Path homologies of digraphs

Alexander Grigor’yan
Nankai University and Bielefeld University

Yau’s MSC, Tsinghua University, November 1, 2 and 8, 2017
Based on a joint work with Yong Lin, Y.Muranov and S.-T.Yau
1 Paths in a finite set

Let $V$ be a finite set. For any $p \geq 0$, an elementary $p$-path is any sequence $i_0, ..., i_p$ of $p + 1$ vertices of $V$ that will be denoted by $i_0...i_p$ or by $e_{i_0...i_p}$. A $p$-path over a field $\mathbb{K}$ is any formal $\mathbb{K}$-linear combinations of elementary $p$-paths, that is, any $p$-path has a form

$$u = \sum_{i_0, i_1, ..., i_p \in V} u^{i_0i_1...i_p} e_{i_0i_1...i_p},$$

where $u^{i_0i_1...i_p} \in \mathbb{K}$.

Denote by $\Lambda_p = \Lambda_p(V)$ the $\mathbb{K}$-linear space of all $p$-paths. For example,

$$\Lambda_0 = \text{span}\{e_i : i \in V\}$$
$$\Lambda_1 = \text{span}\{e_{ij} : i, j \in V\}$$
$$\Lambda_2 = \text{span}\{e_{ijk} : i, j, k \in V\}$$

**Definition.** Define for any $p \geq 1$ a linear boundary operator $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0...i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0...\hat{i}_q...i_p},$$

where $\hat{\cdot}$ means omission of the index. For $p = 0$ set $\partial e_i = 0$. 
For example,
\[ \partial e_{ij} = e_j - e_i \quad \text{and} \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}. \]

We claim that \( \partial^2 = 0 \). Indeed, for any \( p \geq 2 \) we have

\[
\partial^2 e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \ldots \hat{i}_q \ldots i_p} = \\
= \sum_{q=0}^{p} (-1)^q \left( \sum_{r=0}^{q-1} (-1)^r e_{i_0 \ldots \hat{i}_r \ldots \hat{i}_q \ldots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \ldots \hat{i}_q \ldots \hat{i}_r \ldots i_p} \right) = \\
= \sum_{0 \leq r < q \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i}_r \ldots \hat{i}_q \ldots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i}_q \ldots \hat{i}_r \ldots i_p}.
\]

After switching \( q \) and \( r \) in the last sum we see that the two sums cancel out, whence \( \partial^2 e_{i_0 \ldots i_p} = 0 \). This implies \( \partial^2 u = 0 \) for all \( u \in \Lambda_p \).

Hence, we obtain a chain complex \( \Lambda_\ast(V) \):

\[
0 \leftarrow \Lambda_0 \xleftarrow{\partial} \Lambda_1 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Lambda_{p-1} \xleftarrow{\partial} \Lambda_p \xleftarrow{\partial} \ldots
\]
Definition. An elementary \( p \)-path \( e_{i_0...i_p} \) is called regular if \( i_k \neq i_{k+1} \) for all \( k = 0, ..., p-1 \), and irregular otherwise.

Let \( I_p \) be the subspace of \( \Lambda_p \) spanned by irregular \( e_{i_0...i_p} \). We claim that \( \partial I_p \subset I_{p-1} \). Indeed, if \( e_{i_0...i_p} \) is irregular then \( i_k = i_{k+1} \) for some \( k \). We have

\[
\partial e_{i_0...i_p} = e_{i_1...i_p} - e_{i_0i_2...i_p} + ... \\
+ (-1)^k e_{i_0...i_{k-1}i_{k+1}i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1}i_ki_{k+2}...i_p} \\
+ ... + (-1)^p e_{i_0...i_{p-1}}. 
\]

By \( i_k = i_{k+1} \) the two terms in the middle line of (1) cancel out, whereas all other terms are non-regular, whence \( \partial e_{i_0...i_p} \in I_{p-1} \).

Hence, \( \partial \) is well-defined on the quotient spaces \( \mathcal{R}_p := \Lambda_p/I_p \), and we obtain the chain complex \( \mathcal{R}_* (V) : \)

\[
0 \leftarrow \mathcal{R}_0 \leftarrow \mathcal{R}_1 \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow ... 
\]

By setting all irregular \( p \)-paths to be equal to 0, we can identify \( \mathcal{R}_p \) with the subspace of \( \Lambda_p \) spanned by all regular paths. For example, if \( i \neq j \) then \( e_{iji} \in \mathcal{R}_2 \) and

\[
\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}
\]

because \( e_{ii} = 0 \).
2 Paths in a digraph

Definition. A digraph (directed graph) is a pair $G = (V, E)$ of a set $V$ of vertices and a set $E \subset \{V \times V \setminus \text{diag}\}$ of (directed) edges. If $(i, j) \in E$ then we write $i \rightarrow j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary $p$-path $i_0...i_p$ on $V$ is called allowed if $i_k \rightarrow i_{k+1}$ for any $k = 0, ..., p - 1$, and non-allowed otherwise.

Let $A_p = A_p(G)$ be $\mathbb{K}$-linear space spanned by allowed elementary $p$-paths:

$$A_p = \text{span} \left\{ e_{i_0...i_p} : i_0...i_p \text{ is allowed} \right\}.$$

The elements of $A_p$ are called allowed $p$-paths. Since any allowed path is regular, we have $A_p \subset R_p$.

