Path complexes and their homologies

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Abstract

We introduce the notions of a path complex and its homologies. Particular cases of path homologies are simplicial homologies and digraph homologies. We state and prove some properties of path homologies, in particular, the Künneth formulas for Cartesian product and join that happen to be true at the level of chain complexes.

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1 Introduction

The subject of this paper is the notion of a *path complex* that unifies and generalizes the notions of a simplicial complex and a digraph (=directed graph). In short, a path complex P on a finite set V is a collection of paths (=sequences of points) on V such that if a path v belongs to P then a truncated path, that is obtained from v by removing either the first or the last point, is also in P. Given a path complex P, all the paths in P are called *allowed*.

Any simplicial complex S determines naturally a path complex by associating with any simplex from S the sequence of its vertices (see Section 3 for details).

However, the main motivation for considering path complexes comes from digraphs. A digraph G is a pair (V, E) where V is any set and E is a binary relation on V, that is, E is a subset of $V \times V$. If $(a, b) \in E$ then the pair (a, b) is called a directed edge or arrow; this fact is also denoted by $a \to b$. Any digraph naturally gives rise to a path complex where allowed paths are those that go along arrows of the digraph.

One of our key observations is that any path complex P allows to define a chain complex with an appropriate boundary operator that leads to the notion of homology groups of P. We refer to this notion as a *path homology*.

In the case when P arises from a simplicial complex S, the path homology of P coincides with the simplicial homology of S. If P arises from a digraph G then we obtain a new notion: the path homology of a digraph. Although most of the results are presented in this paper for arbitrary path complexes, we always have in mind applications for digraphs. On the other hand, the notion of a path complex provides an alternative viewpoint for the classical results about simplicial complexes.

There has been a number of attempts to define the notion of (co)homology for graphs. At a trivial level, any graph can be regarded as an one-dimensional simplicial complex, so that its simplicial homologies are defined. However, all homology groups in dimension 2 and higher are trivial, which makes this approach uninteresting.

Another way to make a graph into a simplicial complex is to consider all its cliques (=complete subgraphs) as simplexes of the corresponding dimensions (cf. [4], [15]). Then higher dimensional homologies may be non-trivial, but in this approach the notion of graph loses its identity and becomes a particular case of the notion of a simplicial complex. Besides, some desirable functorial properties of such homologies fail, for example, the Künneth formula is not true for Cartesian product of graphs (for example, the Cartesian product of two 4-cycles has trivial H_2 whereas H_1 of 4-cycle is non-trivial).

Yet another approach to homologies of digraphs can be realized via Hochschild homology. Indeed, allowed paths on a digraph have a natural operation of product, which allows to define the notion of a *path algebra* of a digraph. The Hochschild homology of the path algebra is a natural object to consider. However, it was shown in [14] that Hochschild homologies of order ≥ 2 are trivial, which makes this approach not so attractive.

In singular homology theories of graphs one uses predefined "small" graphs as basic cells and defines singular chains as formal sums of the maps of the basic cell into the graph (see, for example, [15], [18]). However, simple examples show that the homology groups obtained in this way, depend essentially on the choice of the basic cells. Besides, such homologies are extremely difficult to compute, even for small graphs, and the functorial properties are not known at all.

The path homologies of digraphs that we present in this paper have the following advantages in comparison with the previously studied notions of graph homologies.

1. The path homologies of all dimensions can be non-trivial; even for planar graphs the path homologies can be non-trivial in dimension 2. Also, the chain complex associated with a path complex has a richer structure than simplicial chain complexes. It contains not only cliques but also binary hypercubes and other interesting subgraphs some of them are reminiscent of polyhedra.

2. The path homologies are easy to compute. For small digraphs their path homologies can be computed by hands, either by definition or by using simple properties. For larger digraphs it can be done using any software package containing operations with matrices, in particular, computation of the rank of a matrix.

3. The path homology theory is compatible with the homotopy theory of digraphs. The latter was introduced by the authors in [9] (a homotopy theory for undirected graphs was developed earlier in [1], [2]), where they proved that the path homologies of digraphs are invariant with respect to homotopy and that the abelization of the fundamental group is isomorphic to the one-dimensional homology group.

4. Path homologies have good functorial properties with respect to graph-theoretical operations, for example, morphisms of digraphs induce homomorphisms of path homologies. Also, the homologies of the Cartesian product of digraphs (as well as of the join) satisfy the Künneth formula (Theorems 5.5 and 6.6 of the present paper).

5. The path homology theory is dual to the *cohomology* theory of digraphs that was introduced by Dimakis and Müller-Hoissen [5], [6] and was further developed in [12]. The latter theory is based on a classification of [3] of exterior derivations on algebras, and the coboundary operator arises naturally as an exterior derivative on the algebra of functions on the vertex set of the digraph. However, in the present paper we do not discuss cohomologies.

We feel that the notion of path homologies (and the dual notion of cohomologies) has a rich mathematical content and hope that it will become a useful tool in various areas of pure and applied mathematics. For example, this notion was employed in [11] to give a new elementary proof of a theorem of Gerstenhaber and Schack [7] that gives a representation of simplicial homology as a Hochschild homology. A link between path homologies of digraphs and cubical homologies was revealed in [10]. Homology and homotopy of digraphs may become use in some graph coloring problems – a simple example of this type has appeared in [9]. On the other hand, it is conceivable that the notion of path homologies of digraphs can be used in practical applications such as coverage verification in sensor networks (cf. [17]), and many others.

Let us briefly describe the structure of the paper and the main results. In Section 2 we introduce the notion of a boundary operator on paths on a finite set V. In Section 3 we define the notions of a path complex, a ∂-invariant path (an element of a chain complex), and the path homologies.

In Section 4 we give some examples of digraphs and ∂-invariant paths there. We state some basic results about path homologies of digraphs, which allow to compute homology groups of simple digraphs (the proofs can be found in [8]).

In Section 5 we introduce the operation *join* of two path complexes and prove the Künneth formula for it (Theorem 5.5). Particular cases of join are operation of taking a cone and suspension of a digraph that behave homologically in the same way as those in the classical algebraic topology.

In Section 6 we introduce the notions of cross product of paths and Cartesian product of path complexes. The latter matches the notion of the Cartesian product of digraphs. We state and prove the Künneth formula for Cartesian product (Theorem 6.6) and give some examples.

Most difficult and interesting results of this paper are Theorems 5.5 and 6.6. In the setting digraphs these theorem were proved in [13], while in the present paper we prove them in a more general setting of path complexes.

2 Paths on a finite set

Let V be an arbitrary non-empty finite set whose elements will be called vertices. For any non-negative integer p, an *elementary p-path* on a set V is any sequence $\{i_k\}_{k=0}^{p}$ of $p+1$ vertices of V (a priori the vertices in the path do not have to be distinct). For $p=-1$, an elementary p-path is an empty set $\emptyset$. The p-path $\{i_k\}_{k=0}^{p}$ will also be denoted simply by $i_0...i_p$, without delimiters between the vertices.

2.1 Boundary operator

Fix a field $\mathbb{K}$ and consider a $\mathbb{K}$-linear space $\Lambda_p = \Lambda_p(V)$ that consists of all formal linear combinations of all elementary p-paths with the coefficients from $\mathbb{K}$. The elements of Λ_p are called *p-paths* on V. An elementary p-path $i_0...i_p$ as an element of Λ_p will be denoted by $e_{i_0...i_p}$. The empty set as an element of Λ_{-1} will be denoted by e.

By definition, the family $\left\{e_{i_0...i_p} : i_0,...,i_p \in V\right\}$ is a basis in Λ_p, and any p-path $v \in \Lambda_p$ has a unique representation in the form

$$v = \sum_{i_0,\ldots,i_p \in V} v^{i_0\ldots i_p} e_{i_0\ldots i_p}, \tag{2.1}$$

where $v^{i_0\cdots i_p} \in \mathbb{K}$. For example, Λ_0 consists of all linear combinations of e_i where $i \in V$, Λ_1 consists of all linear combinations of e_{ij} where $i,j \in V$, etc. Note that, Λ_{-1} consists of all multiples of e so that $\Lambda_{-1} \cong \mathbb{K}$.

For any $p \geq 0$, define the *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ as a linear operator that acts on elementary paths by

$$\partial e_{i_0\ldots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0\ldots \widehat{i_q} \ldots i_p}, \tag{2.2}$$

where the hat $\widehat{i_q}$ means omission of the index i_q. For example, we have

$$\partial e_i = e, \quad \partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}. \tag{2.3}$$

It follows that, for any $v \in \Lambda_p$,

$$(\partial v)^{j_0 \dots j_{p-1}} = \sum_{k \in V} \sum_{q=0}^{p} (-1)^q \, v^{j_0 \dots j_{q-1} k \, j_q \dots j_{p-1}}. \tag{2.4}$$

For example, for any $u \in \Lambda_0$ and $v \in \Lambda_1$ we have

$$\partial u = \sum_{k \in V} u^k \quad \text{and} \quad (\partial v)^i = \sum_{k \in V} \left(v^{ki} - v^{ik} \right).$$

Set also $\Lambda_{-2} = \{0\}$ and define $\partial : \Lambda_{-1} \to \Lambda_{-2}$ to be zero.

Lemma 2.1 *We have $\partial^2 = 0$. Hence, $\Lambda_* = \{\Lambda_p\}$ is a chain complex.*

Proof. The operator ∂^2 acts from Λ_p to Λ_{p-2}, so that the identity $\partial^2 = 0$ makes sense for all $p \geq 0$. In the case $p = 0$ the identity $\partial^2 = 0$ is trivial. For $p \geq 1$, we have by (2.2)

$$\begin{aligned}
\partial^2 e_{i_0 \dots i_p} &= \sum_{q=0}^{p} (-1)^q \, \partial e_{i_0 \dots \widehat{i_q} \dots i_p} \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p} \right) \\
&= \sum_{0 \leq r < q \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \dots i_p} = 0$. This implies $\partial^2 v = 0$ for all $v \in \Lambda_p$. ■

2.2 Join of paths

For all $p, q \geq -1$ and for any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define their *join* $uv \in \Lambda_{p+q+1}$ as follows:

$$(uv)^{i_0 \dots i_p j_0 \dots j_q} = u^{i_0 \dots i_p} v^{j_0 \dots j_q}. \tag{2.5}$$

Clearly, join of paths is a bilinear operation that satisfies the associative law (but is not commutative). It follows from (2.5) that

$$e_{i_0 \dots i_p} e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q}. \tag{2.6}$$

If $p = -2$ and $q \geq -1$ then set $uv = 0 \in \Lambda_{q-1}$. A similar rule applies if $q = -2$ and $p \geq -1$.

Lemma 2.2 (Product rule) *For all $p, q \geq -1$ and $u \in \Lambda_p$, $v \in \Lambda_q$ we have*

$$\partial (uv) = (\partial u) v + (-1)^{p+1} \, u \partial v. \tag{2.7}$$

Proof. It suffices to prove (2.7) for $u = e_{i_0 \dots i_p}$ and $v = e_{j_0 \dots j_q}$. We have

$$\begin{aligned}
\partial (uv) &= \partial e_{i_0 \dots i_p j_0 \dots j_q} = e_{i_1 \dots i_p j_0 \dots j_q} - e_{i_0 i_2 \dots i_p j_0 \dots j_q} + \dots \\
&\quad + (-1)^{p+1} \left(e_{i_0 \dots i_p j_1 \dots j_q} - e_{i_0 \dots i_p j_0 j_2 \dots j_q} + \dots \right) \\
&= \left(\partial e_{i_0 \dots i_p} \right) e_{j_0 \dots j_q} + (-1)^{p+1} \, e_{i_0 \dots i_p} \partial e_{j_0 \dots j_q},
\end{aligned}$$

whence (2.7) follows. ■

2.3 Regular paths

We say that an elementary path $i_0...i_p$ is *regular* if $i_{k-1} \neq i_k$ for all $k = 1, ..., p$, and *non-regular* otherwise. For example, a 2-path iij is non-regular, while a 2-path iji is regular provided $i \neq j$.

For any $p \geq -1$, consider the following subspace of Λ_p spanned by the regular elementary paths:

$$\mathcal{R}_p = \mathcal{R}_p(V) := \operatorname{span}\left\{e_{i_0...i_p} : i_0...i_p \text{ is regular}\right\}.$$

The elements of $\mathcal{R}_p$ are called *regular* p-paths.

We would like to consider the operator ∂ on the spaces $\mathcal{R}_p$. However, ∂ is not invariant on the family $\{\mathcal{R}_p\}$. For example, $e_{iji} \in \mathcal{R}_2$ for $i \neq j$ while

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} \notin \mathcal{R}_1$$

as it has a non-regular term e_{ii}. The same applies to the notion of join of paths: the join of two regular paths does not have to be regular, for example, $e_i e_i = e_{ii}$.

However, it is easy to show that ∂ is invariant on the complementary spaces $\mathcal{N}_p$ spanned by non-regular p-paths, which allows us to extend ∂ to the quotient spaces $\Lambda_p/\mathcal{N}_p$. Then we pull this ∂ back to $\mathcal{R}_p$ using $\mathcal{R}_p \cong \Lambda_p/\mathcal{N}_p$. The operator $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}$ defined in this way is called the *regular boundary operator*. The formula (2.2) remains true for the regular ∂ except that in this case all non-regular terms on the right hand side should be treated as zero. For example, we have for the regular operator ∂

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij} \in \mathcal{R}_1$$

provided $i \neq j$.

Similarly one defines the *regular join*, using the fact that the join of an element of $\mathcal{N}_p$ with any element of Λ_q is in $\mathcal{N}_{p+q+1}$ (see [8] for details). This allows us to define join on the quotients $\Lambda_p/\mathcal{N}_p$ and then pull back to $\mathcal{R}_p$. The formula (2.6) remains true for regular join provided we treat a non-regular path in the right hand side as zero. For example, for the regular join we have $e_{ij}e_{ji} = e_{ijji} = 0$.

It follows from the above constructions that the regular versions of ∂ and join also satisfy $\partial^2 = 0$ and the product rule (2.7), for all $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$. In particular, $\mathcal{R}_* = \{\mathcal{R}_p\}$ is a chain complex.

Let V, V' be two finite set. Any map $f : V \to V'$ induces the map

$$f_* : \Lambda_p(V) \to \Lambda_p(V')$$

by the rule

$$f_*\left(e_{i_0...i_p}\right) = e_{f(i_0)...f(i_p)}.$$

The map f_* evidently commutes with ∂ and, hence, is a morphism $\Lambda_*(V) \to \Lambda_*(V')$ of chain complexes. Since f_* maps non-regular paths to non-regular, it induces a morphism $\mathcal{R}_*(V) \to \mathcal{R}_*(V')$ of chain complexes.

3 Path complexes

3.1 The notion of path complex

Definition 3.1 A *path complex* over a set V is a non-empty collection P of elementary paths on V with the following property:

$$\text{if } \ i_0...i_n \in P \text{ then } i_0...i_{n-1} \in P \text{ and } i_1...i_n \in P. \tag{3.1}$$

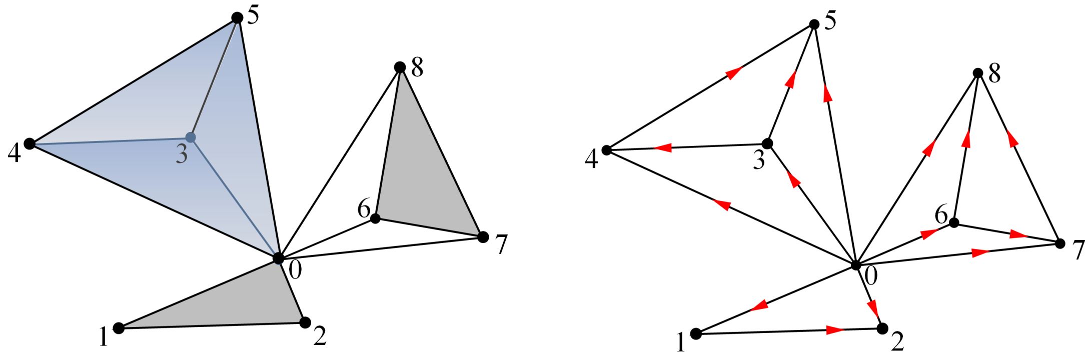


Figure 1: A simplicial complex (left) and a digraph (right)

When a path complex P is fixed, all the paths from P are called *allowed*, whereas the elementary paths that are not in P are called *non-allowed*. Condition (3.1) means that if we remove the first or the last element of an allowed n-path then the resulting $(n-1)$-path is also allowed.

