Lectures on path homology theory of digraphs

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Preface

This text is based on a series of lectures that I delivered at an online joint seminar of Tsinghua University and Bielefeld University in Spring 2022. The purpose of those lectures was to introduce to young researchers a new emerging area of research – the theory of path homology on digraphs and related topics.

There exists a number of ways to define the notion of homology for graphs and digraphs, for example, clique homology ([12], [44]) or singular homology ([5], [44], [49]). However, the path homology has certain advantages as it enjoys adequate functorial properties with respect to graph-theoretical operations, such as morphisms of digraphs, Cartesian products, joins, homotopy etc. The notion of path homology has a rich mathematical content, and I hope that it will become a useful tool in various areas of pure and applied mathematics.

I have tried to keep here the presentation style of the online seminar, which, in particular, featured a wealth of examples and open problems. I give here an overview of the already published results in this field, state and prove some new results, as well as pose some open questions and conjectures.

The material on the following topics is new:
- random digraphs;
- star product and Künneth formula in cohomology;
- intersection form and signature;
while the rest of the material is based on [24], [26], [27], [28], [32], [34], [35], [36], [37], [38].

A complete list of the topics covered is shown in the table of contents.

For further reading I recommend [1], [3], [4], [6], [8], [9], [10], [11], [13], [14], [15], [16], [19], [22], [23], [25], [29], [30], [31], [33], [39], [41], [42], [43], [48], [50].

Acknowledgements. The author is grateful to Chao Chen for his C++ program for computation of path homology groups that was extensively used in this research. Scientific Workplace© of MacKichan Software and Microsoft Excel© were used for other computational purposes.

The author is indebted to S.-T. Yau for initiating and leading the research on this subject as well as for his constant support and encouragement.
The author acknowledges a continued financial support of Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 317210226 - SFB 1283, as well as the hospitality and support of Tsinghua University (Beijing) and Chinese University of Hong Kong during multiple visits there.

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Bielefeld, December 2022