We would like to build a chain complex based on subspaces $A_p$ of $R_p$. However, the spaces $A_p$ are in general not invariant for $\partial$. For example, in the digraph

$$\bullet \rightarrow \bullet \rightarrow \bullet$$

we have $e_{abc} \in A_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin A_1$ because $e_{ac}$ is not allowed.
Consider the following subspace of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p (G) := \{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \}.$$

We claim that $\partial \Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial (\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

**Definition.** The elements of $\Omega_p$ are called $\partial$-invariant $p$-paths or currents.

Hence, we obtain a chain complex $\Omega_* = \Omega_* (G)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \ldots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

**Definition.** Path homologies of $G$ are defined as the homologies of the chain complex $\Omega_* (G)$:

$$H_p (G, \mathbb{K}) = H_p (G) := H_p (\Omega_* (G)) = \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

Betti numbers: $\beta_p (G) := \dim H_p (G)$. The Euler characteristic:

$$\chi (G) = \sum_{p=0}^{\infty} (-1)^p \beta_p (G) = \sum_{p=0}^{\infty} (-1)^p \dim \Omega_p (G).$$
3 Examples of $\partial$-invariant paths

An $1$-path $e_{ab}$ is $\partial$-invariant if and only if it is allowed, that is, $a \rightarrow b$.

A triangle is a sequence of three vertices $a, b, c$ such that $a \rightarrow b \rightarrow c, a \rightarrow c$.
A triangle determines a $2$-path $e_{abc} \in \Omega_2$ because $e_{abc} \in A_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in A_1$.

A snake of length $p \geq 2$ is a sequence of $p + 1$ vertices, say $0, 1, \ldots, p$, such that $i \rightarrow i + 1$ for all $i = 0, \ldots, p - 1$ and $i \rightarrow i + 2$ for all $i = 0, \ldots, p - 2$.

Then a $p$-path $u = e_{01 \ldots p}$ is $\partial$-invariant, because $u \in A_p$ and

$$\partial u = \sum_{q=0}^{p} (-1)^q e_{0\ldots (q-1)\bar{q}(q+1)\ldots p} \in A_{p-1}, \text{ since } q - 1 \rightarrow q + 1.$$
A \textit{p-simplex} is a sequence of \( p + 1 \) vertices, say, 0, 1, ..., \( p \) such that \( i \to j \) for all \( i < j \). Equivalently, a \( p \)-simplex is a directed \textit{clique}. A \( p \)-simplex contains a snake so that the \( p \)-path \( e_{01...p} \) is \( \partial \)-invariant. Since

\[
\partial e_{012...p} = e_{12...p} - e_{02...p} + ... + (-1)^p e_{01...(p-1)},
\]

the boundary of \( p \)-simplex is an alternating sum of \((p-1)\)-simplexes.

An 1-simplex is any arrow \( a \to b \).

A 2-simplex is a triangle as above.

A 3-simplex is shown here:
A square is a sequence of four vertices $a, b, b', c$ such that $a \to b$, $b \to c$, $a \to b'$, $b' \to c$. A square determines a 2-path $u := e_{abc} - e_{ab'}c \in \Omega_2$ because $u \in A_2$ and
\[
\partial u = (e_{bc} - \overline{e_{ac}} + e_{ab}) - (e_{b'c} - \overline{e_{ac}} + e_{ab'}) \\
= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in A_1
\]

A 3-cube is a sequence of 8 vertices, say, 0, 1, 2, 3, 4, 5, 6, 7, connected by arrows as here. A 3-cube determines a $\partial$-invariant 3-path
\[
u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}
\]
Indeed, $\nu \in A_3$ and
\[
\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) \\
- (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in A_2.
\]
An exotic cube is this subgraph containing 9 vertices and 15 edges. It is obtained from 3-cube by “splitting” the vertex 4 into 4, 4′ and adding the edges $4 \to 7$, $4′ \to 7$.

The exotic cube determines the following $\partial$-invariant 3-path:

$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{04′67} - e_{0267}.$$

Indeed, we have $u \in \mathcal{A}_3$ and

$$\partial u = e_{237} - e_{037} + e_{027} - e_{023}$$
$$-e_{137} + e_{037} - e_{017} + e_{013}$$
$$+ e_{157} - e_{057} + e_{017} - e_{015}$$
$$- e_{457} + e_{057} - e_{047} + e_{045}$$
$$+ e_{4′67} - e_{067} + e_{04′7} - e_{04′6}$$
$$- e_{267} + e_{067} - e_{027} + e_{026} \in \mathcal{A}_2.$$
4 Examples of digraphs and spaces $\Omega_p$

Consider the following digraph with 6 vertices and 8 edges:

$\Omega_0 = A_0 = \text{span} \{e_0, e_1, e_2, e_3, e_4, e_5\}$,
$\Omega_1 = A_1 = \text{span} \{e_{01}, e_{02}, e_{13}, e_{23}, e_{24}, e_{53}, e_{54}\}$

Hence, $\dim \Omega_0 = 6$ and $\dim \Omega_1 = 8$

$A_2 = \text{span} \{e_{013}, e_{014}, e_{023}, e_{024}\}$, $\dim A_2 = 4$

However, none of these 2-paths is $\partial$-invariant.

$\Omega_2$ is spanned by two squares:
$\Omega_2 = \text{span} \{e_{013} - e_{023}, e_{014} - e_{024}\}$, $\dim \Omega_2 = 2$.

There are no allowed $p$-paths for any $p \geq 3$.

Hence, $\Omega_p = A_p = \{0\}$ for all $p \geq 3$.

One computes $\dim H_0 = \dim H_1 = 1$ and $\dim H_p = 0$ for $p \geq 2$.