The set of all n-paths from P is denoted by P_n. The set P_{-1} consists of a single empty path e. The elements of P_0 (that is, allowed 0-paths) are called the *vertices* of P. Clearly, P_0 is a subset of V. By the property (3.1), if $i_0...i_n \in P$ then all i_k are vertices of P. Hence, we can (and will) remove from the set V all non-vertices so that $V = P_0$.

Example 3.2 By definition, an abstract finite simplicial complex S is a collection of subsets of a finite vertex set V that satisfies the following property:

$$\text{if } \sigma \in S \text{ then any subset of } \sigma \text{ is also in } S.$$

Let us enumerate the elements of V by distinct reals and identify any subset s of V with the elementary path that consists of the elements of s put in the (strictly) increasing order. Denote by $P(S)$ this collections of elementary paths on V that uniquely determines S. The defining property of a simplex can be restated the following:

$$\text{if } \ v \in P(S) \ \text{ then any subsequence of } \ v \text{ is also in } \ P(S). \tag{3.2}$$

Consequently, the family $P(S)$ satisfies the property (3.1) so that $P(S)$ is a path complex. The allowed n-paths in $P(S)$ are exactly the n-simplexes.

For example, the simplicial complex on Fig. 1(left) has the following path complex:
$P_0 = \{0, 1, ..., 8\}$,
$P_1 = \{01, 02, 03, 04, 05, 06, 07, 08, 12, 34, 35, 45, 67, 68, 78\}$,
$P_2 = \{012, 034, 035, 045, 345, 678\}$,
$P_3 = \{0345\}$.

Example 3.3 Let $G = (V, E)$ be a finite digraph, where V is a finite set of vertices and E is the set of directed edges, that is, $E \subset V \times V$. The fact that $(i, j) \in E$ will also be denoted by $i \to j$.

An elementary n-path $i_0...i_n$ on V is called allowed if $i_{k-1} \to i_k$ for any $k = 1, ..., n$. Denote by $P_n = P_n(G)$ the set of all allowed n-paths. In particular, we have $P_0 = V$ and $P_1 = E$. Clearly, the collection $P = \bigcup_n P_n$ of all allowed paths satisfies the condition (3.1) so that P is a path complex. This path complex is naturally associated with the digraph G and will be denoted by $P(G)$.

For example, a digraph on Fig. 1(right) has the following path complex:

$P_0 = \{0, 1, ..., 8\}$,
$P_1 = \{01, 02, 03, 04, 05, 06, 07, 08, 12, 34, 35, 45, 67, 68, 78\}$,
$P_2 = \{012, 034, 035, 045, 067, 068, 678\}$,
$P_3 = \{0345, 0678\}$.

It is easy to see that a path complex arises from a digraph if and only if it satisfies the following additional condition: if in a path $i_0...i_n$ all pairs $i_{k-1}i_k$ are allowed then the whole path $i_0...i_n$ is allowed.

It is easy to show that a path complex P arises from a simplicial complex if and only if it satisfies the following two properties.

1. Any subsequence of any path from P is also in P (we say in this case that the path complex P is *perfect*).

2. There is an injective real-valued function on the vertex set of P that is strictly monotone increasing along any path from P.

3.2 Homologies of path complex

Given an arbitrary path complex $P = \{P_n\}_{n=0}^{\infty}$ over a finite set V, consider for any integer $n \geq -1$ the $\mathbb{K}$-linear space $\mathcal{A}_n$ that is spanned by all the elementary n-paths from P, that is

$$\mathcal{A}_n = \mathcal{A}_n(P) = \operatorname{span}\{e_{i_0...i_n} : i_0...i_n \in P_n\}.$$

The elements of $\mathcal{A}_n$ are called *allowed* n-paths. By construction, $\mathcal{A}_n$ is a subspace of Λ_n. For example, $\mathcal{A}_p = \Lambda_p$ for $p \leq 0$, while $\mathcal{A}_1$ is spanned by all edges of P and can be smaller than Λ_1.

We would like to restrict the operator ∂ defined on spaces Λ_n to the subspaces $\mathcal{A}_n$. For some path complexes it can happen that $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$, so that the restriction is straightforward. If it is not the case then an additional construction is needed as will be explained below. The inclusion $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$ takes place, for example, for perfect path complexes. In this case we obtain a chain complex

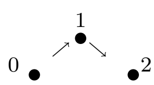
$$0 \leftarrow \mathbb{K} \leftarrow \mathcal{A}_0 \leftarrow ... \leftarrow \mathcal{A}_{n-1} \leftarrow \mathcal{A}_n \leftarrow ... \tag{3.3}$$

whose homology groups are denoted by $\widetilde{H}_n(P), n \geq -1$, and are referred to as the *reduced path homologies* of P. Consider also the truncated complex

$$0 \leftarrow \mathcal{A}_0 \leftarrow ... \leftarrow \mathcal{A}_{n-1} \leftarrow \mathcal{A}_n \leftarrow ... \tag{3.4}$$

whose homology groups are denoted by $H_n(P), n \geq 0$, and are referred to as the *path homologies* of P. For example, this construction works if the path complex P arises from a simplicial complex S. Then the path homology groups of P coincide with the corresponding simplicial homology groups of S.

Now consider the general case when $\partial \mathcal{A}_n$ does not have to be a subspace of $\mathcal{A}_{n-1}$. For example, this is the case for a digraph



where the 2-path e_{012} is allowed, while $\partial e_{012} = e_{12} - e_{02} + e_{01}$ is non-allowed because e_{02} is non-allowed.

For a general path complex P and for any $n \geq -1$, define the following subspace of $\mathcal{A}_n$:

$$\Omega_n = \Omega_n(P) = \{v \in \mathcal{A}_n : \partial v \in \mathcal{A}_{n-1}\}. \tag{3.5}$$

Note that $\Omega_n = \mathcal{A}_n$ for $n \leq 1$ while for $n \geq 2$ the space Ω_n can be actually smaller that $\mathcal{A}_n$. We claim that always $\partial\Omega_n \subset \Omega_{n-1}$. Indeed, if $v \in \Omega_n$ then $\partial v \in \mathcal{A}_{n-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{n-2}$ whence it follows that $\partial v \in \Omega_{n-1}$, which was to be proved.

The elements of Ω_n are called *∂-invariant* n-paths. Thus, we obtain the *augmented* chain complex of ∂-invariant paths:

$$0 \leftarrow \mathbb{K} \leftarrow \Omega_0 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{3.6}$$

where all mappings are given by ∂. Consider also its standard (non-augmented) version

$$0 \leftarrow \Omega_0 \leftarrow ... \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow ... \tag{3.7}$$

The homology groups of (3.7) are referred to as the *path homology groups* of the path complex P and are denoted by $H_n(P), n \geq 0$. The homology groups of (3.6) are called the *reduced path homology groups* of P and are denoted by $\widetilde{H}_n(P), n \geq -1$.

Definition 3.4 A path complex P is called *regular* if it contains no 1-path of the form ii. Equivalently, P is regular if all the paths $i_0...i_n \in P$ are regular.

For example, the path complex of a simplicial complex is always regular. The path complex of a digraph is regular if and only if the digraph is loopless, that is, if the 1-paths ii are not edges.

For a regular path complex the above construction of the spaces Ω_n allows the following variation. As the space $\mathcal{A}_n$ of allowed n-path is in this case a subspace of the space $\mathcal{R}_n$ of regular n-paths, we can replace in (3.5) the non-regular boundary operator ∂ on Λ_n by the regular boundary operator on $\mathcal{R}_n$ as described in Section 2.3. The resulting space Ω_n is referred to as a *regular* space of ∂-invariant paths. Hence, if the path complex P is regular then we can consider also regular versions of the chain complexes (3.6) and (3.7) and the regular versions of homology groups.

If the path complex P is perfect then we obtain $\Omega_n(P) = \mathcal{A}_n(P)$ for all n (in this case there is no difference between regular and non-regular versions). Hence, in this case the chain complex (3.6) is identical to (3.3), and (3.7) is identical to (3.4).

If $P(G)$ is the path complex of a digraph G then we use the notation $\Omega_n(G) := \Omega_n(P(G))$. The corresponding homology groups are denoted by $H_n(G)$, respectively $\widetilde{H}_n(G)$, and are referred to as the *path homologies of the digraph G*.

The Euler characteristic of the path complex is defined by

$$\chi(P) = \sum_{p=0}^{n} (-1)^p \dim H_p(P) \tag{3.8}$$

provided there exists n that $\dim H_p(P) = 0$ for all $p > n$. For a regular path complex P there is a regular and non-regular versions of $\chi(P)$ that do not have to match.

3.3 Some properties of path homologies

Let us state some simple properties of the space $\Omega_n(P)$ and $H_n(P)$.

Proposition 3.5 ([8]) (a) *If* $\dim \Omega_n = 0$ *then* $\dim \Omega_p = 0$ *for all* $p > n$.

(b) *For a regular chain complex* $\{\Omega_*\}$, *the condition* $\dim \Omega_n \leq 1$ *for some* n *implies that* $\dim \Omega_p = 0$ *for all* $p > n$.

Proposition 3.6 ([8]) *For any path complex* P *we have* $\dim H_0(P) = k$, *where* k *is the number of connected components*[1] *of* P. *Moreover,* $H_0(P)$ *is generated by any set* $\{e_{i_1}, ..., e_{i_k}\}$ *of* k *vertices belonging to different connected components.*

In particular, if P *is connected then* $\dim H_0(P) = 1$ *and, hence,* $\dim \widetilde{H}_0(P) = 0$.

Let P be a regular path complex over a set V and P' be a regular path complex over a set V'.

Definition 3.7 We say, that a map $f : V \to V'$ is a morphism of path complexes from P to P' if, for any path $v \in P$, the path $f_*(v)$ either lies in P' or is non-regular.

Proposition 3.8 *Any morphism* $f : V \to V'$ *of path complexes* P *and* P' *induces a morphism of regular chain complexes*

$$f_* : \Omega_*(P) \to \Omega_*(P')$$

and, consequently, a homomorphism of regular homology groups

$$f_* : H_*(P) \to H_*(P').$$

Proof. Any allowed path $v \in \mathcal{A}_n(P)$ is a linear combination of paths $e_{i_0...i_n} \in P$ and, hence, $f_*(v)$ is a linear combination of paths $f_*(e_{i_0...i_n})$ that are either in P' or non-regular. Since non-regular paths are treated as zero, we obtain that $f_*(v) \in \mathcal{A}_n(P')$. If $v \in \Omega_n(P)$ then $\partial v \in \mathcal{A}_{n-1}(P)$ and, hence,

$$\partial(f_*(v)) = f_*(\partial v) \in \mathcal{A}_{n-1}\left(P'\right),$$

which implies $f_*(v) \in \Omega_n(P')$. Hence, f_* is a morphism of regular chain complexes. The second claim is standard. ■

4 Digraphs

4.1 Path homologies on digraphs

In this section we give some examples of ∂-invariant paths on digraphs without loops, that is, edges of the form $a \to a$. If $G = (V, E)$ is a digraph without loops then its path complex $P(G)$ is regular. We deal here with the regular spaces $\Omega_n(G) = \Omega_n(P(G))$ and regular homology groups $H_n(G) = H_n(P(G))$ and $\widetilde{H}_n(G) = \widetilde{H}_n(P(G))$.

[1] A *connected component* of P is any minimal subset U of V that if $i \in U$ then U contains any vertex $j \in V$ such that ij or ji is an allowed 1-path.

Triangles and squares. Let us call by a *triangle* a sequence of three distinct vertices $a, b, c \in V$ such that there are arrows $a \to b, b \to c, a \to c$:

$$\begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

Note that a triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$. The 2-path e_{abc} will also be referred to as a triangle.

Let us call by a *square* a sequence of four distinct vertices $a, b, b', c \in V$ such that there are arrows $a \to b, b \to c, a \to b', b' \to c$:

$$\begin{tikzcd} b \arrow[r] & c \\ a \arrow[u] \arrow[r] & b' \arrow[u] \end{tikzcd}$$

Note that a square determines a 2-path $v := e_{abc} - e_{ab'c} \in \Omega_2$ as $v \in \mathcal{A}_2$ and

$$\partial v = (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.$$

The 2-path v will also be referred to as a square.

A *double edge* is a pair of distinct vertices $a, b \in V$ such that there are arrows $a \to b$ and $b \to a$. It determines a 2-path $e_{aba} \in \Omega_2$ because $e_{aba} \in \mathcal{A}_2$ and

$$\partial e_{aba} = e_{ba} - e_{aa} + e_{ab} = e_{ba} + e_{ab} \in \mathcal{A}_2$$

(since the chain complex $\{\Omega_*\}$ is regular, we have $e_{aa} = 0$). The 2-path e_{aba} will also be referred to as a double edge.

Proposition 4.1 ([9, Prop. 2.9], [8])

(a) Any element of $\Omega_2(G)$ is a linear combination of double edges, triangles, and squares.

(b) Assume that a digraph $G = (V, E)$ contains neither double edges nor squares. Then $\dim \Omega_2(G)$ is equal to the number of distinct triangles in G, and $\dim \Omega_p(G) = 0$ for all $p > 2$.

Consequently, if G contains neither double edges nor triangles nor squares then $\dim \Omega_p(G) = \dim H_p(G) = 0$ for all $p \geq 2$.

In part (a) one cannot relate directly $\dim \Omega_2$ to the number of squares and triangles since there may be a linear dependence between. Indeed, consider the following digraph:

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[r] \arrow[dr] & 2 \arrow[r] & 4 \\ & 3 \arrow[ur] & \end{tikzcd}$$

It contains three squares 0124, 0134, and 0234 which determine three ∂-invariant paths

$$e_{014} - e_{024}, \quad e_{024} - e_{034}, \quad e_{034} - e_{014}.$$

These paths are linearly dependent as their sum is equal to 0. It is easy to see that $\dim \Omega_2 = 2$. For this digraph all reduced homologies are trivial.

In the presence of squares one may have non-trivial Ω_p for arbitrary p as one can see from numerous examples in the next sections.

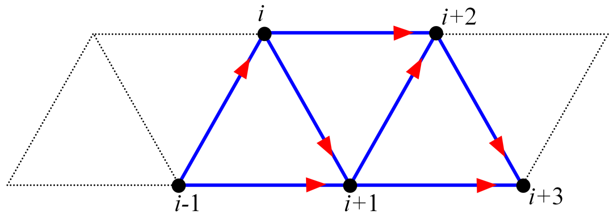


Figure 2: A snake

Snake. A *snake* of length p is a digraph with $p+1$ vertices, say $0, 1, ..., p$, and with the arrows $i \to (i+1)$ and $i \to (i+2)$ (see Fig. 2). In particular, any triple $i\,(i+1)\,(i+2)$ is a triangle.

A snake of length p contains a ∂-invariant p-path $v = e_{01...p}$. Indeed, this path is obviously allowed, its boundary

$$\partial v = \sum_{k=0}^{p} (-1)^k \, e_{0...\widehat{k}...p}$$

is also allowed (because $(k-1)\,(k+1)$ is an arrow), whence $v \in \Omega_p$.

Simplex-digraph. Let us define for any $n \geq 0$ a *simplex-digraph* Sm_n as follows: its set of vertices is $\{0, 1, ..., n\}$ and the arrows are $i \to j$ for all $i < j$. For example, we have

$$\mathrm{Sm}_1 = {}^0\bullet \;\to\; \bullet^1, \qquad \mathrm{Sm}_2 = \begin{array}{ccc} & \overset{2}{\bullet} & \\ \nearrow & & \nwarrow \\ {}_0\bullet & \to & \bullet_1 \end{array},$$

and Sm_3 is shown on Fig. 3.

