Path homology and join of digraphs

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## 1 Paths in a finite set

Let $V$ be a finite set. For any $p \geq 0$, an *elementary $p$-path* is any sequence $i_0, \ldots, i_p$ of $p + 1$ vertices of $V$.

Fix a field $K$ and denote by $\Lambda_p = \Lambda_p(V, K)$ the $K$-linear space that consists of all formal $K$-linear combinations of elementary $p$-paths in $V$. Any element of $\Lambda_p$ is called a $p$-path.

An elementary $p$-path $i_0, \ldots, i_p$ as an element of $\Lambda_p$ will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

\[ \Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \quad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle, \quad \text{etc} \]

Define also an elementary $(-1)$-path as the unity $e$ of $K$ so that

\[ \Lambda_{-1} = \langle e \rangle = K. \]

Any $p$-path $u$ can be written in a form

\[ u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p}, \]

where $u^{i_0 i_1 \ldots i_p} \in K$. 
**Definition.** Define for any $p \geq 0$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$
\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \hat{i}_q \ldots i_p},
$$

where $\hat{}$ means omission of the index.

For example,

$$
\partial e_i = e, \quad \partial e_{ij} = e_j - e_i, \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}, \quad \text{etc.}
$$

**Lemma 1.1** $\partial^2 = 0$.

**Proof.** Indeed, for any $p \geq 1$ we have

$$
\partial^2 e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \ldots \hat{i}_q \ldots i_p} = \sum_{q=0}^{p} (-1)^q \left( \sum_{r=0}^{q-1} (-1)^r e_{i_0 \ldots \hat{i}_r \ldots \hat{i}_q \ldots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \ldots \hat{i}_q \ldots \hat{i}_r \ldots i_p} \right)
$$

$$
= \sum_{0 \leq r < q \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i}_r \ldots \hat{i}_q \ldots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} e_{i_0 \ldots \hat{i}_q \ldots \hat{i}_r \ldots i_p}.
$$
After switching $q$ and $r$ in the last sum we see that the two sums cancel out, whence
\[ \partial^2 e_{i_0...i_p} = 0. \] This implies \( \partial^2 u = 0 \) for all \( u \in \Lambda_p. \)

Hence, we obtain a chain complex \( \Lambda_*(V) \):

\[
0 \leftarrow \Lambda_{-1} \xleftarrow{\partial} \Lambda_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Lambda_{p-1} \xleftarrow{\partial} \Lambda_p \xleftarrow{\partial} \ldots
\]

**Definition.** An elementary \( p \)-path \( e_{i_0...i_p} \) is called *regular* if \( i_k \neq i_{k+1} \) for all \( k = 0, \ldots, p-1 \), and irregular otherwise.

Let \( I_p \) be the subspace of \( \Lambda_p \) spanned by irregular \( e_{i_0...i_p} \). We claim that \( \partial I_p \subset I_{p-1} \). Indeed, if \( e_{i_0...i_p} \) is irregular then \( i_k = i_{k+1} \) for some \( k \). We have

\[
\partial e_{i_0...i_p} = e_{i_1...i_p} - e_{i_0i_2...i_p} + \ldots + (-1)^k e_{i_0...i_{k-1}i_{k+1}i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1}i_ki_{k+2}...i_p} + \ldots + (-1)^p e_{i_0...i_{p-1}}. \quad (1.1)
\]

By \( i_k = i_{k+1} \) the two terms in the middle line of (1.1) cancel out, whereas all other terms are non-regular, whence \( \partial e_{i_0...i_p} \in I_{p-1} \).

Hence, \( \partial \) is well-defined on the quotient spaces \( \mathcal{R}_p := \Lambda_p/I_p \), and we obtain the chain complex \( \mathcal{R}_*(V) \):

\[
0 \leftarrow \mathcal{R}_0 \xleftarrow{\partial} \mathcal{R}_1 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \mathcal{R}_{p-1} \xleftarrow{\partial} \mathcal{R}_p \xleftarrow{\partial} \ldots
\]
By setting all irregular $p$-paths to be equal to 0, we can identify $R_p$ with the subspace of $\Lambda_p$ spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in R_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$.

## 2 Chain complex and path homology of a digraph

**Definition.** A digraph (directed graph) is a pair $G = (V, E)$ of a set $V$ of vertices and a set $E \subset \{V \times V \setminus \text{diag}\}$ of arrows (directed edges). If $(i, j) \in E$ then we write $i \rightarrow j$.