In fact, $H_0 = \text{span} \{e_0\}$, $H_1 = \text{span} \{e_{13} - e_{53} + e_{54} - e_{14}\}$.

The Euler characteristic: $\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 = 6 - 8 + 2 = 0$. 
Consider the following octahedral digraph with 6 vertices and 12 edges:

\[ \Omega_0 = A_0 = \text{span}\ \{e_0, e_1, e_2, e_3, e_4, e_5\}. \]
\[ \Omega_1 = A_1 = \text{span}\{e_{01}, e_{02}, e_{04}, e_{05}, e_{13}, e_{14}, e_{15}, e_{23}, e_{24}, e_{34}, e_{52}, e_{53}\}. \]
Hence, \( \dim \Omega_0 = 6, \ \dim \Omega_1 = 12. \)
\[ A_2 = \text{span}\ \{e_{013}, e_{014}, e_{015}, e_{023}, e_{024}, e_{052}, e_{053}, e_{134}, e_{152}, e_{153}, e_{234}, e_{523}, e_{524}, e_{534}\}. \]

Space \( \Omega_2 \) is spanned by 8 triangles:
\( e_{014}, e_{015}, e_{024}, e_{052}, e_{134}, e_{153}, e_{234}, e_{523} \)
and 3 squares:
\( e_{013} - e_{023}, \ e_{013} - e_{053}, \ e_{524} - e_{534}. \)
Hence, \( \dim \Omega_2 = 8 + 3 = 11. \)

Space \( \Omega_3 \) is spanned by five \( \partial \)-invariant 3-paths:
\( e_{0153}, \ e_{0523}, \ e_{5234}, \ e_{0134} - e_{0234}, \ e_{0534} - e_{0134} - e_{0524}. \)
Hence, \( \dim \Omega_3 = 5. \)

\[ \Omega_4 = \text{span}\ \{e_{05234}\}. \] Hence, \( \dim \Omega_4 = 1. \)

There is only 1 allowed 5-path \( e_{015234} \) but it is not \( \partial \)-invariant. Hence, \( \Omega_p = \{0\} \ \forall p \geq 5. \)
The Euler characteristic is
\[ \chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 - \dim \Omega_3 + \dim \Omega_4 = 6 - 12 + 11 - 5 + 1 = 1. \]

One can show that \( \dim H_0 = 1 \) and \( \dim H_p = 0 \) for all \( p \geq 1 \), which confirms \( \chi = 1 \).

Here is a verification of the \( \partial \)-invariance of five 3-paths and the 4-path:

\[ \partial e_{0153} = e_{153} - e_{053} + e_{013} - e_{015} \in A_2 \]
\[ \partial e_{0523} = e_{523} - e_{023} + e_{053} - e_{052} \in A_2 \]
\[ \partial e_{5234} = e_{234} - e_{534} + e_{524} - e_{523} \in A_2 \]
\[ \partial (e_{0134} - e_{0234}) = e_{134} - e_{034} + e_{014} - e_{013} - e_{234} + e_{034} - e_{024} + e_{023} = e_{134} + e_{014} - e_{013} - e_{234} - e_{024} + e_{023} \in A_2 \]
\[ \partial (e_{0534} - e_{0134} - e_{0524}) = e_{534} - e_{034} + e_{054} - e_{053} - e_{134} + e_{034} - e_{014} + e_{013} - e_{524} + e_{024} - e_{054} + e_{052} = e_{534} - e_{053} - e_{134} - e_{014} + e_{013} - e_{524} + e_{024} + e_{052} \in A_2 \]
\[ \partial e_{05234} = e_{5234} - e_{0234} + e_{0534} - e_{0524} + e_{0523} \in A_3 \]
5 Cross product of paths

Given two finite sets $X, Y$, consider their product

$$Z = X \times Y = \{(a, b) : a \in X \text{ and } b \in Y\}.$$ 

Let $z = z_0z_1...z_r$ be a regular elementary $r$-path on $Z$, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that $z$ is stair-like if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1}z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

Given a stair-like path $z$ on $Z$, define its projection onto $X$ as an elementary path $x$ on $X$ obtained from $z$ by removing $Y$-components in all the vertices of $z$ and then by collapsing in the resulting sequence any subsequence of repeated vertices to one vertex. In the same way define projection of $z$ onto $Y$ and denote it by $y$.

Projections $x = x_0...x_p$ and $y = y_0...y_q$ are regular elementary paths, and $p + q = r$. 

![Diagram](image.png)
Every vertex \((x_i, y_j)\) of path \(z\) can be represented as a point \((i, j)\) of \(\mathbb{Z}^2\) so that path \(z\) is represented by a staircase \(S(z)\) in \(\mathbb{Z}^2\) connecting points \((0, 0)\) and \((p, q)\).

Define the \textit{elevation} \(L(z)\) of \(z\) as the number of cells in \(\mathbb{Z}_+^2\) below the staircase \(S(z)\).

For given elementary regular paths \(x\) on \(X\) and \(y\) on \(Y\), denote by \(\Sigma_{x,y}\) the set of all stair-like paths \(z\) on \(Z\) whose projections on \(X\) and \(Y\) are respectively \(x\) and \(y\).