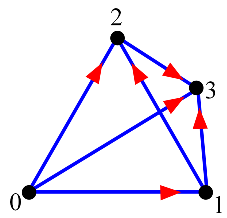


Figure 3: A 3-simplex digraph Sm_3

Since a simplex contains a snake as a subgraph, the n-path $v = e_{01...n}$ is ∂-invariant on Sm_n.

Star-shaped digraphs. We say that a digraph G is *star-shaped* if there is a vertex a (called a star center) such that there is an arrow $a \to b$ for all $b \neq a$. Similarly, a digraph G is called inverse star-shaped if if there is a vertex a (called a star center) such that there is an arrow $b \to a$ for all $b \neq a$

For example, any simplex-digraph is star-shaped and inverse star-shaped.

Proposition 4.2 (A Poincaré lemma) *If G is a (inverse) star-shaped digraph, then all reduced homologies $\widetilde{H}_n(G)$ are trivial.*

The proof can be found in [8]. Alternatively, Proposition 4.2 is an easy consequence of Theorem 5.5, as will be explained below in Section 5.2.

It follows from Proposition 4.2 that all reduced homologies of Sm_n are trivial.

Cycles. We say that a digraph $G=(V,E)$ is a *cycle-graph* if it is connected (as an undirected graph) and every vertex has the degree 2. For a cycle-graph we have $\dim H_0(G)=1$ and $\dim\Omega_0(G)=|V|=|E|=\dim\Omega_1(G)$.

Proposition 4.3 ([8], [9, Ex. 2.8]) *Let G be a cycle-graph. Then*

$$\dim\Omega_p(G)=0\ \ \forall p\geq 3\ \ \textit{and}\ \dim H_p(G)=0\ \forall p\geq 2.$$

If G is a triangle or a square then

$$\dim\Omega_2(G)=1,\ \dim H_1(G)=0,\quad \chi(G)=1$$

whereas otherwise

$$\dim\Omega_2(G)=0,\quad \dim H_1(G)=1,\quad \chi(G)=0.$$

In the latter case, the spanning element of $H_1(G)$ is the 1-path σ such that

$$\sigma^{i(i+1)}=\begin{cases}1, & \textit{if } i(i+1) \textit{ is an edge}\\ -1, & \textit{if } (i+1)i \textit{ is an edge,}\end{cases}\tag{4.1}$$

and all other components of σ vanish.

Möbius band. Consider a (undirected) graph G on Fig. 4 with 6 vertices and 12 edges.

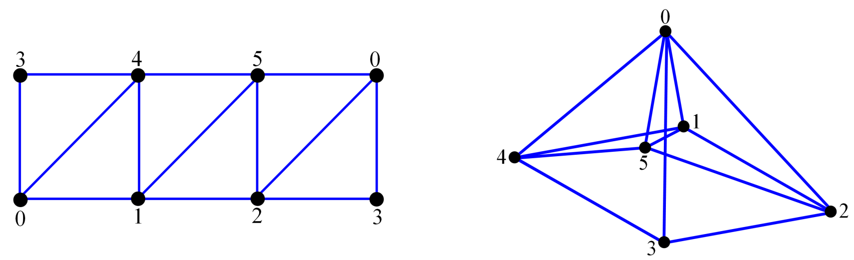


Figure 4: Graph G in two representations: embedded on the Möbius band (left) and in $\mathbb{R}^3$ (right).

As an one-dimensional simplicial complex, G has simplicial homologies $H_*(C_*(G))$. On the other hand, let us introduce arbitrarily a set D of directions on the edges of G, so that (G,D) is a digraph and, hence, has the digraph homologies $H_*(G,D)$. Let us show that for, any choice of D,

$$H_1(C_*(G))\neq H_1(G,D).\tag{4.2}$$

Let Ω_* be the chain complex of the digraph (G,D). In particular, $\dim\Omega_0=6$ that is the number of vertices, and $\dim\Omega_1=12$ that is the number of edges. By homological algebra, we have the following universal identity

$$\dim H_1(\Omega)-\dim H_0(\Omega)=\dim\Omega_1-\dim\Omega_0-\dim\partial\Omega_2$$

and an analogous identity for the simplicial homologies. Since the graph G is connected, we have $\dim H_0(\Omega) = 1$. It follows that

$$\dim H_1(\Omega) = 7 - \dim \partial\Omega_2.$$

A similar formula holds for the simplicial homologies:

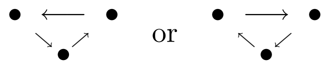
$$\dim H_1(C_*(G)) = 7 - \dim \partial C_2(G) = 7,$$

since $C_2(G)$ is trivial.

It remains to show that the space $\partial\Omega_2$ is non-trivial for any choice D of the edge directions, which will yield

$$\dim H_1(G, D) \leq 6$$

and, hence, (4.2). For that it suffices to verify that there is at least one triangle abc in (G, D) since then $e_{abc} \in \Omega_2$ and $\partial e_{abc} \neq 0$. Indeed, let us try to define directions D on the edges of G so that (G, D) contains no triangles. Then any undirected triangle in G must become one of the two cycles



Given a direction of the edge 03, this requirement determines uniquely the directions of all other edges (cf. Fig. 5), up to the edge 23. However, with any direction on 23 the sequence 023 will become a triangle, which finishes the proof.

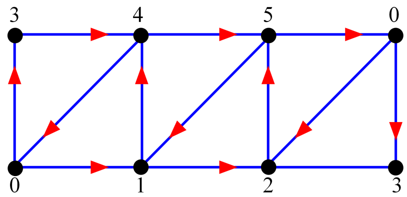


Figure 5: Any direction of the edge 23 will create a triangle

Connected sum. A digraph $G = (V, E)$ is called the *connected sum* of digraphs $G' = (V', E')$ and $G'' = (V'', E'')$ if $V = V' \cup V''$, $E = E' \cup E''$ and $V' \cap V''$ consists of a single vertex.

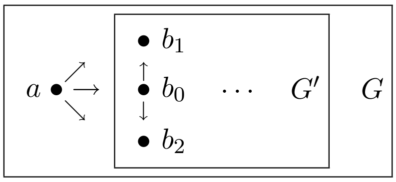
Proposition 4.4 ([12]) *If G is a connected sum of G' and G'' then*

$$\widetilde{H}_*(G) \cong \widetilde{H}_*(G') \oplus \widetilde{H}_*(G'').$$

For example, the digraph G on the right panel of Fig. 1 is a connected sum of a triangle 012 and two 3-simplexes 0678, 0345. Since all reduced homologies of simplexes are trivial, we obtain that all the reduced homology groups of G are trivial.

4.2 Homologies of subgraphs

Proposition 4.5 ([8], [9]) *Suppose that a digraph G has a vertex a with n outcoming arrows $a \to b_0, a \to b_1, ..., a \to b_{n-1}$ and no incoming arrows. Assume also that there are arrows $b_0 \to b_i$ for all $i \geq 1$:*



Denote by G' the digraph that is obtained from G by removing the vertex a with all adjacent edges. Then $H_*(G) \cong H_*(G')$.

The same is true if a vertex a has n incoming arrows $b_0 \to a, b_1 \to a, ..., b_{n-1} \to a$ and no outcoming arrows, while there are arrows $b_i \to b_0$ for all $i \geq 1$.

Corollary 4.6 *Let a digraph G be a tree* (that is, the underlying undirected graph is a tree). *Then $H_p(G) = 0$ for all $p \geq 1$.*

Example 4.7 Consider a digraph G as shown in Fig. 6.

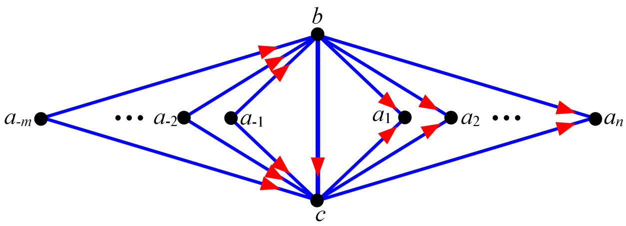
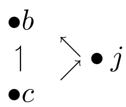


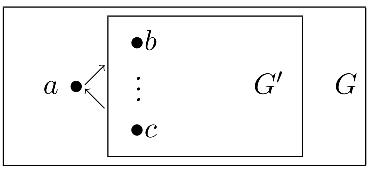
Figure 6: A digraph with many triangles and squares

Each of the vertices a_k satisfies the hypotheses of Proposition 4.5 with $n = 2$ (either with incoming or outcoming arrows). Removing successively the vertices a_k, we see that all the homologies of G are the same as those of the remaining digraph ${}^b\bullet \to \bullet^c$. Since it is a star-shaped digraph, we obtain $\dim H_0 = 1$ and $\dim H_p = 0$ for all $p \geq 1$. In particular, $\chi = 1$.

A pair cb of distinct vertices on a digraph is called a *semi-edge* if $c \not\to b$ but there is a vertex j such that $c \to j$ and $j \to b$ as on the diagram:



Proposition 4.8 ([8]) *Let the field $\mathbb{K}$ has characteristic 0. Suppose that a digraph (V, E) has a vertex a such that there is only one outcoming arrow $a \to b$ from a and only one incoming arrow $c \to a$, where $b \neq c$. Denote by G' the digraph that is obtained from G by removing the vertex a and the adjacent edges $a \to b, c \to a$:*



Then the following is true.

(a) *For any* $p \geq 2$,

$$\dim H_p(G) = \dim H_p(G'). \tag{4.3}$$

(b) *If* cb *is an edge or a semi-edge in* G' *then* (4.3) *is satisfied also for* $p = 0, 1$, *that is, for all* $p \geq 0$.

(c) *If* cb *is neither edge nor semi-edge in* G', *but* b, c *belong to the same connected component of* G' *then* $\dim H_1(G) = \dim H_1(G') + 1$ *and* $\dim H_0(G) = \dim H_0(G')$.

(d) *If* b, c *belong to different connected components of* G' *then* $\dim H_1(G) = \dim H_1(G')$ *and* $\dim H_0(G) = \dim H_0(G') - 1$.

Consequently, in the case (b), $\chi(G) = \chi(G')$, whereas in the cases (c) and (d), $\chi(G) = \chi(G') - 1$.

Example 4.9 Consider the digraphs

$$G = \begin{array}{ccccc} & & {\scriptstyle b} & & \\ & & \bullet & & \\ {\scriptstyle a}\bullet & \begin{array}{c}\nearrow\\ \nwarrow\end{array} & \downarrow & \begin{array}{c}\nwarrow\\ \nearrow\end{array} & \bullet{\scriptstyle d} \\ & & \bullet & & \\ & & {\scriptstyle c} & & \end{array} \quad \text{and} \quad G' = \begin{array}{ccc} {\scriptstyle b} & & \\ \bullet & & \\ \downarrow & \begin{array}{c}\nwarrow\\ \nearrow\end{array} & \bullet{\scriptstyle d} \\ \bullet & & \\ {\scriptstyle c} & & \end{array}$$

Since cb is semi-edge in G' we have case (b) so that all homologies of G and G' are the same. Removing further vertex d we obtain a digraph ${}^{b}\bullet \to \bullet^{c}$ that will be denoted by G''. It is a star-shaped digraph with $\dim H_p(G'') = 0$ for $p \geq 1$. Since cb is neither edge nor semi-edge in G'', but the digraph is connected, we conclude by case (c) that

$$H_p\left(G'\right) = H_p\left(G''\right) \text{ for } p \geq 2,$$

and

$$\dim H_1\left(G'\right) = \dim H_1\left(G''\right) + 1 = 1.$$

It follows that $\dim H_p(G) = 0$ for $p \geq 2$ and $\dim H_1(G) = 1$.

Example 4.10 Consider a digraph on Fig. 7 (an *anti-snake*).

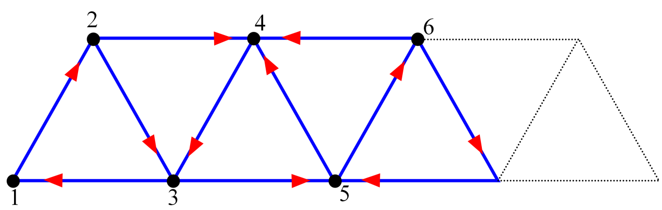


Figure 7: An anti-snake

We start building this digraph with $1 \to 2$. Since 21 is neither edge nor semi-edge, adding a path $2 \to 3 \to 1$ increases $\dim H_1$ by 1 and preserves other homologies. Since 23 is an edge, adding a path $2 \to 4 \to 3$ preserves all homologies. Since 34 is neither edge nor semi-edge, adding a path $3 \to 5 \to 4$ increases $\dim H_1$ by 1 and preserves other homologies. Similarly, adding a path $5 \to 6 \to 4$ preserves all homologies.

One can repeat this pattern arbitrarily many times. By doing so we construct a digraph with a prescribed positive integer value of $\dim H_1$ while keeping $\dim H_p = 0$ for all $p \geq 2$. Consequently, the Euler characteristic χ can take arbitrary negative integer values.

Example 4.11 Consider a digraph on Fig. 1(right). By Proposition 4.5, we can remove the vertices 5 and 8 (and their adjacent edges) without change of homologies. Then by the same proposition we can remove 4 and 7. By Proposition 4.8 we can remove the vertex 1. The resulting digraph with the vertices $0, 2, 3, 6$ is star-shaped, so that by Proposition 4.2 the homology groups H_p are trivial for all $p \geq 1$, while $\dim H_0 = 1$.

5 Join of path complexes

In this and next sections we use slightly different way of denoting the path spaces associated with a given path complex as we will have to consider path complexes on more than one set. Given a finite set V, denote by $P(V)$ a path complex on V. The space $\mathcal{A}_n(P(V))$ of all allowed n-paths will be denoted shortly by $\mathcal{A}_n(V)$. Similarly, the space $\Omega_n(P(V))$ of all ∂-invariant n-paths will be denoted by $\Omega_n(V)$. Similar notation will apply to all other relevant notions including path homologies $H_n(V)$, etc.

In this section the range of n is $n \geq -1$ so that we use the augmented chain complexes (3.6).

5.1 Definition and examples of join

Definition 5.1 Given two disjoint finite sets X, Y and their path complexes $P(X), P(Y)$, set $Z = X \sqcup Y$ and define a path complex $P(Z)$ as follows: $P(Z)$ consists of all paths of the form uv where $u \in P(X)$ and $v \in P(Y)$. The path complex $P(Z)$ is called a *join* of $P(X), P(Y)$ and is denoted by $P(Z) = P(X) * P(Z)$.

The operation $*$ on the path complexes is obviously non-commutative but associative. An example of the path $uv \in P(Z)$ is shown on Fig. 8(left). Note that each of u, v can be empty so that all allowed paths on X and Y will also be allowed on Z.

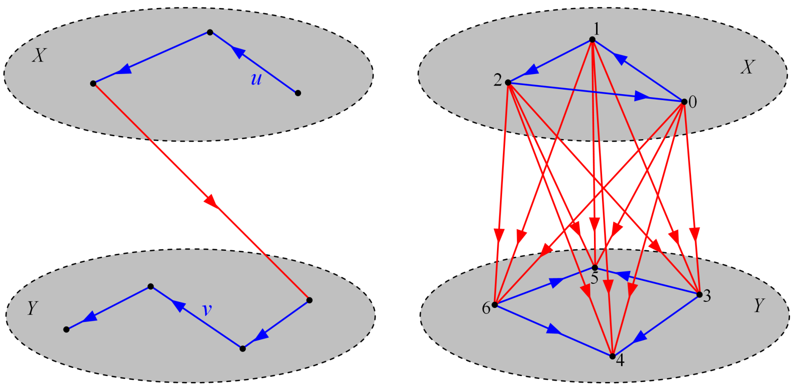


Figure 8: Join of two paths (left) and join of two digraphs (right)

Example 5.2 Let X, Y be two digraphs with disjoint sets of vertices. Consider the digraph Z whose the set of vertices is $X \sqcup Y$, while the set of edges of Z consists of all the edges of X and Y, as well as of all the edges $x \to y$ for all $x \in X$ and $y \in Y$. The digraph Z is called a *join* of X and Y and is denoted by $X * Y$. An example of a join of two digraphs is shown on Fig. 8(right).

