamma \frac{\partial w}{\partial \nu} dS \ge 0, \qquad \int_\Gamma \frac{\partial v}{\partial \nu} dS \ge \frac{1}{K} \int_\Gamma \frac{\partial U}{\partial \nu} dS.$$

We prove that

$$\int_\Gamma \frac{\partial v}{\partial \nu} dS = \operatorname{cap}(\partial B_1, \partial B_2; G).$$

This, of course, follows from Green's formula of Proposition 2, but in proving the latter we used Proposition 3. An alternative approach is as follows. For each regular value t of v, put

$$p_t = \int_{\{v=t\}} |\nabla v| dS = \int_{\{v=t\}} \frac{\partial v}{\partial \nu} dS.$$

By Federer's formula [14] we have

$$\operatorname{cap}(\partial B_1, \partial B_2; G) = \int_G |\nabla v|^2 dV = \int_0^1 p_t\, dt.$$

On the other hand, for $0 < t < 1$ the function p_t is certainly independent of t, since for $t_2 > t_1$

$$\int_{\{v=t_2\}} \frac{\partial v}{\partial \nu} dS - \int_{\{v=t_1\}} \frac{\partial v}{\partial \nu} dS = \int_{\{t_2 > v > t_1\}} \Delta v\, dV = 0.$$

Therefore, $p_t = \operatorname{cap}(\partial B_1, \partial B_2; G)$. Finally, if t is a regular value of v such that $0 < t < \inf_\Gamma v$, then once again by the usual Green's formula

$$\int_\Gamma \frac{\partial v}{\partial \nu} dS = \int_{\{v=t\}} \frac{\partial v}{\partial \nu} dS = \operatorname{cap}(\partial B_1, \partial B_2; G).$$

REMARK. If l is the distance between ∂B_1 and ∂B_2, then the following capacity estimate is obvious:

$$\operatorname{cap}(\partial B_1, \partial B_2; G) \le \operatorname{meas} G / l^2$$

(it is obtained by taking the distance to ∂B_2 with factor l^{-1} as the admissible function in the definition of capacity). Therefore, under the conditions of Proposition 3,

$$\int_\Gamma \frac{\partial U}{\partial \nu} dS \le \frac{K \operatorname{meas} G}{l^2}. \tag{2}$$

In [16] an analogous estimate is proved for $\int_\Gamma |\partial U/\partial \nu| dS$. Its proof is much more complicated than the proof of Proposition 3. In the majority of applications, the theorem of [16] is used in the form (2).

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