\textbf{Definition.} Define the \textit{cross product} of the paths \(e_x\) and \(e_y\) as a path \(e_x \times e_y\) on \(Z\) as follows:

\[
e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z. \tag{2}
\]

Then extend the cross product by linearity to all paths \(u \in \mathcal{R}_p(X)\) and \(v \in \mathcal{R}_q(Y)\) so that \(u \times v \in \mathcal{R}_{p+q}(Z)\).
Example. Let us denote the vertices on $X$ by letters $a, b, c$ etc and the vertices on $Y$ by integers $1, 2, 3$, etc so that the vertices on $Z$ can be denoted as $a1, b2$ etc as the fields on the chessboard. Then we have

$$e_a \times e_{12} = e_{a1a2}, \quad e_{ab} \times e_1 = e_{a1b1}$$

$$e_{ab} \times e_{12} = e_{a1b1b2} - e_{a1a2b2}$$

$$e_{ab} \times e_{123} = e_{a1b1b2b3} - e_{a1a2b2b3} + e_{a1a2a3b3}$$

$$e_{abc} \times e_{123} = e_{a1b1c1c2c3} - e_{a1b1b2c2c3} + e_{a1b1b2b3c3} + e_{a1a2b2c2c3} - e_{a1a2b2b3c3} + e_{a1a2a3b3c3}$$

**Proposition 1** If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p,q \geq 0$, then

$$\partial (u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v).$$
Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs $X$ and $Y$, define their Cartesian product as a digraph $Z = X \Box Y$ as follows:

- the set of vertices of $Z$ is $X \times Y$, that is, the vertices of $Z$ are the couples $(a, b)$ where $a \in X$ and $b \in Y$;

- the edges in $Z$ are of two types: $(a, b) \to (a', b)$ where $a \to a'$ (a horizontal edge) and $(a, b) \to (a, b')$ where $b \to b'$ (a vertical edge):

$$
\begin{align*}
&b' \bullet \quad \ldots \quad (a, b') \quad \longrightarrow \quad (a', b') \quad \ldots \\
&\uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
&b \bullet \quad \ldots \quad (a, b) \quad \longrightarrow \quad (a', b) \quad \ldots \\
\end{align*}
$$

$$
\begin{align*}
&Y \quad \frac{\ldots}{\ldots} \quad \bullet \quad \longrightarrow \quad \bullet \quad \ldots \\
&x \quad \text{a} \quad \longrightarrow \quad \text{a'} \quad \ldots
\end{align*}
$$

It follows that any allowed elementary path in $Z$ is stair-like.
Moreover, any regular elementary path on $Z$ is allowed if and only if it is stair-like and its projections onto $X$ and $Y$ are allowed.

It follows from definition (2) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \Rightarrow u \times v \in \mathcal{A}_{p+q}(Z).$$

(3)

Furthermore, the following is true.

**Proposition 2** If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.

**Proof.** $u \times v$ is allowed by (3). Since $\partial u$ and $\partial v$ are allowed, by (3) also $\partial u \times v$ and $u \times \partial v$ are allowed. By the product rule, $\partial (u \times v)$ is also allowed. Hence, $u \times v \in \Omega_{p+q}(Z)$. 

**Theorem 3 (Main Theorem)** Then any $\partial$-invariant path $w$ on $Z = X \square Y$ admits a representation in the form

$$w = \sum_{i=1}^{k} u_i \times v_i$$

for some finite $k$, where $u_i$ and $v_i$ are $\partial$-invariant paths on $X$ and $Y$, respectively.
Theorem 4 (Küneth formula) Let $X, Y$ be two finite digraphs and $Z = X \Box Y$. Then we have the following isomorphism of the chain complexes:

\[
\Omega_* (Z) \cong \Omega_* (X) \otimes \Omega_* (Y).
\]  

(4)

It is given by the map $u \otimes v \mapsto u \times v$ with $u \in \Omega_* (X)$ and $v \in \Omega_* (Y)$.

A more detailed version of (4) is the following: for any $r \geq 0$,

\[
\Omega_r (Z) \cong \bigoplus_{\{p,q \geq 0: p+q=r\}} \left( \Omega_p (X) \otimes \Omega_q (Y) \right).
\]

(5)

By an abstract theorem of Küneth, we obtain from (4)

\[
H_* (Z) \cong H_* (X) \otimes H_* (Y),
\]

that is, for any $r \geq 0$,

\[
H_r (Z) \cong \bigoplus_{\{p,q \geq 0: p+q=r\}} \left( H_p (X) \otimes H_q (Y) \right).
\]

(6)

Consequently,

\[
\beta_r (Z) = \sum_{\{p,q \geq 0: p+q=r\}} \beta_p (X) \beta_q (Y).
\]
Example. Consider the digraph $Z = X \square Y$ where $X$ is an interval and $Y$ is a square:

\[
X = a \cdot \rightarrow b \quad \text{and} \quad Y = \begin{array}{c}\uparrow \\
0 \rightarrow 1\end{array}
\]

$Z$ has 8 vertices $(i, j)$ where $i = a, b$, $j = 0, 1, 2, 3$. Let us enumerate them: $(a, i) \equiv i$ and $(b, i) \equiv i + 4$.

We see that $Z$ is a 3-cube:

We have:

\[
\begin{aligned}
\Omega_1 (X) &= \text{span} \{e_{ab}\}, \quad \Omega_p (X) = 0 \text{ for } p \geq 2, \\
\Omega_1 (Y) &= \text{span} \{e_{01}, e_{13}, e_{23}, e_{02}\}, \quad \Omega_2 (Y) = \text{span} \{e_{013} - e_{023}\}, \\
\Omega_q (Y) &= 0 \text{ for } q \geq 3.
\end{aligned}
\]

By (5) we obtain

\[
\Omega_3 (Z) \cong \Omega_1 (X) \otimes \Omega_2 (Y) = \text{span} \{e_{ab} \times e_{013} - e_{ab} \times e_{023}\}.
\]
\[ e_{ab} \times e_{013} = e_{a0b0b1b3} - e_{a0a1b1b3} + e_{a0a1a3b3} = e_{0457} - e_{0157} + e_{0137} \]

and

\[ e_{ab} \times e_{023} = e_{0467} - e_{0267} + e_{0237} \]

Hence, we obtain

\[ \Omega_3 (Z) = \text{span} \{ e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237} \} \]

that is the \( \partial \)-invariant 3-path associated with 3-cube.