Let $P(Z)$ be the path complex arising from the digraph structure of Z. Then it is obvious from the definition that $P(Z)$ is the join of $P(X)$ and $P(Y)$ so that $P(X*Y) = P(X)*P(Y)$. Hence, the operation of joining of digraphs is compatible with the operation of joining path complexes.

Example 5.3 Let X and Y be the vertex sets of finite simplicial complexes $S(X)$ and $S(Y)$. Let us construct a simplicial complex $S(Z)$ with the vertex set $Z = X \sqcup Y$ as follows. Assuming that $|X| = n$ and $|Y| = m$, embed the set X (together with all simplexes from $S(X)$) into a hyperplane $h^{n-1} \subset \mathbb{R}^{n+m-1}$ and Y — into a hyperplane $h^{m-1} \subset \mathbb{R}^{n+m-1}$, where the hyperplanes h^{n-1}, h^{m-1} are orthogonal and non-intersecting. For any two simplexes $\sigma_1 \in S(X)$ and $\sigma_2 \in S(Y)$, define their join $\sigma_1 * \sigma_2$ as the convex hull of σ_1 and σ_2 embedded in $\mathbb{R}^{n+m-1}$ as above (see Fig. 9).

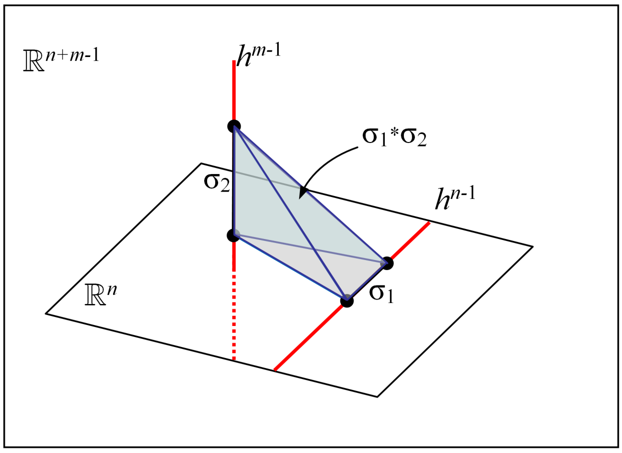


Figure 9: A join $\sigma_1 * \sigma_2$ of two one-dimensional simplexes σ_1, σ_2 (case $n = m = 2$)

Due to a general position of σ_1 and σ_2, the join $\sigma_1 * \sigma_2$ is also a simplex. Then $S(Z)$ is a collection of all simplexes $\sigma_1 * \sigma_2$ with $\sigma_1 \in S(X)$ and $\sigma_2 \in S(Y)$. We refer to $S(Z)$ as a join of simplicial complexes $S(X), S(Y)$ and denote it by $S(X) * S(Y)$.

Equivalently, one can define $S(Z)$ in an abstract way without embedding into a Euclidean space. Indeed, considering simplexes as sequences of vertices, we can say that $S(Z)$ consists of all simplexes of the form $[x_0, ..., x_p, y_0, ..., y_q]$ where $[x_0, ..., x_p] \in S(X)$ and $[y_0, ..., y_q] \in S(Y)$. It is clear that $S(Z)$ is a simplicial complex as it satisfies the defining property (3.2). It is also obvious that the path complexes $P(X), P(Y), P(Z)$ of the simplicial complexes $S(X), S(Y), S(Z)$, respectively, satisfy $P(Z) = P(X) * P(Y)$. Hence, the operation of joining of simplicial complexes is compatible with the operation of joining path complexes.

Proposition 5.4 *Let $P(X)$ and $P(Y)$ be two path complexes and let $P(Z) = P(X)*P(Y)$. If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $uv \in \Omega_{p+q+1}(Z)$. Moreover, the operation $u, v \mapsto uv$ of join extends to that for the homology classes $u \in \widetilde{H}_p(X)$ and $v \in \widetilde{H}_q(Y)$ so that $uv \in \widetilde{H}_{p+q+1}(Z)$.*

Proof. If u and v are allowed then uv is allowed on Z by definition. In particular, if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $uv \in \mathcal{A}_{p+q+1}(Z)$. Let us show that $\partial(uv) \in \mathcal{A}_{p+q}(Z)$, which would imply $uv \in \Omega_{p+q+1}(Z)$. Indeed, we have by (2.7)

$$\partial(uv) = (\partial u)v + (-1)^{p+1} u(\partial v). \tag{5.1}$$

Since ∂u and ∂v are also allowed, we obtain that the right hand side here is allowed, whence the claim follows.

If u, v are cyles, then by (5.1) the join uv is a cycle for Z. We are left to verify that the homology class of uv depends only on the homology classes of u and v. For that it suffices to prove that if either u or v is a boundary then so is uv. Indeed, if $u = dw$ then

$$\partial (wv) = (\partial w) v + (-1)^p w (\partial v) = uv$$

so that uv is a boundary. ■

5.2 Path homologies of join

Before we state the main theorem, let us recall some notations from homological algebra. Let $\{A_p\}_{p \geq p_0}$ be a sequence of finite dimensional linear spaces over $\mathbb{K}$ enumerated by an integer parameter p. Denote by $A_\bullet$ the direct sum of all A_p, that is

$$A_\bullet = \bigoplus_{p \geq p_0} A_p$$

so that $A_\bullet$ is a graded linear space. If $\{A_p\}$ is a chain complex with the boundary operator ∂_A then ∂_A extends linearly to an operator in $A_\bullet$ that respects a graded structure. It will be convenient identify $A_\bullet$ with the chain complex $A_* = \{A_p\}$ as $A_\bullet$ contains the same information as A_*. The sequence of homologies $\{H_p (A_\bullet)\}$ of the chain complex $A_\bullet$ gives rise to a graded linear space $H_\bullet (A_\bullet)$.

Given two graded linear spaces $A_\bullet$ and $B_\bullet$ as above, define their tensor product by

$$A_\bullet \otimes B_\bullet = \bigoplus_{p,q} (A_p \otimes B_q),$$

where $A_p \otimes B_q$ is the tensor product over $\mathbb{K}$ of the linear spaces A_p and B_q. In other words, $A_\bullet \otimes B_\bullet = C_\bullet$ where

$$C_r = \bigoplus_{\{p,q: p+q=r\}} (A_p \otimes B_q).$$

If $A_\bullet$ and $B_\bullet$ are chain complexes with the boundary operators ∂_A and ∂_B, respectively, then define the boundary operator ∂_C in $C_\bullet$ by

$$\partial_C (u \otimes v) = (\partial_A u) \otimes v + (-1)^p u \otimes (\partial_B v) \tag{5.2}$$

for all $u \in A_p$ and $v \in B_q$. It is well-known that $\partial_C^2 = 0$ so that $C_\bullet$ with ∂_C is a chain complex. Furthermore, by a theorem of Künneth, we have the following identity for homologies:

$$H_\bullet (C_\bullet) \cong H_\bullet (A_\bullet) \otimes H_\bullet (B_\bullet) \tag{5.3}$$

that is,

$$H_r (C_\bullet) \cong \bigoplus_{\{p,q: p+q=r\}} H_p (A_\bullet) \otimes H_q (B_\bullet)$$

(see [16]). Given a graded linear space $A_\bullet$, define a graded space $A'_\bullet$ by

$$A'_n := A_{n-1}.$$

If $A_\bullet$ is a chain complex then also $A'_\bullet$ is a chain complex with the same boundary operator.

Given a regular path complex $P(V)$ on a finite set V, we consider as before the spaces $\mathcal{R}_n (V), \mathcal{A}_n (V)$ and $\Omega_n (V)$, where $n \geq -1$. Then we have the chain complexes $\mathcal{R}_\bullet (V)$, $\mathcal{R}'_\bullet (V)$, $\Omega_\bullet (V)$, $\Omega'_\bullet (V)$ with the regular boundary operator ∂ and a graded space $\mathcal{A}_\bullet (V)$.

Theorem 5.5 *Let X, Y be two finite non-empty sets and $P(X)$ and $P(Y)$ be regular path complexes on X and Y, respectively. Set $Z = X \sqcup Y$ and consider the join path complex $P(Z) = P(X) * P(Y)$. Then we have the following isomorphism of the chain complexes:*

$$\Omega_\bullet(Z) \cong \Omega'_\bullet(X) \otimes \Omega_\bullet(Y), \tag{5.4}$$

where the mapping $\Omega'_\bullet(X) \otimes \Omega_\bullet(Y) \to \Omega_\bullet(Z)$ is given by $u \otimes v \mapsto uv$.

It follows from (5.4) that, for any $r \geq -1$,

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)) \tag{5.5}$$

and, for any $r \geq 0$,

$$\widetilde{H}_r(Z) \cong \bigoplus_{\{p,q \geq 0: p+q=r-1\}} \left(\widetilde{H}_p(X) \otimes \widetilde{H}_q(Y)\right) \tag{5.6}$$

(a Künneth formula for join).

The identity (5.6) gives easily the proof of Proposition 4.2. Indeed, let G be a star-shaped digraph with a star center a. Denote by G' the digraph that is obtained from G by removing the vertex a and all adjacent edges. Then $G = \{a\} * G'$, and by (5.6) we obtain $\widetilde{H}_r(G) \cong \{0\}$ for all $r \geq 0$ because $\widetilde{H}_p(\{a\}) \cong \{0\}$ for all $p \geq 0$. If G is an inverse star-shaped digraph then $G = G' * \{a\}$ and again $\widetilde{H}_r(G) \cong \{0\}$.

Example 5.6 Consider the digraph $Z = X * Y$ as on Fig. 8(right). In this case we have by Proposition 4.3 that all homologies $\widetilde{H}_p(X)$ and $\widetilde{H}_q(Y)$ are trivial except for

$$\begin{aligned} H_1(X) &= \operatorname{span}\{e_{01} + e_{12} + e_{20}\}, \\ H_1(Y) &= \operatorname{span}\{e_{35} - e_{65} + e_{64} - e_{34}\}. \end{aligned}$$

Therefore, all $\widetilde{H}_r(Z)$ are trivial except for $H_3(Z)$ that is generated by a single element

$$e_{0135} - e_{0165} + e_{0164} - e_{0134} + e_{1235} - e_{1265} + e_{1264} - e_{1234} + e_{2035} - e_{2065} + e_{2064} - e_{2034}.$$

5.3 Cone and suspension

A *cone* over a digraph X is a digraph $\operatorname{Cone} X$ that is obtained from X by adding one more vertex a and all the edges of the form $b \to a$ for all $b \in X$. The vertex a is called the cone vertex. Clearly, we have $\operatorname{Cone} X = X * Y$ where Y consists of a single vertex a.

Proposition 5.7 *For any digraph X, we have for any $r \geq 0$*

$$\Omega_r(\operatorname{Cone} X) \cong \Omega_r(X) \oplus \Omega_{r-1}(X), \tag{5.7}$$

where the isomorphism is given by the map $u, v \mapsto u + ve_a$, where $u \in \Omega_r(X)$, $v \in \Omega_{r-1}(X)$ and a is the cone vertex. Furthermore, all the reduced homologies of $\operatorname{Cone} X$ are trivial.

Proof. Since $\operatorname{Cone} X = X * Y$ with $Y = \{a\}$, the isomorphism (5.7) follows from (5.5), $\Omega_{-1}(Y) = \operatorname{span}\{1_{\mathbb{K}}\}$, $\Omega_0(Y) = \operatorname{span}\{e_a\}$ and $\Omega_q(Y) = \{0\}$ for $q \geq 1$. Since all the homologies $\widetilde{H}_q(Y)$ are trivial, it follows from (5.6) that all homologies $\widetilde{H}_r(Z)$ are also trivial. The latter follows also from Proposition 4.2 since $\operatorname{Cone} X$ is inverse star-shaped. ■

Example 5.8 Clearly, a simplex-digraph Sm_n can be regarded as a cone over Sm_{n-1} (cf. Section 4.1). Since $\Omega_0(\mathrm{Sm}_0)$ is spanned by a 0-path e_0, we obtain by induction from (5.7) that $\Omega_n(\mathrm{Sm}_n)$ is spanned by a path $e_{01\dots n}$.

Definition 5.9 A *suspension* over a digraph X is a digraph $\mathrm{Sus}\,X$ that is obtained from X by adding two vertices a, b and all the edges $c \to a$ and $c \to b$ for all $c \in X$. The vertices a, b are called the suspension vertices.

Clearly, we have $\mathrm{Sus}\,X = X * Y$ where $Y = \{a, b\}$ is a digraph that consists of two vertices a, b and no edges.

Proposition 5.10 *For any digraph X we have, for any $r \geq 0$,*

$$\Omega_r(\mathrm{Sus}\,X) \cong \Omega_r(X) \oplus \Omega_{r-1}(X) \oplus \Omega_{r-1}(X), \tag{5.8}$$

where the isomorphism is given by the map $u, v, w \mapsto u + ve_a + we_b$, where $u \in \Omega_r(X)$, $v, w \in \Omega_{r-1}(X)$ and a, b are the suspension vertices. Furthermore, we have

$$\widetilde{H}_r(\mathrm{Sus}\,X) \cong \widetilde{H}_{r-1}(X), \tag{5.9}$$

where the isomorphism is given by the map $u \mapsto u(e_a - e_b)$, $u \in \widetilde{H}_{r-1}(X)$. Consequently, we have

$$\chi(\mathrm{Sus}\,X) = 2 - \chi(X). \tag{5.10}$$

Proof. Let Y as above. The isomorphism (5.8) follows from (5.5) because $\Omega_{-1}(Y) = \mathrm{span}\{1_{\mathbb{K}}\}$, $\Omega_0(Y) = \mathrm{span}\{e_a, e_b\}$ and $\Omega_q(Y) = \{0\}$ for $q \geq 1$. Since $\widetilde{H}_q(Y) = \{0\}$ for all $q \neq 0$ and $\widetilde{H}_0(Y) = \mathrm{span}\{e_a - e_b\}$, (5.9) follows from (5.6). Finally, setting $Z = \mathrm{Sus}\,X$ and using (5.9), we obtain

$$\begin{aligned} \chi(Z) &= 1 + \sum_{r \geq 1} (-1)^r \dim H_r(Z) = 1 + \sum_{r \geq 1} (-1)^r \dim \widetilde{H}_{r-1}(X) \\ &= 1 - \sum_{s \geq 0} (-1)^s \dim \widetilde{H}_s(X) = 2 - \sum_{s \geq 0} (-1)^s \dim H_s(X) = 2 - \chi(X), \end{aligned}$$

which proves (5.10). ■

In particular, having examples of digraphs X with arbitrary negative integer values of χ (cf. Example 4.10), we obtain examples of digraphs $\mathrm{Sus}\,X$ with arbitrary positive integer values of χ.

Example 5.11 Let S be any cycle-graph that is neither triangle nor square; it will be considered as an analog of a circle. Define S_n inductively by $S_1 = S$ and $S_{n+1} = \mathrm{Sus}\,S_n$. Then S_n can be regarded as n-dimensional sphere-graph. Since $\chi(S) = 0$ by Proposition 4.3, it follows that $\chi(S_n) = 0$ if n is odd and $\chi(S_n) = 2$ if n is even. Proposition 5.10 implies that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial H_n for an arbitrary n.

For example, the octahedron digraph *Oct* on Fig 10 is S_2 based on the cycle S with the vertices $0, 1, 2, 3$. It follows that *Oct* has non-trivial $H_2(Oct)$ despite the fact that this digraph is obviously planar.

Let v be an 1-path on S that spans $H_1(S)$ (see Section 4.1). If S_{n+1} is a suspension of S_n on the vertices a_n, b_n then we obtain by induction that the spanning element of $H_n(S_n)$ is

$$u = v(e_{a_1} - e_{b_1})(e_{a_2} - e_{b_2}) \dots (e_{a_{n-1}} - e_{b_{n-1}}).$$

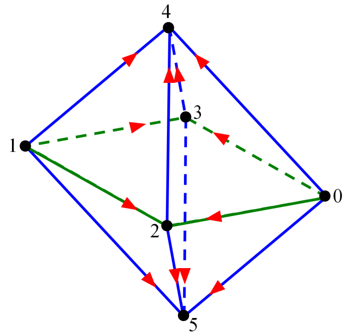


Figure 10: The octahedron digraph Oct

For the cycle S on Fig. 10 we have by Proposition 4.3 $v = e_{12} - e_{02} + e_{03} - e_{13}$, which implies that the spanning element of $H_2(Oct)$ is

$$u = v(e_4 - e_5) = e_{124} - e_{024} + e_{034} - e_{134} - e_{125} + e_{025} - e_{035} + e_{135}.$$

Obviously, each term in this sum corresponds to one of the eight faces of the octahedron, and the sum u represents in some sense the surface of the octahedron.