**Definition.** Let $G = (V, E)$ be a digraph. An elementary $p$-path $i_0...i_p$ on $V$ is called allowed if $i_k \rightarrow i_{k+1}$ for any $k = 0, ..., p - 1$, and non-allowed otherwise.

Let $A_p = A_p(G)$ be $K$-linear space spanned by allowed elementary $p$-paths:

$$A_p = \langle e_{i_0...i_p} : i_0...i_p \text{ is allowed} \rangle.$$

The elements of $A_p$ are called allowed $p$-paths. Since any allowed path is regular, we have $A_p \subset R_p$. 
We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general not invariant for $\partial$. For example, in the digraph

$$\bullet \rightarrow \bullet \rightarrow \bullet$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because $e_{ac}$ is not allowed.

**Definition.** A $p$-path $u$ is called $\partial$-invariant if $u \in \mathcal{A}_p$ and $\partial u \in \mathcal{A}_{p-1}$.

The space of $\partial$-invariant paths is denoted by $\Omega_p$:

$$\Omega_p = \{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \}.$$ 

Important: $\partial \Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial (\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Hence, we obtain a chain complex $\Omega_* = \Omega_* (G)$:

$$0 \leftarrow \Omega_{-1} \stackrel{\partial}{\leftarrow} \Omega_0 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \ldots$$

Note that $\Omega_{-1} = \mathbb{K}$, $\Omega_0 = \mathcal{A}_0 = \langle e_i, i \in V \rangle$ and $\Omega_1 = \mathcal{A}_1 = \langle e_{ij}, i \rightarrow j \rangle$, while in general $\Omega_p \subset \mathcal{A}_p$. 


3 Examples of $\partial$-invariant paths

A *triangle* is a sequence of three vertices $a, b, c$ such that
\[ a \to b \to c, \ a \to c. \]
It determines 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and
\[ \partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1. \]

A *square* is a sequence of four vertices $a, b, b', c$ such that
\[ a \to b, \ b \to c, \ a \to b', \ b' \to c. \]
It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and
\[ \partial u = (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1 \]

*In general, $\Omega_2$ has a basis that consists of triangles and squares and double arrows $e_{aba}$.***
A *p-simplex* (or *p-clique*) is a sequence of $p+1$ vertices, say, $0, 1, ..., p$, such that

\[ i \rightarrow j \iff i < j. \]

It determines a $p$-path $e_{01...p} \in \Omega_p$.

1-simplex is $\bullet \rightarrow \bullet$, 2-simplex is a triangle.
Here is a 3-simplex:

A *3-cube* is a sequence of 8 vertices $0, 1, 2, 3, 4, 5, 6, 7$, connected by arrows as here:
It determines a $\partial$-invariant 3-path

\[ u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3 \]

because $u \in \mathcal{A}_3$ and

\[
\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267})
- (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2
\]
4 Homology groups

Alongside the chain complex

\[ 0 \xleftarrow{\partial} \Omega_{-1} \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \ldots \]  \hspace{1cm} (4.1)

consider also a \textit{truncated} chain complex

\[ 0 \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \ldots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \ldots \]  \hspace{1cm} (4.2)

The homology groups of (4.2) are called the \textit{path homology groups} of the digraph \( G \) and denoted by \( \tilde{H}_p \), that is,

\[ H_p = \ker \partial|_{\Omega_p} / \text{Im} \partial|_{\Omega_{p+1}}. \]

The homology groups of (4.1) are called the \textit{reduced} path homology groups of \( G \) and are denoted by \( \tilde{H}_p \). We have

\[ \tilde{H}_p = H_p \text{ for } p \geq 1 \text{ and } \tilde{H}_0 = H_0/\mathbb{K}. \]

Define the Betti numbers \( \beta_p = \dim H_p \) and the reduced Betti numbers \( \tilde{\beta}_p = \dim \tilde{H}_p \) so that

\[ \tilde{\beta}_p = \beta_p \text{ for } p \geq 1 \text{ and } \tilde{\beta}_0 = \beta_0 - 1. \]
It is known that $\beta_0$ is equal to the number of connected components of $G$. In particular, if $G$ is connected then $\tilde{\beta}_0 = 0$.

If $G = X \sqcup Y$ - a disjoin union of two digraphs $X, Y$ then

$$\beta_r (X \sqcup Y) = \beta_r (X) + \beta_r (Y)$$

and

$$\tilde{\beta}_r (X \sqcup Y) = \tilde{\beta}_r (X) + \tilde{\beta}_r (Y) + 1\{r = 0\}.$$

In what follows, for a vector space $S$ over $\mathbb{K}$ we write $|S| = \dim_{\mathbb{K}} S$.