Define \( n \)-cube as follows:

\[ n \text{-cube} = I \square I \square \ldots \square I, \]

where \( I = ^a \bullet \longrightarrow ^b \). Similarly one shows that \( \Omega_n (n \text{-cube}) \) is spanned by a single \( n \)-path that is an alternating sum of \( n! \) elementary \( n \)-paths connecting the vertices \( 0 \) and \( 2^n - 1 \). This corresponds to partitioning of a solid \( n \)-dim cube into \( n! \) simplexxes.
Example. Consider the digraph $Z = X □ Y$ where

$$X = \begin{array}{c}
\bullet \quad \bullet \\
\downarrow & \downarrow \\
\quad & \quad \\
\uparrow & \uparrow \\
0 & 2 \\
\bullet & \bullet \\
\end{array}$$

and $Y = \begin{array}{c}
\bullet \\
\uparrow \\
\bullet \\
0 \\
\quad \\
\end{array}$

One can show that

$$H_1(X) = \text{span}\{e_{ab} + e_{bc} + e_{ca}\}, \quad H_p(X) = 0 \text{ for } p \geq 2$$

$$H_1(Y) = \text{span}\{-e_{10} + e_{02} + e_{23} - e_{13}\}, \quad H_q(Y) = 0 \text{ for all } q \geq 2$$

By (6) we obtain

$$H_1(Z) = H_0(X) \otimes H_1(Y) + H_1(X) \otimes H_0(Y) \cong \mathbb{K}^2,$$

$$H_2(Z) \cong H_1(X) \otimes H_1(Y) = \text{span}\{(e_{ab} + e_{bc} + e_{ca}) \times (-e_{10} + e_{02} + e_{23} - e_{13})\} \cong \mathbb{K},$$

and $H_r(Z) = 0$ for all $r \geq 2$. 

21
7 Homotopy of digraphs

For vertices $a, b$ of a digraph, write $a \equiv b$ if either $a \to b$ or $a = b$. Let $X$ and $Y$ be two digraphs.

Definition. A mapping $f : X \to Y$ called a digraph map (or morphism) if

$$a \to b \text{ on } X \implies f(a) \equiv f(b) \text{ on } Y.$$ 

Any digraph map $f : X \to Y$ induces a linear map

$$f_* : A_p(X) \to A_p(Y), \quad f_* (e_{i_0 \ldots i_p}) = e_{f(i_0) \ldots f(i_p)}.$$

It is easy to check that $f_* \partial = \partial f_*$, which implies that $f_*$ provides a morphism of chain complexes $f_* : \Omega_p(X) \to \Omega_p(Y)$ and, consequently, a homomorphism of homology groups $f_* : H_p(X) \to H_p(Y)$.

Definition. For any $n \geq 1$ define a line digraph $I_n$ as any digraph with $n + 1$ vertices \{0, 1, \ldots, n\} and such that, for any $i = 0, \ldots, n - 1$ holds either $i \to (i + 1)$ or $(i + 1) \to i$, and there is no other arrow.
Definition. Let $X, Y$ be two digraphs. Two digraph maps $f, g: X \to Y$ are called \textit{homotopic} if there exists a line digraph $I_n$ and a digraph map $\Phi: X \square I_n \to Y$ such that

$$\Phi|_{X \times \{0\}} = f \quad \text{and} \quad \Phi|_{X \times \{n\}} = g.$$ 

In this case we write $f \simeq g$. The map $\Phi$ is called a \textit{homotopy} between $f$ and $g$.

Definition. Two digraphs $X$ and $Y$ are called \textit{homotopy equivalent} if there exist digraph maps

$$f: X \to Y, \quad g: Y \to X$$

such that

$$f \circ g \simeq \text{id}_Y, \quad g \circ f \simeq \text{id}_X.$$  

In this case we write $X \simeq Y$.

Theorem 5  \textit{(i)} Let $f, g: X \to Y$ be two digraph maps. If $f \simeq g$ then they induce the identical maps of homology groups:

$$f_*: H_p(X) \to H_p(Y) \quad \text{and} \quad g_*: H_p(X) \to H_p(Y).$$

\textit{(ii)} If the digraphs $X$ and $Y$ are homotopy equivalent, then $H_*(X) \cong H_*(Y)$.
In particular, if a digraph $X$ is contractible, that is, if $X \simeq \{\ast\}$, then all the homology groups of $X$ are trivial except for $H_0$.

We say that a digraph $Y$ is a subgraph of $X$ if the set of vertices of $Y$ is a subset of that of $X$ and the arrows of $Y$ are all those arrows of $X$ whose adjacent vertices belong to $Y$.

**Definition.** Let $X$ be a digraph and $Y$ be its subgraph. A retraction of $X$ onto $Y$ is a digraph map $r : X \to Y$ such that $r|_Y = \text{id}_Y$.

**Theorem 6** Let $r : X \to Y$ be a retraction of a digraph $X$ onto a subgraph $Y$. Assume that

\[
either x \equiv r(x) \text{ for all } x \in X \text{ or } r(x) \equiv x \text{ for all } x \in X. \tag{9}\]

Then $X \simeq Y$ and, consequently, $H_\ast(X) \simeq H_\ast(Y)$.