Applying Proposition 4.3 to compute the homology groups of S and then Proposition 5.10, we obtain

$$\dim H_0(Oct) = 1,\ \dim H_1(Oct) = 0,\ \ \dim H_2(Oct) = 1,\ \dim H_p(Oct) = 0\ \text{ for } p \geq 3. \tag{5.11}$$

Example 5.12 Consider a digraph G on Fig. 11(left).

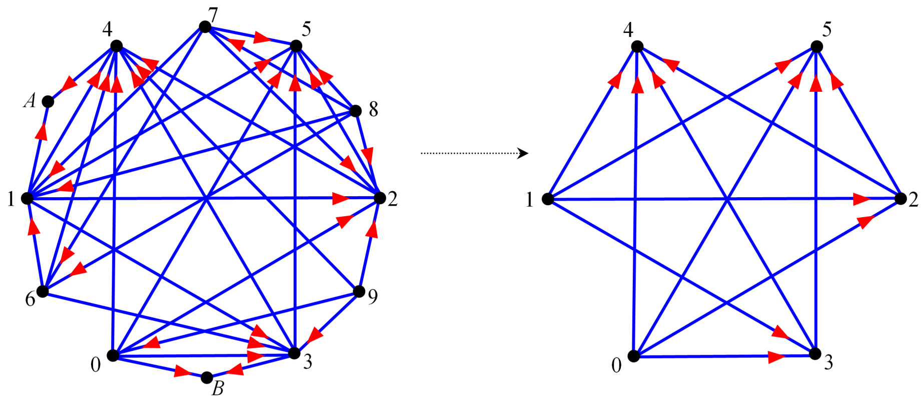


Figure 11: A digraph G with 12 vertices and 32 edges (left) and its reduction to the digraph G' (right)

Removing successively the vertices $A, B, 8, 9, 6, 7$ by Proposition 4.5, we obtain a digraph G' as on Fig. 11(right) with the vertex set $\{0, 1, 2, 3, 4, 5\}$ that has the same homologies as G. The digraph G' is clearly the same as Oct on Fig. 10. Hence, we obtain by (5.11) that $\dim H_2(G) = 1$ while $H_p(G) = \{0\}$ for $p = 1$ and $p > 2$. The spanning element of $H_2(G)$ is hence

$$u = e_{124} - e_{024} + e_{034} - e_{134} - e_{125} + e_{025} - e_{035} + e_{135}.$$

In other words, this 2-path u determines a 2-dimensional "hole" in G given by the octahedron. Note that on Fig. 11 this octahedron is hardy visible, but it can be determined purely algebraically using the above tools.

5.4 Some properties of ∂-invariant paths on joins

We prove here some auxiliary results needed for the proof of Theorem 5.5. For a finite set V, denote by $R(V)$ the path complex on V consisting of all regular elementary paths on V. Then, for any $n \geq -1$, $R_n(V)$ denotes the set of all regular elementary n-paths on V, As before, $\mathcal{R}_n(V)$ is the space of all finite $\mathbb{K}$-linear combinations of the paths from $R_n(V)$.

Let X, Y be two finite non-empty sets and $P(X)$ and $P(Y)$ be regular path complexes on X and Y, respectively. Set $Z = X \sqcup Y$ and consider the join of path complexes $P(Z) = P(X) * P(Y)$.

Lemma 5.13 *Any $w \in \Omega_\bullet(Z)$ admits a representation*

$$w = \sum_{x \in P(X)} e_x a^x = \sum_{y \in P(Y)} b^y e_y, \tag{5.12}$$

where $a^x \in \Omega_\bullet(Y)$ and $b^y \in \Omega_\bullet(X)$ are uniquely determined.

Proof. Since any allowed elementary path on X is a join of elementary paths on X and Y, we see that any $w \in \mathcal{A}_\bullet(Z)$ admits a representation

$$w = \sum_{x \in P(X),\ y \in P(Y)} c^{xy} e_x e_y, \tag{5.13}$$

where the coefficients $c^{xy} \in \mathbb{K}$ are uniquely determined. It follows from (5.13) that

$$w = \sum_{x \in P(X)} e_x a^x, \tag{5.14}$$

where

$$a^x = \sum_{y \in P(Y)} c^{xy} e_y \in \mathcal{A}_\bullet(Y).$$

Clearly, a^x are uniquely determined.

Assume now that $w \in \Omega_\bullet(Z)$ and show that $a^x \in \Omega_\bullet(Y)$. Let us define the coefficients $\delta^x_{x'} \in \{0, 1, -1\}$ by

$$\partial e_x = \sum_{x' \in R(X)} \delta^{x'}_x e_{x'}. \tag{5.15}$$

Also, if $x \in P_p(X)$ then set $\varepsilon_x = (-1)^{p+1}$. Using (5.14) and the product rule (2.7) we obtain

$$\partial w = \sum_{x \in P(X)} (\partial e_x) a^x + \varepsilon_x e_x (\partial a^x) = \sum_{x \in P(X)} \sum_{x' \in R(X)} \delta^{x'}_x e_{x'} a^x + \sum_{x \in P(X)} \varepsilon_x e_x \partial a^x.$$

Switching in the double sum the notations x and x' and interchanging the summation signs, we obtain

$$
\begin{aligned}
\partial w &= \sum_{x\in R(X)}\sum_{x'\in P(X)} \delta^x_{x'} e_x a^{x'} + \sum_{x\in P(X)} \varepsilon_x e_x \partial a^x \\
&= \sum_{x\in P(X)} e_x \left(\sum_{x'\in P(X)} \delta^x_{x'} a^{x'} + \varepsilon_x \partial a^x \right) \qquad (5.16)\\
&\quad + \sum_{x\in R(X)\setminus P(X)} e_x \left(\sum_{x'\in P(X)} \delta^x_{x'} a^{x'} \right). \qquad (5.17)
\end{aligned}
$$

Note that any elementary path of the full expansion of the sum (5.17) has a non-allowed X-part, while that of (5.16) has the allowed X-part. Therefore, there is no cross cancellation between the elementary paths of (5.16) and (5.17). Since their sum ∂w is allowed, it follows that the sum (5.17) consisting only of non-allowed paths, must vanish.

On the other hand, since $\partial w \in \Omega_* (Z)$, we have analogously to (5.14) a representation

$$
\partial w = \sum_{x\in P(X)} e_x \widetilde{a}^x \ ,
$$

where $\widetilde{a}^x \in \mathcal{A}_* (Y)$. Comparison with (5.16) yields

$$
\widetilde{a}^x = \sum_{x'\in P(X)} \delta^x_{x'} a^{x'} + \varepsilon_x \partial a^x .
$$

Since $a^{x'} \in \mathcal{A}_* (Y)$, it follows that $\partial a^x \in \mathcal{A}_* (Y)$, which proves that $a^x \in \Omega_* (Y)$.

The second identity in (5.12) is proved similarly. ■

Let V be a finite set. If $u \in \mathcal{R}_n (V)$ and $x \in R_m (V)$ then we denote by $u^x \in \mathbb{K}$ the coefficient of x -component of u if $n = m$ and set $u^x = 0 \in \mathbb{K}$ if $n \neq m$. Let us introduce in $\mathcal{A}_p (V)$ the $\mathbb{K}$-scalar product as follows: for all $u, v \in \mathcal{A}_p (V)$ we put

$$
[u, v] := \sum_{x\in P(V)} u^x v^x, \qquad (5.18)
$$

where as before u^x and v^x are the coefficients of the components of u and v, respectively. If $\mathbb{K} = \mathbb{R}$ then $[\cdot, \cdot]$ is a proper scalar product, but for a general field $\mathbb{K}$ there is no positivity property (in fact, it can happen that $[u, u] = 0$). Set also

$$
\Omega^\perp_p (V) = \{ u \in \mathcal{A}_p (V) : [u, v] = 0 \text{ for all } v \in \Omega_p (V) \} . \qquad (5.19)
$$

If $\mathbb{K} = \mathbb{R}$ then $\Omega^\perp_p$ is an orthogonal complement of Ω_p in $\mathcal{A}_p$ and $\mathcal{A}_p = \Omega_p \oplus \Omega^\perp_p$.

For a general $\mathbb{K}$, this is not true, as Ω_p and $\Omega^\perp_p$ may have a non-trivial intersection. However, for any field $\mathbb{K}$, it is still true that

$$
\dim \Omega_p + \dim \Omega^\perp_p = \dim \mathcal{A}_p
$$

(see [13, Lemma 6.1]).

Lemma 5.14 *If $u \in \Omega^\perp_p (X)$ and $v \in \mathcal{A}_q (Y)$ then $uv \in \Omega^\perp_r (Z)$ where $r = p+q+1$. Similarly, if $u \in \mathcal{A}_p (X)$ and $v \in \Omega^\perp_q (Y)$ then $uv \in \Omega^\perp_r (Z)$.*

Proof. To prove the first claim, we need to show that $[uv, w] = 0$ for any $w \in \Omega_r(Z)$. By Lemma 5.13, w is a sum of the joins $\varphi\psi$ where $\varphi \in \Omega_\bullet(X)$ and $\psi \in \mathcal{A}_\bullet(Y)$. Hence, it suffices to prove that

$$[uv, \varphi\psi] = 0, \tag{5.20}$$

assuming that $\varphi \in \Omega_{p'}(X)$ and $\psi \in \mathcal{A}_{q'}(Y)$. If $p' + q' + 1 \neq r$ then uv and $\varphi\psi$ do not have common elementary paths in their expansions, and (5.20) is trivially satisfied. Assuming $p' + q' + 1 = r$, we obtain

$$[uv, \varphi\psi] = \sum_{z \in P_r(Z)} (uv)^z (\varphi\psi)^z = \sum_{x \in P_p(X), y \in Pq(Y)} u^x v^y \varphi^x \psi^y.$$

If $p' \neq p$ then $\varphi^x = 0$ and again (5.20) holds trivially. Finally, if $p' = p$ and, hence, $q' = q$, then we obtain

$$[uv, \varphi\psi] = \sum_{x \in P_p(X)} u^x \varphi^x \sum_{y \in P_q(Y)} v^y \psi^y = [u, \varphi][v, \psi] = 0,$$

because $[u, \varphi] = 0$ by assumption $u \in \Omega_p^\perp(X)$. The second claim is proved similarly. ■

5.5 Proof of the Künneth formula for join

The main technical part of the proof of Theorem 5.5 is contained in the following theorem.

Theorem 5.15 *Let $P(X)$ and $P(Y)$ be two regular path complexes and let $P(Z) = P(X) * P(Y)$ be their join. Then any ∂-invariant path w on Z admits a representation in the form*

$$w = \sum_{i=1}^{k} u_i v_i \tag{5.21}$$

for some finite k, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

The proof of Theorem 5.15 will be given at the end of Section 6.5 because it is similar to the proof of an analogous property for Cartesian products of path complexes (Theorem 6.12 below).

Proof of Theorem 5.5. Let us first show how (5.5) and (5.6) follow from (5.4). By definition (5.4) means that

$$\Omega_r(Z) \cong \bigoplus_{\{p \geq 0, q \geq -1 : p+q=r\}} \left(\Omega'_p(X) \otimes \Omega_q(Y)\right),$$

whence (5.5) follows by changing $p - 1$ to p. The isomorphism (5.4) of the chain complexes $\Omega_\bullet(Z)$ and $\Omega'_\bullet(X) \otimes \Omega_\bullet(Y)$ implies that their homologies are also isomorphic. On the other hand, by the Künneth theorem (5.3), we obtain

$$H_\bullet\left(\Omega'_\bullet(X) \otimes \Omega_\bullet(Y)\right) \cong H_\bullet\left(\Omega'_\bullet(X)\right) \otimes H_\bullet(\Omega_\bullet(Y)),$$

whence

$$H_\bullet(\Omega_\bullet(Z)) \cong H_\bullet\left(\Omega'_\bullet(X)\right) \otimes H_\bullet(\Omega_\bullet(Y)).$$

More explicitly this means that, for any $r \geq -1$,

$$\begin{aligned} H_r(\Omega_\bullet(Z)) &\cong \bigoplus_{\{p' \geq 0, q \geq -1 : p'+q=r\}} \left(H_{p'}\left(\Omega'_\bullet(X)\right) \otimes H_q(\Omega_\bullet(Y))\right) \\ &= \bigoplus_{\{p, q \geq -1 : p+q=r-1\}} \left(H_p(\Omega_\bullet(X)) \otimes H_q(\Omega_\bullet(Y))\right). \end{aligned}$$

Since the homology group $H_{-1}(\Omega_\bullet)$ is always trivial, the condition $p,q \geq -1$ can be replaced here by $p,q \geq 0$. Finally, using that $H_p(\Omega_\bullet(X)) = \widetilde{H}_p(X)$ and $H_q(\Omega_\bullet(Y)) = \widetilde{H}_q(Y)$ are the reduced homologies, we obtain (5.6).

Now we concentrate on the proof of (5.4). We use the graded spaces $\{\mathcal{R}_\bullet\}$, $\{\mathcal{A}_\bullet\}$, $\{\Omega_\bullet\}$ associated with the path complexes $P(X), P(Y)$ and $P(Z)$. If $\{W_\bullet\}$ is one of these spaces then set

$$W_\bullet(X,Y) = W'_\bullet(X) \otimes W_\bullet(Y).$$

Then (5.4) can be restated as follows:

$$\Omega_\bullet(Z) \cong \Omega_\bullet(X,Y).$$

To prove this, we will construct explicitly a mapping

$$\Phi : \Omega_r(X,Y) \to \Omega_r(Z)$$

that will be isomorphism of linear spaces and will commute with the boundary operator ∂.

Consider first a larger the chain complex

$$\mathcal{R}_\bullet(X,Y) = \mathcal{R}'_\bullet(X) \otimes \mathcal{R}_\bullet(Y)$$

and define for any $r \geq -1$ the linear mapping

$$\Phi : \mathcal{R}_r(X,Y) \to \mathcal{R}_r(Z)$$

as follows: for all $u \in \mathcal{R}'_p(X)$ and $v \in \mathcal{R}_q(Y)$ with $p+q=r$, set

$$\Phi(u \otimes v) = uv,$$

where uv is the join of u and v on Z (note that X and Y are subsets of Z).

It follows from Lemma 2.2 that, for u, v as above,

$$\partial(uv) = (\partial u)v + (-1)^p u\partial v. \tag{5.22}$$

Here the operator ∂ is the boundary operator on $\mathcal{R}_\bullet(Z)$, but in the expressions ∂u and ∂v it coincides with the boundary operators on $\mathcal{R}_\bullet(X)$ and $\mathcal{R}_\bullet(Y)$, respectively. By (5.2) we have for the operator ∂ on $\mathcal{R}_\bullet(X,Y)$

$$\partial(u \otimes v) = (\partial u) \otimes v + (-1)^p u \otimes \partial v.$$

The comparison with (5.22) shows that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{R}_{r-1}(X,Y) & \overset{\partial}{\leftarrow} & \mathcal{R}_r(X,Y) \\
\downarrow^{\Phi} & & \downarrow^{\Phi} \\
\mathcal{R}_{r-1}(Z) & \overset{\partial}{\leftarrow} & \mathcal{R}_r(Z)
\end{array}$$

Hence, the mapping Φ is a homomorphism of chain complexes $\mathcal{R}_\bullet(X,Y)$ and $\mathcal{R}_\bullet(Z)$.