A retraction that satisfies (9) is called a deformation retraction.

**Example.** Let us show that $n$-cube is contractible. Indeed, a natural projection of $n$-cube onto $(n - 1)$-cube is a deformation retraction. Hence, by induction we obtain $n$-cube $\simeq \{\ast\}$.
Example. Consider the digraph $X$ as here.

Let $Y$ be its subgraph with the vertex set $\{1, 3, 4\}$. Consider a retraction $r : X \to Y$ given by $r(0) = 1$, $r(2) = 3$. It is easy to see that $r$ is a deformation retraction, whence $X \simeq Y$. Then we obtain

$$H_1(X) \cong H_1(Y) = \text{span} \{e_{13} + e_{34} + e_{41}\} \cong \mathbb{K}$$

and $H_p(X) = \{0\}$ for $p \geq 2$. 
8 Summary

Fix a finite set $V$ and a field $\mathbb{K}$. For any $p \geq 0$, set $\mathcal{R}_p = \text{span}_\mathbb{K}\{e_{i_0...i_p} : i_0...i_p \text{ is regular}\}$, where “regular” means that $i_k \neq i_{k+1}$ for all $k$. There is a boundary operator $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}$ such that $\partial^2 = 0$.

Let $G = (V, E)$ be a digraph. Set $\mathcal{A}_p = \text{span}_\mathbb{K}\{e_{i_0...i_n} : i_0...i_p \text{ is allowed}\} \subset \mathcal{R}_p$, where “allowed” means that $i_k \to i_{k+1}$ for all $k$.

Spaces of $\partial$-invariant paths: $\Omega_p = \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}$.

Chain complex $\Omega_\ast (G)$: $0 \leftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \ldots$.

Path homology: $H_p (G) = \ker \partial|_{\Omega_p} / \text{Im} \partial|_{\Omega_{p+1}}$.

**Theorem 4** $\Omega_\ast (X \Box Y) \cong \Omega_\ast (X) \otimes \Omega_\ast (Y)$ and $H_\ast (X \Box Y) \cong H_\ast (X) \otimes H_\ast (Y)$

A mapping $f : X \to Y$ is called a digraph map if $a \to b$ in $X$ implies $f (a) \equiv f (b)$ in $Y$.

We have also defined *homotopy equivalence* $X \simeq Y$ of two digraphs.

**Theorem 5** If $X \simeq Y$ then $H_\ast (X) \cong H_\ast (Y)$.
Theorem 6  If $Y$ is a subgraph of $X$ then $X \simeq Y$ provided there exists a deformation retraction $r: X \to Y$, that is:

(i) $r|_Y = \text{id}$;
(ii) $r$ is a digraph map;
(iii) either $x \equiv r(x)$ for all $x \in X$ or $r(x) \equiv x$ for all $x \in X$.

For example, consider digraphs:
The left hand side digraph is contractible as there is a sequence of two deformation retractions reducing it to $\{\ast\}$:
$r_1(4) = r_1(5) = 3$
$r_2(1) = r_2(2) = 3$

The right hand side digraph differs only by one arrow $3 \to 1$, but it is not contractible because $H_2 \neq \{0\}$

$H_2 = \text{span} \{e_{124} + e_{234} + e_{314} - e_{125} - e_{235} - e_{315}\}$
9 Undirected graphs

If $G = (V, E)$ is an undirected graph then it can be turned into a digraph by allowing both arrows $x \rightarrow y$ and $y \rightarrow x$ whenever $x \sim y$. All the above results can be reformulated for undirected graphs in an obvious way.

**Example.** Fix integers $1 \leq k \leq n$ and a set $S$ of $n$ elements. The *Johnson graph* $J(n, k)$ is the graph whose vertices are $k$-subsets of $S$, and the edges are defined as follows: two $k$-subsets are connected by an edge if their intersection contains $k - 1$ elements of $S$.

Let us describe $J(4, 2)$. Taking $S = \{1, 2, 3, 4\}$, we see that the vertices of $J(4, 2)$ are the pairs $43, 42, 41, 32, 31, 31$. The graph $J(4, 2)$ has twelve edges:
Johnson graphs are a special class of undirected graphs defined from systems of sets. The vertices of the Johnson graph $J(n, k)$ are the $k$-element subsets of an $n$-element set; two vertices are adjacent when the intersection of the two vertices (subsets) contains $(k - 1)$-elements.¹ Both Johnson graphs and the closely related Johnson scheme are named after Selmer M. Johnson.

### Contents

1. Special cases
2. Graph-theoretic properties
3. Automorphism group
4. Intersection array
5. Eigenvalues and Eigenvectors
6. Relation to Johnson scheme
7. Open Problems
8. References
9. External links

### Special cases

The Johnson graph $J(5, 2)$

<table>
<thead>
<tr>
<th>Named after</th>
<th>Selmer M. Johnson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertices</td>
<td>$\binom{n}{k}$</td>
</tr>
<tr>
<td>Edges</td>
<td>$\frac{k(n - k)}{2} \binom{n}{k}$</td>
</tr>
</tbody>
</table>
Proposition 7  For all \( n > k \geq 1 \) we have \( J(n, k) \simeq J(n - 1, k) \).

Consequently, \( J(n, k) \simeq J(n - 1, k) \simeq \ldots \simeq J(k, k) = \{\ast\} \), and all the homology groups of \( J(n, k) \) are trivial.