Let us verify that Φ is in fact a monomorphism. Indeed, the basis in $\mathcal{R}_r(X,Y)$ consists of all elements of the form $e_x \otimes e_y$ where $x \in R_p(X), y \in R_q(Y)$ with $p+q=r$. Since $\Phi(e_x \otimes e_y) = e_{xy}$ and all such paths e_{xy} are linearly independent in $\mathcal{R}_r(Z)$, we see that Φ is injective.

Next, observe that

$$\Phi(\mathcal{A}_r(X,Y)) = \mathcal{A}_r(Z).$$

Indeed, the basis in $\mathcal{A}_r(X,Y)$ consists of all elements of the form $e_x \otimes e_y$ where $x \in R_p(X), y \in R_q(Y)$ with $p+q=r$, while the basis in $\mathcal{A}_r(Z)$ consists of the paths e_{xy} with the same set of x, y, whence the claim follows. In particular, the linear spaces $\mathcal{A}_r(X,Y)$ and $\mathcal{A}_r(Z)$ are isomorphic.

Finally, let us prove that, for all $r \geq -1$,

$$\Phi(\Omega_r(X,Y)) = \Omega_r(Z),$$

which will finish the proof of (5.4). The inclusion

$$\Phi(\Omega_r(X,Y)) \subset \Omega_r(Z)$$

is trivial because by Proposition 5.4 $u \in \Omega'_p(X)$ and $v \in \Omega_q(Y)$ with $p+q=r$ imply $uv \in \Omega_r(Z)$. The opposite inclusion

$$\Phi(\Omega_r(X,Y)) \supset \Omega_r(Z)$$

follows from Theorem 5.15. Indeed, any $w \in \Omega_r(Z)$ admits a representation in the form

$$w = \sum_i u_i v_i$$

where u_i and v_i are ∂-invariant paths on X and Y, respectively. It follows that

$$\Phi\left(\sum_i u_i \otimes v_i\right) = \sum_i u_i v_i = w$$

and, hence, $w \in \Phi(\Omega_r(X,Y))$. ■

6 Cartesian product of path complexes

In this section we slightly redefine the sequence $\{\mathcal{R}_n(V)\}$ of spaces of regular paths on a finite set V. Namely, instead of the previous convention $\mathcal{R}_{-1} = \operatorname{span}\{e\}$, we set now $\mathcal{R}_{-1} = \{0\}$. In other words, the index n has now the range $n \geq 0$ instead of $n \geq -1$ in Section 5.

All path complexes in this section are regular, and we always use a regular standard chain complex $\{\Omega_n\}_{n\geq 0}$ given in (3.7) and the associated homology groups $\{H_n\}_{n\geq 0}$.

6.1 Cross product of paths

Given two finite sets X, Y, consider their Cartesian product $Z = X \times Y$. Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (x_k, y_k)$ with $x_k \in X$ and $y_k \in Y$. We say that the path z is *step-like* if, for any $k = 1, ..., r$, either $x_{k-1} = x_k$ or $y_{k-1} = y_k$. In fact, exactly one of these conditions holds as z is regular.

Any step-like path z on Z determines by projection regular elementary paths x on X and y on Y. More precisely, x is obtained from z by taking the sequence of all X-components of the vertices of z and then by collapsing in it any subsequence of repeated vertices to one vertex. The same rule applies to y. By construction, the projections x and y are *regular* elementary paths on X and Y, respectively. If the projections of $z = z_0 ... z_r$ are $x = x_0 ... x_p$ and $y = y_0 ... y_q$ then $p + q = r$ (cf. Fig. 12(left)).

Every vertex $z_k = (x_i, y_j)$ of a step-like path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that the whole path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting the points $(0,0)$ and (p, q). Define the *elevation* $L(z)$ of the path z as the number of cells in $\mathbb{Z}^2_+$ below the staircase $S(z)$ (the shaded area on Fig. 12(right)).

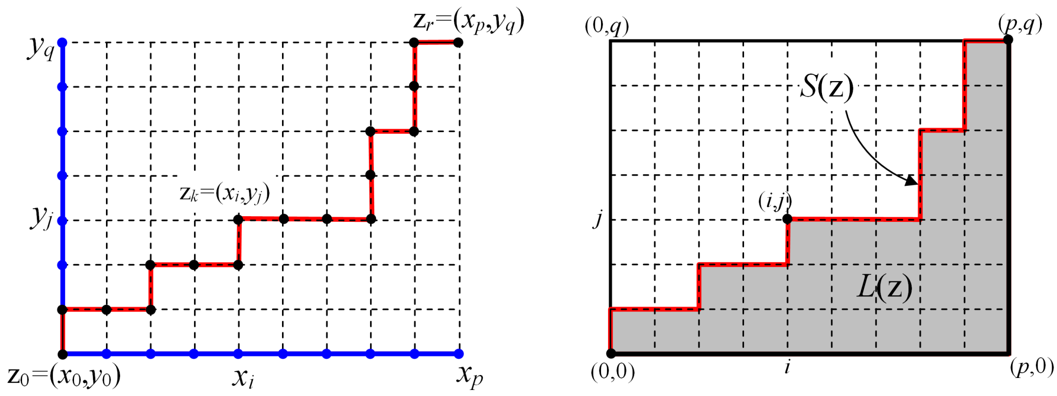


Figure 12: Left: a step-like path z and its projections x and y. Right: a staircase $S(z)$ and its elevation $L(z)$ (here $L(z) = 30$).

Definition 6.1 Given paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with some $p, q \geq 0$, define a path $u \times v$ on Z by the following rule: for any step-like elementary $(p+q)$-path z on Z, the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y, \tag{6.1}$$

where x and y are the projections of z onto X and Y, respectively, and u^x and v^y are the corresponding components of u and v. For non-step-like paths z set $(u \times v)^z = 0$.

The path $u \times v$ is called the *cross product* of u and v. It follows that $u \times v \in \mathcal{R}_{p+q}(Z)$.

For given elementary regular p-path x on X and q-path y on Y, denote by $\Pi_{x,y}$ the set of all step-like paths z on Z whose projections on X and Y are x and y, respectively. It follows from (6.1) that

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z. \tag{6.2}$$

It is not difficult to see that the cross product is associative.

Example 6.2 Let us denote the vertices of X by the letters $a, b, c, \ldots$ and the vertices of Y by the integers $0, 1, 2, \ldots$ so that the vertices of Z can be denoted as chessboard fields, for example, $a0, b1$ etc. Then we have

$$\begin{aligned} e_{abc} \times e_{012} &= e_{a0b0c0c1c2} - e_{a0b0b1c1c2} + e_{a0b0b1b2c2} \\ &\quad + e_{a0a1b1c1c2} - e_{a0a1b1b2c2} + e_{a0a1a2b2c2} \end{aligned}$$

as one can see on Fig. 13.

From now on and throughout this section we use the regular boundary operator ∂ acting on the chain complex $\{\mathcal{R}_n\}_{n \geq 0}$ (note the difference with Section 5 where we used $\{\mathcal{R}_n\}_{n \geq -1}$).

It turns out that the boundary operator ∂ satisfies the product rule with respect to the cross product.

Proposition 6.3 (Product rule) *If* $u \in \mathcal{R}_p(X)$ *and* $v \in \mathcal{R}_q(Y)$ *where* $p, q \geq 0$, *then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v). \tag{6.3}$$

The proof of this statement is rather involved and can be found in [13, Prop. 4.4].

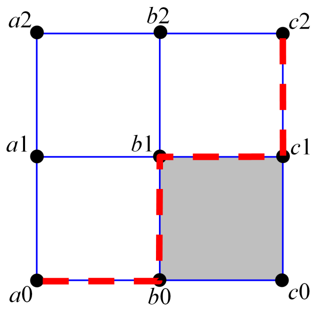


Figure 13: The staircase $a0b0b1c1c2$ has elevation 1. Hence, $e_{a0b0b1c1c2}$ enters the product $e_{abc} \times e_{012}$ with the negative sign.

6.2 Path homologies of Cartesian product

Definition 6.4 Given two finite sets X and Y with path complexes $P(X)$ and $P(Y)$, respectively, define on the set $Z = X \times Y$ a path complex $P(Z)$ as follows: the elements of $P(Z)$ are step-like paths on Z whose projections on X and Y belong to $P(X)$ and $P(Y)$, respectively. The path complex $P(Z)$ is called the *Cartesian product* of the path complexes $P(X)$ and $P(Y)$ and is denoted by $P(X) \Box P(Y)$.

In short: a step-like path z on Z is allowed if and only if its projections on X and Y are allowed. In particular, if x and y are elementary allowed paths on X and Y, respectively, then all the paths $z \in \Pi_{x,y}$ are allowed on Z. It clearly follows from (6.2) that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z).$$

Furthermore, the following is true.

Proposition 6.5 *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.*

Proof. Indeed, ∂u and ∂v are allowed, whence also $\partial u \times v$ and $u \times \partial v$ are allowed, whence $\partial(u \times v)$ is allowed by the product rule (6.3). It follows that $u \times v \in \Omega_{p+q}(Z)$. ∎

The next theorem is one of the main results of this paper. It gives a complete description of ∂-invariant paths on Z.

Theorem 6.6 *Let $P(X)$ and $P(Y)$ be two regular path complexes. Then for their Cartesian product $P(Z) = P(X) \Box P(Y)$ the following isomorphism of chain complexes holds:*

$$\Omega_\bullet(Z) \cong \Omega_\bullet(X) \otimes \Omega_\bullet(Y) \tag{6.4}$$

where the mapping $\Omega_\bullet(X) \otimes \Omega_\bullet(Y) \to \Omega_\bullet(Z)$ is given by $u \otimes v \mapsto u \times v$.

A more detailed version of (6.4) is the following: for any $r \geq 0$,

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq 0: p+q=r\}} (\Omega_p(X) \otimes \Omega_q(Y)). \tag{6.5}$$

Consequently, we obtain the Künneth formula

$$H_\bullet(Z) \cong H_\bullet(X) \otimes H_\bullet(Y), \tag{6.6}$$

that is, for any $r \geq 0$,

$$H_r(Z) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} \left(H_p(X) \otimes H_q(Y)\right). \tag{6.7}$$

The proof of Theorem 6.6 will be given in Section 6.5 after a necessary preparation in Section 6.4. Before that we consider some examples of Cartesian products.

Let X be a digraph. For simplicity of notation, we denote the set of vertices of X by the same letter X, and the set of edges denote by E_X. Given two digraphs X and Y, their Cartesian product is the digraph $Z = X \Box Y$ where the set of vertices of Z is the Cartesian product of the sets of vertices of X and Y, while the set E_Z of edges is defined as follows: $(x, y) \rightarrow (x', y')$ if and only if either $x \rightarrow x'$ and $y = y'$, or $y \rightarrow y'$ and $x = x'$:

$$\begin{array}{cccccc}
y' \bullet & \ldots & \overset{(x,y')}{\bullet} & \longrightarrow & \overset{(x',y')}{\bullet} & \ldots \\
\uparrow & & \uparrow & & \uparrow & \\
y \bullet & \ldots & \overset{(x,y)}{\bullet} & \longrightarrow & \overset{(x',y)}{\bullet} & \ldots \\
 & & & & & \\
{}^{Y} / {}_{X} & \ldots & \underset{x}{\bullet} & \longrightarrow & \underset{x'}{\bullet} & \ldots
\end{array}$$

Clearly, any allowed path on Z is step-like, and its projections onto X and Y are also allowed. Hence, the path complex of the digraph Z is the Cartesian product of the path complexes of the digraphs X and Y.

Example 6.7 Let $Z = X \Box Y$ where X is a 3-cycle and Y is a square, that is,

$$X = \begin{array}{ccc} & \overset{b}{\bullet} & \\ \nearrow & & \searrow \\ {}^{a}\bullet & \leftarrow & \bullet^{c} \end{array} \quad \text{and} \quad Y = \begin{array}{ccc} 2\bullet & \longrightarrow & \bullet 3 \\ \uparrow & & \uparrow \\ 0\bullet & \longrightarrow & \bullet 1 \end{array}.$$

We have

$$\Omega_0(X) = \text{span}\{e_a, e_b, e_c\}, \ \Omega_1(X) = \text{span}\{e_{ab}, e_{bc}, e_{ca}\}, \ \Omega_p(X) = \{0\} \ \text{ for } p \geq 2$$

and

$$\begin{aligned} \Omega_0(Y) &= \text{span}\{e_0, e_1, e_2, e_3\}, \ \Omega_1(Y) = \text{span}\{e_{01}, e_{13}, e_{23}, e_{02}\}, \\ \Omega_2(Y) &= \text{span}\{e_{013} - e_{023}\}, \ \ \Omega_q(Y) = \{0\} \ \text{ for } q \geq 3. \end{aligned}$$

Hence, we obtain by (6.5)

$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y)$$

and

$$\Omega_3(Z) = \text{span}\{e_{ab} \times (e_{013} - e_{023}), e_{bc} \times (e_{013} - e_{023}), e_{ca} \times (e_{013} - e_{023})\}.$$

Similarly one computes $\Omega_r(Z)$ for other values of r.

By Proposition 4.3, we have

$$H_1(X) = \text{span}\{e_{ab} + e_{bc} + e_{ca}\}, \ H_p(X) = \{0\} \ \text{ for } p \geq 2$$

and

$$H_0(Y) = \text{span}\{e_0\}, \ \ H_q(Y) = \{0\} \ \text{ for all } q \geq 1.$$

By (6.7) we obtain

$$H_1(Z) \cong H_1(X) \otimes H_0(Y)$$

and

$$H_1(Z) = \text{span}\{(e_{ab} + e_{bc} + e_{ca}) \times e_0\}.$$

It follows also from (6.7) that $H_r(Z) = \{0\}$ for all $r \geq 2$.

6.3 Cylinders and cubes

For any digraph X, the *cylinder* over X is the digraph

$$\operatorname{Cyl} X := X \Box \left\{ {}^{0}\bullet \to \bullet^{1} \right\}.$$

Assuming that the vertices of X are enumerated by $0, 1, ..., n-1$, we can enumerate the vertices of $\operatorname{Cyl} X$ by $0, 1, ..., 2n-1$ using the following rule: $(x, 0)$ is assigned the number x, while $(x, 1)$ is assigned $x + n$.

Every regular p-path v on X has two copies on $\operatorname{Cyl} X$: $v_{(0)} = v \times e_0$ and $v_{(1)} = v \times e_1$. Moreover, v gives rise to the following $(p+1)$-path on $\operatorname{Cyl} X$: $v_{(01)} = v \times e_{01}$, that is called *lifting* of v. For example, if $v = e_{i_0 \ldots i_p}$ then

$$v_{(01)} = e_{i_0 \ldots i_p} \times e_{01} = \sum_{k=0}^{p} (-1)^{p-k} \, e_{i_0 \ldots i_k (i_k+n) \ldots (i_p+n)}. \tag{6.8}$$

By Proposition 6.5, if v is ∂-invariant, then $v_{(0)}, v_{(1)}, v_{(01)}$ are also ∂-invariant.

Proposition 6.8 *For any digraph X and for any $r \geq 0$, we have*

$$\Omega_r (\operatorname{Cyl} X) \cong \Omega_r (X) \oplus \Omega_r (X) \oplus \Omega_{r-1} (X),$$

where the isomorphism is given by the map $u, v, w \mapsto u_{(0)} + v_{(1)} + w_{(01)}$, for $u, v \in \Omega_r (X)$ and $w \in \Omega_{r-1} (X)$. Furthermore, we have

$$H_r (\operatorname{Cyl} X) \cong H_r (X),$$

where the isomorphism is given by the map $u \mapsto u_{(0)}$ for $u \in H_r (X)$.

Proof. All claims follow directly from Theorem 6.6 and the knowledge of Ω_* and H_* of the digraph $Y = \{ {}^{0}\bullet \to \bullet^{1} \}$. ■

Define for any non-negative integer n the n-cube digraph by

$$\mathrm{Cube}_n = \operatorname{Cyl} \mathrm{Cube}_{n-1}, \quad \mathrm{Cube}_0 = \{0\}.$$

For example, $\mathrm{Cube}_1 = \{ {}^{0}\bullet \to \bullet^{1} \}$, Cube_2 is a square:

$$\begin{array}{ccc} {}^{2}\bullet & \longrightarrow & \bullet^{3} \\ \uparrow & & \uparrow \\ {}^{0}\bullet & \longrightarrow & \bullet^{1} \end{array}$$

and Cube_3 is shown in Fig. 14.