For the proof, assume that \( J(n, k) \) is constructed over the set \( S = \{1, \ldots, n - 1, n\} \), so that graph \( J(n - 1, k) \) is a subgraph of \( J(n, k) \). Then there exists a deformation retraction \( r : J(n, k) \to J(n - 1, k) \). Here is a deformation retraction \( r : J(4, 2) \to J(3, 2) \):
In general, we construct $r$ as follows. Any vertex $a$ of $J(n,k)$ is represented by a monotone decreasing sequence $a = a_1 a_2 ... a_k$ of integers from \{1, ..., $n$\}: $n \geq a_1 > a_2 > ... > a_k \geq 1$. Define $r(a) = a' = a_1' a_2' ... a_k'$ where

$$a_1' = \min (a_1, n - 1), \quad a_2' = \min (a_2, n - 2), \quad ... \quad a_k' = \min (a_k, n - k).$$

Then $n - 1 \geq a_1' > a_2' > ... > a_k' \geq 1$, so that $a'$ is a vertex of $J(n - 1, k)$. We claim that $r : J(n,k) \to J(n-1,k)$ is a deformation retraction.

(i) If $a \in J(n-1,k)$ then $r(a) = a$ because $a_1 \leq n - 1, a_2 \leq n - 2, ..., a_k \leq n - k$, which implies $a_i' = a_i$.

(ii) If $a \sim b$ in $J(n,k)$ then $r(a) \sim r(b)$ or $r(a) = r(b)$ because sequences $a_1 ... a_k$ and $b_1 ... b_k$ have $k - 1$ common elements, whence it follows that $a'$ and $b'$ have at least $k - 1$ common elements.

(iii) If $a \in J(n,k) \setminus J(n-1,k)$ then $r(a) \sim a$. In this case $a_1 = n$. Assume $a_2 \leq n - 2$. Then $a_3 \leq n - 3, ..., a_k \leq n - k$, which implies

$$a_1' = n - 1, \quad a_2' = a_2, ..., \quad a_k' = a_k$$

that is, $r(a) = (n - 1) a_2 ... a_k$ and $r(a) \sim a$. The case $a_2 = n - 1$ is a bit more involved.
10  $C$-homotopy of loops

For any digraph $G$ and a vertex $\ast$ of $G$, denote by $G^{\ast}$ a based digraph.

Definition. A loop on $G^{\ast}$ is a digraph map $\varphi: I_n \to G$ such that $\varphi(0) = \varphi(n) = \ast$.

Here $I_n$ is any line digraph with any $n \geq 0$.

Definition. Consider in $G^{\ast}$ two loops $\varphi: I_n \to G$ and $\psi: I_m \to G$. An one-step direct $C$-homotopy from $\varphi$ to $\psi$ is a digraph map $h: I_n \to I_m$ such that

(a) $h(0) = 0$, $h(n) = m$ and $h(i) \leq h(j)$ whenever $i \leq j$;

(b) $\varphi(i) \overset{C}{\equiv} \psi(h(i))$ for all $i \in I_n$.

If in (b) holds $\varphi(i) \overset{C}{\equiv} \psi(h(i))$ for all $i \in I_n$ then $h$ is called an one-step inverse $C$-homotopy.

We denote an one-step direct $C$-homotopy with $\varphi \overset{C}{\to} \psi$ and the one-step inverse $C$-homotopy with $\varphi \overset{C}{\leftarrow} \psi$. 
**Example.** On the next diagram we have $\varphi \xrightarrow{C} \psi$.

Condition (b) means that $\varphi$ and $\psi$ provide a digraph map from the digraph on the left panel to $G$.

**Definition.** We call two loops $\varphi, \psi$ $C$-homotopic and write $\varphi \simeq_{C} \psi$ if there exists a finite sequence $\{\varphi_k\}_{k=0}^{m}$ of loops in $G^*$ such that $\varphi_0 = \varphi$, $\varphi_m = \psi$ and, for any $k = 0, ..., m - 1$, holds $\varphi_k \xrightarrow{C} \varphi_{k+1}$ or $\varphi_k \xleftarrow{C} \varphi_{k+1}$.

Obviously, $C$-homotopy is an equivalence relation. A loop $\varphi$ is called contractible if $\varphi \simeq e$ where $e : I_0 \to G$ is a trivial loop.
The following theorem gives an efficient way of verifying if two loops are $C$-homotopic.

Any loop $\varphi: I_n \to G$ defines a sequence $\theta_{\varphi} = \{\varphi(i)\}_{i=0}^n$ of vertices of $G$. We consider $\theta_{\varphi}$ as a word over the alphabet $V$.

**Theorem 8** Two loops $\varphi: I_n \to G$ and $\psi: I_m \to G$ are $C$-homotopic if and only if $\theta_{\psi}$ can be obtained from $\theta_{\varphi}$ by a finite sequence of the following word transformations (or inverses to them):

(i) ...$abc$... $\mapsto$ ...$ac$... where $a, b, c$ is a triangle in $G$ or any permutation of a triangle.

(ii) ...$abc$... $\mapsto$ ...$adc$... where $a, b, c, d$ is a square in $G$ or any cyclic permutation of a square or an inverse cyclic permutation of a square.

(iii) ...$abcd$... $\mapsto$ ...$ad$... where $a, b, c, d$ is as in (ii).

(iv) ...$aba$... $\mapsto$ ...$a$... if $a \to b$ or $b \to a$.

(v) ...$aa$... $\mapsto$ ...$a$...
Examples

1. Consider a triangular loop
   \( \varphi : (0 \to 1 \to 2 \leftarrow 3) \to G \)

   It is contractible because
   
   \[ \theta_\varphi = abca \quad \text{\( (i) \)} \quad \sim \quad aca \quad \text{\( (iv) \)} \quad \sim \quad a. \]

2. Consider a square loop
   \( \varphi : (0 \to 1 \to 2 \leftarrow 3 \leftarrow 4) \to G \)

   It is contractible because
   
   \[ \theta_\varphi = abcd\quad \text{\( (iii) \)} \quad \sim \quad ada \quad \text{\( (iv) \)} \quad \sim \quad a. \]
3. Consider the loops $\varphi : I_5 \rightarrow G$ and $\psi : I_3 \rightarrow G$ as on p. 33. It is shown here how to transform $\theta_\varphi$ to $\theta_\psi$ by means of Theorem 8: using successively transformations $(i)^-$, $(i), (ii)$ and $(iii)$.
11 Fundamental group

The $C$-homotopy equivalence class of a loop $\varphi : I_n \to G$ will be denoted by $[\varphi]$. For any two loops $\varphi : I_n \to G$ and $\psi : I_m \to G$ define their concatenation $\varphi \vee \psi : I_{n+m} \to G$ by

$$\varphi \vee \psi(i) = \begin{cases} 
\varphi(i), & 0 \leq i \leq n \\
\psi(i - n), & n \leq i \leq n + m.
\end{cases}$$

Then the product $[\varphi] \cdot [\psi] := [\varphi \vee \psi]$ of equivalence classes is then well-defined.

**Theorem 9** (a) The set of all equivalence classes $[\varphi]$ with the above product is a group with the neutral element $[e]$. It is denoted by $\pi_1(G^*)$.

(b) Any based digraph map $f : X^* \to Y^*$ induces a group homomorphism

$$\pi_1(f) : \pi_1(X^*) \to \pi_1(Y^*), \quad (\pi_1(f))[\varphi] = [f \circ \varphi].$$

(c) If $f, g : X^* \to Y^*$ are two digraph maps then $f \simeq g$ implies $\pi_1(f) = \pi_1(g)$.

(d) If $X, Y$ are connected and $X \simeq Y$ then $\pi_1(X^*) \cong \pi_1(Y^*)$. 
Theorem 10  For any based connected digraph $G^*$ we have an isomorphism

$$\pi_1(G^*) / [\pi_1(G^*), \pi_1(G^*)] \cong H_1(G, \mathbb{Z}),$$

where $[\pi_1(G^*), \pi_1(G^*)]$ is a commutator subgroup.
12 Application to graph coloring

An illustration of the theory of digraph homotopy, we give here a new proof of the classical lemma of Sperner, using the notion the fundamental group of digraphs.

Consider a triangle $ABC$ on the plane $\mathbb{R}^2$ and its triangulation $T$. Assume that the set of vertices of $T$ is colored in three colors 1, 2, 3 so that:

- the vertex $A$ is colored in 1, $B$ in 2, $C$ in 3;
- each vertex on the side $AB$ is colored in 1 or 2, on the side $AC$ in 1 or 3, on the side $BC$ in 2 or 3.

**Lemma of Sperner.** Under the above conditions, there exists in $T$ a 3-color triangle, that is, a triangle, whose vertices are colored with three different colors.
Let us first modify the triangulation $T$ so that there are no vertices on the sides $AB, AC, BC$ except for $A, B, C$. If $X \in AB$ then move $X$ a bit inside of $ABC$. A new triangle $XYZ$ arises, where $Y, Z$ are former neighbors of $X$ on $AB$. However, since $X, Y, Z$ are colored in two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from all sides of $ABC$.

Consider the triangulation $T$ as a graph and make it into a digraph $G$ as follows. If $a, b$ are two vertices on $T$ and $a \sim b$ then choose direction between $a, b$ using the colors of $a, b$ and the following rule:

$$1 \rightarrow 2, \ 2 \rightarrow 3, \ 3 \rightarrow 1$$

$$1 \leftrightarrow 1, \ 2 \leftrightarrow 2, \ 3 \leftrightarrow 3$$

Denote by $S$ the following colored digraph and define a mapping $f : G \rightarrow S$ to preserve colors of vertices. Then $f$ is a digraph map by the choice of arrows in $G$.

Consider a 3-loop $\varphi$ on $G^*$ (with $* = A$) with the word

$$\theta_\varphi = ABCA.$$

For the loop $f \circ \varphi$ on $S$ we have $\theta_{f \circ \varphi} = 1231$. This loop is not contractible because none of the transformations of Theorem 8 can be applied to the word $1231$. By Theorem 9(b), the loop $\varphi$ is also not contractible and, hence, $\pi_1(G^*) \neq \{0\}$.
Assume now that there is no 3-color triangle in $T$. Then each triangle from $T$ looks in $G$ like

\[
\begin{array}{c}
\bullet \leftrightarrow \bullet \\
\downarrow \\
\bullet \leftrightarrow \bullet \\
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet \leftrightarrow \bullet \\
\downarrow \\
\bullet \leftrightarrow \bullet \\
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet \leftrightarrow \bullet \\
\downarrow \\
\bullet \leftrightarrow \bullet \\
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet \leftrightarrow \bullet \\
\downarrow \\
\bullet \leftrightarrow \bullet \\
\end{array}
\]

In particular, each of them contains a triangle in the sense of Theorem 8. Using the partition of $G$ into the triangles and transformations $(\text{ii})$ and $(\text{iv})$ of Theorem 8, we contract any loop on $G$ to the empty word, which contradicts to $\pi_1(G') \neq \{0\}$.