Lifting a ∂-invariant 1-path $v_1 = e_{01}$ on 1-cube, we obtain the following ∂-invariant 2-path on 2-cube: $v_2 = e_{013} - e_{023}$. Lifting further v_2, we obtain the following ∂-invariant 3-path on the 3-cube:

$$v_3 = e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}.$$

We obtain by induction a ∂-invariant n-path v_n on Cube_n that is a lifting of a ∂-invariant $(n-1)$-path v_{n-1} on Cube_{n-1}. It is easy to see that v_n is an alternating sum of $n!$ elementary terms, corresponding to partitioning of a geometric n-cube into $n!$ simplexes. It follows from Proposition 6.8 that $\Omega_n (\mathrm{Cube}_n) = \operatorname{span}(v_n)$ so that the path v_n represents the n-cube. Proposition 6.8 also implies that homology groups of Cube_n are trivial except for H_0.

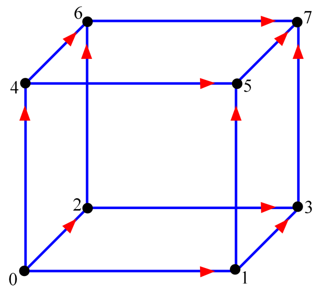


Figure 14: A 3-cube

6.4 Some properties of ∂-invariant paths on products

Here we prove some lemma needed for the proof of Theorem 6.6. Given a regular path complex $P(V)$ on a finite set V, we consider the spaces $\mathcal{R}_n(V), \mathcal{A}_n(V)$ and $\Omega_n(V)$ with $n \geq 0$, as well as their direct sums $\mathcal{R}_\bullet(V)$, $\mathcal{A}_\bullet(V)$, $\Omega_\bullet(V)$.

In all statements we consider two regular paths complexes $P(X)$, $P(Y)$ and their Cartesian product $P(Z) = P(X) \Box P(Y)$ where $Z = X \times Y$.

Lemma 6.9 *Any path $w \in \Omega_\bullet(Z)$ admits a representation*

$$w = \sum_{x \in P(X),\ y \in P(Y)} c^{xy} (e_x \times e_y) \tag{6.9}$$

with some coefficients $c^{xy} \in \mathbb{K}$ (only finitely many coefficients are non-vanishing). Furthermore, the coefficients c^{xy} are uniquely determined by w.

Proof. Let us first show the uniqueness of c^{xy}, which is equivalent to the linear independence of the family $\{e_x \times e_y\}$ across all $x \in P(X)$ and $y \in P(Y)$. Indeed, assume that, for some scalars c^{xy},

$$\sum_{x \in P(X), y \in P(Y)} c^{xy} e_x \times e_y = 0,$$

and prove that $c^{xy} = 0$ for any couple x, y as in the summation. Fix such a couple x, y and choose one $z \in \Pi_{x,y}$. Then by (6.1)

$$\left(e_{x'} \times e_{y'}\right)^z = \begin{cases} (-1)^{L(z)}, & x' = x \text{ and } y' = y, \\ 0, & \text{otherwise,} \end{cases}$$

which implies that

$$\left(\sum_{x' \in P(X), y' \in P(Y)} c^{x'y'} e_{x'} \times e_{y'} \right)^z = (-1)^{L(z)} c^{xy}$$

and, hence, $c^{xy} = 0$.

Let us show existence of the representation (6.9) for any $w \in \Omega_r(Z)$ and any $r \geq 0$. As before, for any elementary r-path z on Z, w^z denotes the e_z-coordinate of w. If z is an elementary r'-path with $r' \neq r$ then set $w^z = 0$. For any $x \in P(X)$ and $y \in P(Y)$ choose some $z \in \Pi_{x,y}$ and set

$$c^{xy} = (-1)^{L(z)} w^z. \tag{6.10}$$

Let us first show that the value of c^{xy} in (6.10) is independent of the choice of $z \in \Pi_{x,y}$. Set $z = i_0...i_r$. Let k be an index such that one of the couples $i_{k-1}i_k$, i_ki_{k+1} is vertical and the other is horizontal. If $i_{k-1} = (a, b)$ and $i_{k+1} = (a', b')$ where $a, a' \in X$ and $b, b' \in Y$, then i_k is either (a', b) or (a, b'). Denote the other of these two vertices by i'_k, as, for example, on the diagram:

$$\begin{array}{ccccccc}
\vdots & & & & & \vdots & \\
b' \bullet & & & \overset{i'_k}{\bullet} & \longrightarrow & \overset{i_{k+1}}{\bullet} & \cdots \\
\uparrow & & & \uparrow & & \uparrow & \\
b \bullet & & \cdots & \overset{i_{k-1}}{\bullet} & \longrightarrow & \overset{i_k}{\bullet} & \\
\vdots & & & \vdots & & & \\
\shortparallel & & & & & & \\
y & & & & & & \\
 & x= & \cdots & \underset{a}{\bullet} & \longrightarrow & \underset{a'}{\bullet} & \cdots
\end{array}$$

Replacing in the path $z = i_0...i_r$ the vertex i_k by i'_k, we obtain the path $z' = i_0...i_{k-1}i'_ki_{k+1}...i_r$ that clearly belongs to $\Pi_{x,y}$ and, hence, is allowed. Since the $(r-1)$-path $i_0...i_{k-1}i_{k+1}...i_r$ is regular but non-allowed (as it is not step-like), while ∂w is allowed, we have

$$(\partial w)^{i_0...i_{k-1}i_{k+1}...i_r} = 0. \tag{6.11}$$

On the other hand, we have by (2.4)

$$(\partial w)^{i_0...i_{k-1}i_{k+1}...i_r} = \sum_{j \in Z} \left(\sum_{m=0}^{k-1} (-1)^m \, w^{i_0...i_{m-1}ji_m...i_{k-1}i_{k+1}...i_r} \right. \tag{6.12}$$

$$+ (-1)^k \, w^{i_0...i_{k-1}ji_{k+1}...i_r} \tag{6.13}$$

$$\left. + \sum_{m=k+2}^{r+1} (-1)^{m-1} \, w^{i_0...i_{k-1}i_{k+1}...i_{m-1}ji_m...i_r} \right). \tag{6.14}$$

All the components of w in the sums (6.12) and (6.14) vanish since they correspond to non-allowed paths, while w is allowed. The path $i_0...i_{k-1}ji_{k+1}...i_r$ in the term (6.13) is also non-allowed unless $j = i_k$ or $j = i'_k$ (note that i_k and i'_k are uniquely determined by i_{k-1} and i_{k+1}). Hence, the only non-zero terms in (6.12)-(6.14) are $w^{i_0...i_{k-1}i_ki_{k+1}...i_r} = w^z$ and $w^{i_0...i_{k-1}i'_ki_{k+1}...i_r} = w^{z'}$. Combining (6.11) and (6.12)-(6.14), we obtain

$$0 = w^z + w^{z'}.$$

Since $L(z') = L(z) \pm 1$, it follows that

$$(-1)^{L(z')} \, w^{z'} = (-1)^{L(z)} \, w^z. \tag{6.15}$$

The transformation $z \mapsto z'$ described above, allows us to obtain from a given $z \in \Pi_{x,y}$ in a finite number of steps any other path in $\Pi_{x,y}$. Since the quantity $(-1)^{L(z)} w^z$ does not change under this transformation, it follows that it does not depend on a particular choice of $z \in \Pi_{x,y}$, which was claimed. Hence, the coefficients c^{xy} are well-defined by (6.10).

Finally, let us show that the equality (6.9) holds with the coefficients c^{xy} from (6.10). By (6.2) we have

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} \, e_z.$$

Using (6.10) we obtain

$$\begin{aligned}\sum_{x\in P(X),\ y\in P(Y)} c^{xy}\left(e_x\times e_y\right) &= \sum_{x\in P(X),\ y\in P(Y)} c^{xy}\sum_{z\in\Pi_{x,y}}(-1)^{L(z)}e_z\\ &= \sum_{x\in P(X),\ y\in P(Y)}\ \sum_{z\in\Pi_{x,y}} w^z e_z\\ &= \sum_{z\in P(Z)} w^z e_z = w,\end{aligned}$$

which finishes the proof. ∎

Corollary 6.10 *Any path $w\in\Omega_\bullet(Z)$ admits representations*

$$w=\sum_{x\in P(X)} e_x\times a^x=\sum_{y\in P(Y)} b^y\times e_y \tag{6.16}$$

where $a^x\in\Omega_\bullet(Y)$ and $b^y\in\Omega_\bullet(X)$ are uniquely determined.

Proof. It follows from (6.9) that

$$w=\sum_{x\in P(X)} e_x\times a^x$$

where

$$a^x=\sum_{y\in P(Y)} c^{xy}e_y\in\mathcal{A}_\bullet(Y).$$

It is obvious that a^x are uniquely determined as so are the coefficients c^{xy}. Let us show that, in fact, $a^x\in\Omega_\bullet(Y)$. Let us define the coefficients $\delta^x_{x'}\in\{0,1,-1\}$ by

$$\partial e_x=\sum_{x'\in R(X)}\delta_x^{x'}e_{x'}. \tag{6.17}$$

Also, if $x\in P_p(X)$ then set $\varepsilon_x=(-1)^p$. We have by the product rule (6.3) and by (6.17)

$$\begin{aligned}\partial w &= \sum_{x\in P(X)}\partial e_x\times a^x+\varepsilon_x e_x\times\partial a^x\\ &= \sum_{x\in P(X)}\sum_{x'\in R(X)}\delta_x^{x'}e_{x'}\times a^x+\sum_{x\in P(X)}\varepsilon_x e_x\times\partial a^x\\ &= \sum_{x\in R(X)}\sum_{x'\in P(X)}\delta_{x'}^{x}e_{x}\times a^{x'}+\sum_{x\in P(X)}\varepsilon_x e_x\times\partial a^x\\ &= \sum_{x\in P(X)} e_x\times\left(\sum_{x'\in P(X)}\delta_{x'}^{x}a^{x'}+\varepsilon_x\partial a^x\right) \qquad (6.18)\\ &\quad+\sum_{x\in R(X)\setminus P(X)} e_x\times\left(\sum_{x'\in P(X)}\delta_{x'}^{x}a^{x'}\right). \qquad (6.19)\end{aligned}$$

Every elementary path on Z that is present in the full expansion of the sums (6.18) and (6.19) has the X-projection equal to x. Since in (6.18) x is allowed, while in (6.19) – not, there is no cross cancellation of the elementary paths in (6.18) and (6.19). Since every elementary path in (6.19) is non-allowed, while the sum ∂w of (6.18) and (6.19) is allowed, we see that the sum in (6.19) vanishes.

On the other hand, since $\partial w \in \Omega_\bullet(Z)$, we have by Lemma 6.9 a representation

$$\partial w = \sum_{x \in P(X)} e_x \times \widetilde{a}^x,$$

where $\widetilde{a}^x \in \mathcal{A}_\bullet(Y)$. Comparison with (6.18) shows that

$$\widetilde{a}^x = \sum_{x' \in P(X)} \delta^x_{x'} a^{x'} + \varepsilon_x \partial a^x.$$

Since $a^{x'} \in \mathcal{A}_\bullet(Y)$, it follows that $\partial a^x \in \mathcal{A}_\bullet(Y)$, which proves that $a^x \in \Omega_\bullet(Y)$. The second identity in (6.16) is proved similarly. ■

In the next lemma we use the $\mathbb{K}$-scalar product $[\cdot,\cdot]$ of paths that was introduced in Section 5.5 (see (5.18) and (5.19)).

Lemma 6.11 *If $u \in \Omega_p^\perp(X)$ and $v \in \mathcal{A}_q(Y)$ then $u \times v \in \Omega_r^\perp(Z)$ where $r = p+q$. Similarly, if $u \in \mathcal{A}_p(X)$ and $v \in \Omega_q^\perp(Y)$ then $u \times v \in \Omega_r^\perp(Z)$.*

Proof. We need to prove that, for any $w \in \Omega_r(Z)$,

$$[u \times v, w] = 0, \tag{6.20}$$

assuming that $u \in \Omega_p^\perp(X)$ (the second claim is proved similarly). We have:

$$\begin{aligned}
[u \times v, w] &= \sum_{z \in P_r(Z)} (u \times v)^z w^z \\
&= \sum_{z \in P_r(Z)} (-1)^{L(z)} u^x v^y w^z \quad (x, y \text{ are projections of } z) \\
&= \sum_{x \in P_p(X)} \sum_{u \in P_q(Y)} \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} u^x v^y w^z.
\end{aligned}$$

By Corollary 6.10, the path w is a sum of the terms $\varphi \times \psi$ where $\varphi \in \Omega_\bullet(X)$ and $\psi \in \mathcal{A}_\bullet(Y)$, so that it suffices to prove (6.20) for $w = \varphi \times \psi$. Let $\varphi \in \Omega_p(X)$ and, hence, $\psi \in \mathcal{A}_q(Y)$. Then we have by (6.1)

$$w^z = (-1)^{L(z)} \varphi^x \psi^y$$

and, hence,

$$[u \times v, w] = \sum_{x \in P_p(X)} \sum_{y \in P_q(Y)} \sum_{z \in \Pi_{x,y}} u^x \varphi^x v^y \psi^y.$$

Since

$$\sum_{x \in P_p(X)} u^x \varphi^x = [u, \varphi] = 0,$$

we obtain (6.20). If $\varphi \in \Omega_{p'}$ with $p' \neq p$, then $w^z = 0$ for any $z \in \Pi_{x,y}$ with $x \in P_p(X)$, and (6.20) is trivially satisfied. ■

6.5 Proof of the Künneth formula for product

Here we prove Theorem 6.6. The major part of the proof of Theorem 6.6 is contained in the following theorem 6.12 that is similar to Theorem 5.15 for join. Since the proofs of Theorems 6.12 and 5.15 are practically identical, we have preferred to give a detailed proof of Theorem 6.12 for the product and sketch of the proof of Theorem 5.15 for join at the end of this section.

Theorem 6.12 *Let $P(X)$ and $P(Y)$ be two regular path complexes and let $P(Z) = P(X)\Box P(Y)$ be their Cartesian product. Then any ∂-invariant path w on Z admits a representation in the form*

$$w = \sum_{i=1}^{k} u_i \times v_i \tag{6.21}$$

for some finite k, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

Proof. The representation (6.21) is simple in a special case when the path complexes $P(X)$ and $P(Y)$ are perfect, that is, when all allowed paths are ∂-invariant. Indeed, by Lemma 6.9, any $w \in \Omega_r(Z)$ admits a representation in the form (6.9), where e_x and e_y are allowed paths on X and Y, respectively. By the assumption of the perfectness of $P(X)$ and $P(Y)$, the paths e_x and e_y are ∂-invariant, so that (6.9) implies (6.21).

For arbitrary path complexes $P(X)$ and $P(Y)$, the previous argument does not work since $e_x \times e_y$ does not have to be ∂-invariant. Hence, we need a more elaborate strategy. Given two subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$, denote by $U \times V$ the subspace of $\mathcal{A}_r(Z)$ that is spanned by all products $u \times v$ with $u \in U$ and $v \in V$. For any $r \geq 0$ set

$$\widetilde{\Omega}_r(Z) = \sum_{p+q=r} \Omega_p(X) \times \Omega_q(Y), \tag{6.22}$$

that is, $\widetilde{\Omega}_r(Z)$ is the space of paths on Z that is spanned by all paths of the form $u \times v$ where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with some $p, q \geq 0$ such that $p + q = r$. By Proposition 6.5, we have $u \times v \in \Omega_r(Z)$ whence it follows that

$$\widetilde{\Omega}_r(Z) \subset \Omega_r(Z).$$

The existence of the representation (6.21) is equivalent to the opposite inclusion, that is, to the identity

$$\widetilde{\Omega}_r(Z) = \Omega_r(Z).$$

Clearly, it suffices to show that

$$\dim \Omega_r(Z) \leq \dim \widetilde{\Omega}_r(Z). \tag{6.23}$$

Consider also the space

$$\widetilde{\mathcal{A}}_r(Z) = \sum_{p+q=r} \mathcal{A}_p(X) \times \mathcal{A}_q(Y).$$

By definition of the cross product, all the paths in $\widetilde{\mathcal{A}}_r(Z)$ are allowed, that is,

$$\widetilde{\mathcal{A}}_r(Z) \subset \mathcal{A}_r(Z).$$

By Lemma 6.9, any path from $\Omega_r(Z)$ is a linear combination of paths $e_x \times e_y$ with allowed x, y, which means that

$$\Omega_r(Z) \subset \widetilde{\mathcal{A}}_r(Z).$$

In particular, we have also

$$\widetilde{\Omega}_r(Z) \subset \widetilde{\mathcal{A}}_r(Z).$$

Fix some triple p, q, r with $p + q = r$ and consider the spaces (cf. (5.19)):

- $\Omega_p^{\perp}(X)$ — an orthogonal complement of $\Omega_p(X)$ in $\mathcal{A}_p(X)$;
- $\Omega_q^{\perp}(Y)$ — an orthogonal complement of $\Omega_q(Y)$ in $\mathcal{A}_q(Y)$;
- $\Omega_r^{\perp}(Z)$ — an orthogonal complement of $\Omega_r(Z)$ in $\widetilde{\mathcal{A}}_r(Z)$ (warning: not in $\mathcal{A}_r(Z)$!)

Consider first the case when the field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{Q}$. In this case, a linear space with a $\mathbb{K}$-scalar product is represented as a direct sum of a subspace with its orthogonal complement. For each $u \in \mathcal{A}_p(X)$ consider a decomposition

$$u = u_\Omega + u_\perp \tag{6.24}$$

where $u_\Omega \in \Omega_p(X)$ and $u_\perp \in \Omega_p^{\perp}(X)$, and a similar decomposition $v = v_\Omega + v_\perp$ for $v \in \mathcal{A}_q(Y)$. Then we have

$$u \times v = u_\Omega \times v_\Omega + u_\Omega \times v_\perp + u_\Omega \times v_\perp + u_\perp \times v_\perp.$$

Here $u_\Omega \times v_\Omega \in \widetilde{\Omega}_r(Z)$, while by Lemma 6.11 all other terms in the right hand side belong to $\Omega_r^{\perp}(Z)$, whence it follows that

$$u \times v \in \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z).$$

Since $\widetilde{\mathcal{A}}_r(Z)$ is spanned by the products $u \times v$ where u, v are allowed, we obtain that

$$\widetilde{\mathcal{A}}_r(Z) = \widetilde{\Omega}_r(Z) + \Omega_r^{\perp}(Z).$$

Comparing with the decomposition

$$\widetilde{\mathcal{A}}_r(Z) = \Omega_r(Z) \oplus \Omega_r^{\perp}(Z),$$

we obtain (6.23).

Consider now the most general case of an arbitrary field $\mathbb{K}$. Let us introduce the following notation:

$$\begin{aligned} a_p &= \dim \mathcal{A}_p(X), \ a_q = \dim \mathcal{A}_q(Y), \ a_r = \dim \widetilde{\mathcal{A}}_r(Z), \\ \omega_p &= \dim \Omega_p(X), \ \omega_q = \dim \Omega_q(Y), \ \omega_r = \dim \Omega_r(Z), \end{aligned}$$

and observe that

$$\dim \Omega_p^{\perp}(X) = a_p - \omega_p, \ \dim \Omega_q^{\perp}(Y) = a_q - \omega_q, \ \dim \Omega_r^{\perp}(Z) = a_r - \omega_r. \tag{6.25}$$

Let us prove that

$$a_r = \sum_{p+q=r} a_p a_q. \tag{6.26}$$

Indeed, $\mathcal{A}_p(X)$ is spanned by all elementary paths e_x with $x \in P_p(X)$ and $\mathcal{A}_q(Y)$ is spanned by all elementary paths e_y with $y \in P_q(Y)$. Therefore, $\widetilde{\mathcal{A}}_r(Z)$ is spanned by all products $e_x \times e_y$ where x, y as above are considered for all possible p, q such that $p + q = r$. The number of such products $e_x \times e_y$ is equal to the right hand side of (6.26), so that the identity (6.26) follows from the linear independence of the family $\{e_x \times e_y\}$ (cf. Lemma 6.9).

It follows from the above argument that

$$\dim(\mathcal{A}_p(X) \times \mathcal{A}_q(Y)) = a_p a_q \tag{6.27}$$

and that

$$\widetilde{\mathcal{A}}_r(Z) = \bigoplus_{p+q=r} \left(\mathcal{A}_p(X) \times \mathcal{A}_q(Y)\right). \tag{6.28}$$

Before we can proceed further, let us prove two claims about properties of subspaces of $\mathcal{A}_p(X)$ and $\mathcal{A}_q(Y)$.

Claim 1. *For any two subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$, we have*

$$\dim(U \times V) = \dim U \dim V. \tag{6.29}$$

Indeed, let $u_1, u_2, ..u_k$ be a basis in U and $v_1, ...v_l$ be a basis in V. Then $U \times V$ is spanned by all products $u_i \times v_j$, so that

$$\dim(U \times V) \le kl. \tag{6.30}$$

Let us complement the basis $\{u_i\}$ to a basis in $\mathcal{A}_p(X)$ by adding additional paths $u_1', ..., u_{k'}'$, and, similarly, complement $\{v_j\}$ to a basis in $\mathcal{A}_q(Y)$ by adding $v_1', ..., v_{l'}'$. Set $U' = \operatorname{span}\{u_i'\}$ and $V' = \operatorname{span}\left\{v_j'\right\}$. Then

$$\mathcal{A}_p(X) \times \mathcal{A}_q(Y) = \left(U + U'\right) \times \left(V + V'\right) = U \times V + U \times V' + U' \times V + U' \times V', \tag{6.31}$$

whence by (6.27) and (6.30) we have

$$\begin{aligned} a_p a_q &\le \dim(U \times V) + \dim\left(U \times V'\right) + \dim\left(U' \times V\right) + \dim\left(U' \times V'\right) \\ &\le kl + kl' + k'l + k'l'. \end{aligned} \tag{6.32}$$

The right hand side here is equal to $(k + k')(l + l') = a_p a_q$, which implies that we must have the equality case in (6.32), in particular, $\dim(U \times V) = kl$, which proves (6.29).

Claim 2. *For any two subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$, we have*

$$(U \times \mathcal{A}_q(Y)) \cap (\mathcal{A}_p(X) \times V) = U \times V. \tag{6.33}$$

Indeed, it follows from Claim 1, that the sum at the right hand side of (6.31) is direct and, hence,

$$U \times \mathcal{A}_q(Y) = U \times \left(V \oplus V'\right) = (U \times V) \oplus \left(U \times V'\right)$$

and

$$A_p(X) \times V = \left(U \oplus U'\right) \times V = (U \times V) \oplus \left(U' \times V\right),$$

whence (6.33) follows.

By Lemma 6.11, we have

$$\Omega_p^{\perp}(X) \times \mathcal{A}_q(Y) \subset \Omega_r^{\perp}(Z)$$

and

$$\mathcal{A}_p(X) \times \Omega_q^{\perp}(Y) \subset \Omega_r^{\perp}(Z)$$

so that

$$\sum_{p+q=r} \left[(\Omega_p^{\perp}(X) \times \mathcal{A}_q(Y)) + (\mathcal{A}_p(X) \times \Omega_q^{\perp}(Y))\right] \subset \Omega_r^{\perp}(Z) \tag{6.34}$$

(see Fig. 15).

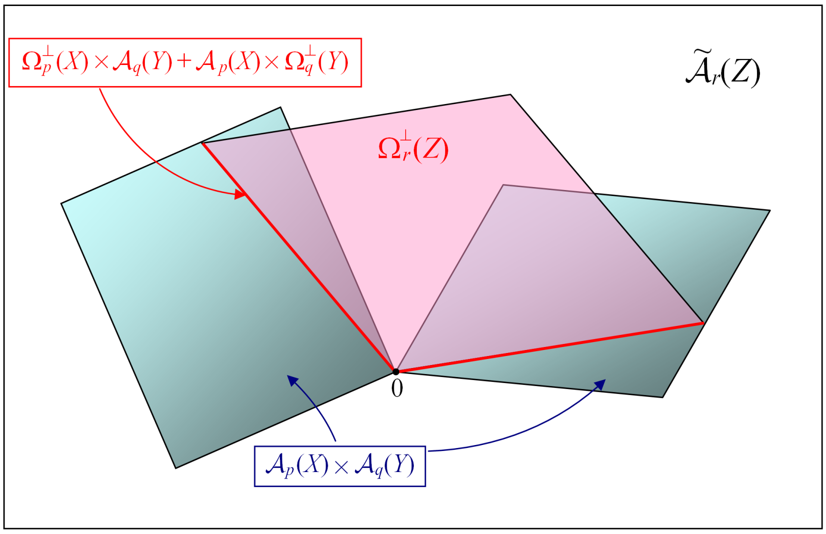


Figure 15: Space $\widetilde{\mathcal{A}}_r(Z)$ and its subspaces $\Omega_r^\perp(Z)$, $\mathcal{A}_p(X)\times\mathcal{A}_q(Y)$ (two instances), and $\Omega_p^\perp(X)\times\mathcal{A}_q(Y)+\mathcal{A}_p(X)\times\Omega_q^\perp(Y)$.

Note that the space in the square brackets in (6.34) is a subspace of $\mathcal{A}_p(X)\times\mathcal{A}_q(Y)$. It follows from (6.28) that the sum $\sum$ in (6.34) is direct, which implies an inequality

$$\sum_{p+q=r}\dim\left[\left(\Omega_p^\perp(X)\times\mathcal{A}_q(Y)\right)+\left(\mathcal{A}_p(X)\times\Omega_q^\perp(Y)\right)\right]\leq\dim\Omega_r^\perp(Z)\,. \tag{6.35}$$

By Claim 2, the subspaces $\Omega_p^\perp(X)\times\mathcal{A}_q(Y)$ and $\mathcal{A}_p(X)\times\Omega_q^\perp(Y)$ have intersection $\Omega_p^\perp(X)\times\Omega_q^\perp(Y)$, whence

$$\begin{aligned}&\dim\left[\left(\Omega_p^\perp(X)\times\mathcal{A}_q(Y)\right)+\left(\mathcal{A}_p(X)\times\Omega_q^\perp(Y)\right)\right]\\=\;&\dim(\Omega_p^\perp(X)\times\mathcal{A}_q(Y))+\dim(\mathcal{A}_p(X)\times\Omega_q^\perp(Y))-\dim(\Omega_p^\perp(X)\times\Omega_q^\perp(Y)).\end{aligned}\tag{6.36}$$

Using (6.25), we obtain that the right hand side of (6.36) is equal to

$$(a_p-\omega_p)\,a_q+a_p\,(a_q-\omega_q)-(a_p-\omega_p)\,(a_q-\omega_q)=a_pa_q-\omega_p\omega_q.$$

Substituting this into (6.35) yields

$$\sum_{p+q=r}(a_pa_q-\omega_p\omega_q)\leq a_r-\omega_r,$$

which together with (6.26) implies that

$$\omega_r\leq\sum_{p+q=r}\omega_p\omega_q.$$

Finally, we are left to observe that, by (6.22),

$$\sum_{p+q=r}\omega_p\omega_q=\dim\widetilde{\Omega}_r(Z)\,,$$

which finishes the proof of inequality (6.23). ∎

Proof of Theorem 6.6. The isomorphism (6.6) follows from (6.4) and the abstract Künneth theorem (5.3), so we only need to prove (6.4). Consider the tensor product of the graded linear spaces

$$\mathcal{A}_\bullet(X,Y) := \mathcal{A}_\bullet(X) \otimes \mathcal{A}_\bullet(Y)$$

and a linear mapping

$$\Phi : \mathcal{A}_r(X,Y) \to \mathcal{A}_r(Z)$$

defined on the basis by

$$\Phi(e_x \otimes e_y) = e_x \times e_y$$

for all $x \in P_p(X)$ and $y \in P_q(Y)$ with $p+q=r$. In fact, we have

$$\Phi(\mathcal{A}_r(X,Y)) = \widetilde{\mathcal{A}}_r(Z)$$

where $\widetilde{\mathcal{A}}_r(Z)$ is defined in (6.28). It follows from the argument in the proof of Theorem 6.12 that the mapping Φ is injective.

Consider now the tensor product of the chain complexes

$$\Omega_\bullet(X,Y) := \Omega_\bullet(X) \otimes \Omega_\bullet(Y),$$

that is, set for any $r \geq 0$

$$\Omega_r(X,Y) = \bigoplus_{\{p,q\geq 0:p+q=r\}} (\Omega_p(X) \otimes \Omega_q(Y))$$

and define the boundary operator ∂ on $\Omega_r(X,Y)$ by (5.2). It follows from the definition of Φ and $\widetilde{\Omega}_r(Z)$ that

$$\Phi(\Omega_r(X,Y)) = \widetilde{\Omega}_r(Z).$$

Since by Theorem 6.12

$$\widetilde{\Omega}_r(Z) = \Omega_r(Z), \tag{6.37}$$

we obtain that the mapping Φ provides a linear isomorphism of the spaces $\Omega_\bullet(X,Y)$ and $\Omega_\bullet(Z)$. Moreover, Φ commutes with ∂, which follows from (5.2) and the product rule of Proposition 6.3. Hence, Φ is an isomorphism of the chain complexes $\Omega_\bullet(X,Y)$ and $\Omega_\bullet(Z)$, which finishes the proof. $\blacksquare$

Proof of Theorem 5.15. The proof of Theorem 5.15 is obtained from the proof of Theorem 6.12 by "search and replace" operation. Indeed, we need only to make the following changes in the proof of Theorem 6.12:

- Remove everywhere the sign $\times$ of cross product, so that the cross product $u \times v$ of two paths u on X and v on Y will be replaced by their join uv. The same applies to the cross product $U \times V$ of subspaces $U \subset \mathcal{A}_p(X)$ and $V \subset \mathcal{A}_q(Y)$: it is replaced by the join UV that is by the space spanned by all joins uv with $u \in U$ and $v \in V$.
- Replace everywhere $\mathcal{A}_p(X)$ by $\mathcal{A}'_p(X)$ and $\Omega_p(X)$ by $\Omega'_p(X)$.
- Replace the (implicitly used) range $p \geq 0, q \geq 0$ of the parameters p,q by $p \geq 0, q \geq -1$.

Let us verify that after these changes the proof remains valid. For that we only need to trace the places where the properties of the cross product were used and replace them by the corresponding properties (and references) of join. Here is the list of the properties of cross product that were used in the proof of Theorem 6.12, and their replacements for join.

1. If $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{A}_q(Y)$ then $u \times v \in \mathcal{A}_{p+q}(Z)$, which follows immediately from the definition of the cross product. The same property is true for join: if $u \in \mathcal{A}'_p(X)$ and $v \in \mathcal{A}_q(Y)$ then $uv \in \mathcal{A}_{p+q}(Z)$, which is also a trivial consequence of the definition.

2. Proposition 6.5: if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$. It should be replaced by Proposition 5.4: if $u \in \Omega'_p(X)$ and $v \in \Omega_q(Y)$ then $uv \in \Omega_{p+q}(Z)$.

3. Lemma 6.9: any path $w \in \Omega_r(Z)$ is a unique linear combination of the products $e_x \times e_y$, where x is an allowed path on X and y — that on Y. It should be replaced by the following property of join: any path $w \in \mathcal{A}_r(Z)$ is a unique linear combination of joins $e_x e_y$ with x and y as above, which is a trivial consequence of the definition of join of path complexes.

4. Lemma 6.11: if $u \in \Omega_p^{\perp}(X)$ and $v \in \mathcal{A}_q(Y)$ then $u \times v \in \Omega_{p+q}^{\perp}(Z)$. It should be replaced by Lemma 5.14: if $u \in \Omega_p'^{\perp}(X)$ and $v \in \mathcal{A}_q(Y)$ then $uv \in \Omega_{p+q}^{\perp}(Z)$.

By these observations we finish the proof. ■

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