5 Examples of spaces $\Omega_p$ and $H_p$

A linear digraph of $n$ vertices:

$|\Omega_0| = n, \quad |\Omega_1| = n - 1$,

$\Omega_p = \{0\}$ for $p \geq 2$,

$\tilde{H}_p = \{0\}$ for all $p \geq 0$.  

$\bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \ldots \leftarrow \bullet \rightarrow \bullet$

$0 \rightarrow 1 \rightarrow 2 \leftarrow 3 \rightarrow \ldots \leftarrow n-2 \rightarrow n-1$
A triangle as a digraph:
\[ \Omega_1 = \langle e_{01}, e_{02}, e_{12} \rangle, \quad \Omega_2 = \langle e_{012} \rangle, \quad \Omega_p = \{0\} \text{ for } p \geq 3 \]
\[ \ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12} \rangle \]
but \[ e_{01} - e_{02} + e_{12} = \partial e_{012} \]
so that \( H_1 = \{0\} \). We have \( \tilde{H}_p = \{0\} \) for all \( p \geq 0 \).

A square as a digraph:
\[ \Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23} \rangle, \quad \Omega_2 = \langle e_{013} - e_{023} \rangle, \quad \Omega_p = \{0\} \text{ for } p \geq 3 \]
\[ \ker \partial|_{\Omega_1} = \langle e_{01} + e_{13} - e_{02} - e_{23} \rangle \]
but \[ e_{01} + e_{13} - e_{02} - e_{23} = \partial (e_{013} - e_{023}) \]
so that \( H_1 = \{0\} \). We have \( \tilde{H}_p = \{0\} \) for all \( p \geq 0 \).

A hexagon:
\[ |\Omega_0| = |\Omega_1| = 6, \quad \Omega_p = \{0\} \text{ for all } p \geq 2. \]
\[ H_1 = \langle e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50} \rangle, \quad \tilde{H}_p = \{0\} \text{ for } p \neq 1. \]
The same is true for any cyclic digraph (directed polygon) that is neither triangle nor square:
\[ |H_1| = 1 \text{ and } \tilde{H}_p = \{0\} \text{ for all } p \neq 1. \]
Octahedron: $|\Omega_0| = 6, \quad |\Omega_1| = 12$

Space $\Omega_2$ is spanned by 8 triangles:

$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$

$|\Omega_2| = 8, \quad \Omega_p = \{0\} \text{ for all } p \geq 3$

$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \rangle$

$|H_2| = 1, \quad \tilde{H}_p = \{0\} \text{ for all } p \neq 2.$

Octahedron with different orientation:

$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$

$\Omega_3 = \langle e_{0234} - e_{0134}, e_{0235} - e_{0135} \rangle$

$|\Omega_2| = 9, \quad |\Omega_3| = 2, \quad \Omega_p = \{0\} \text{ for all } p \geq 4.$

$\ker \partial|\Omega_2 = \langle u, v \rangle \text{ where}$

$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$

$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$

but $H_2 = \{0\}$ because

$u = \partial (e_{0234} - e_{0134}) \text{ and } v = \partial (e_{0235} - e_{0135})$

So, $\tilde{H}_p = \{0\} \text{ for all } p \geq 0.$
A 3-cube:

We have $|\Omega_0| = 8$, $|\Omega_1| = 12$.
Space $\Omega_2$ is spanned by 6 squares:

$$\Omega_2 = \langle e_{013} - e_{023}, \ e_{015} - e_{045}, \ e_{026} - e_{046}, \ e_{137} - e_{157}, \ e_{237} - e_{267}, \ e_{457} - e_{467} \rangle$$

hence, $|\Omega_2| = 6$.

Space $\Omega_3$ is spanned by one 3-cube:

$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \rangle$$

hence, $|\Omega_3| = 1$.

$\Omega_p = \{0\}$ for all $p \geq 4$ and $\tilde{H}_p = \{0\}$ for all $p \geq 0$. 
6 A join of two digraphs

Given two digraphs $X, Y$, define their join $X \ast Y$ as follows: take first a disjoint union $X \sqcup Y$ and add arrows from any vertex of $X$ to any vertex of $Y$.

For example,

$$\{0, 1\} \ast \{2, 3\} = \begin{array}{ccc}
3 & \leftarrow & 1 \\
0 & \rightarrow & 2
\end{array} \quad \text{and} \quad \begin{array}{ccc}
3 & \leftarrow & 1 \\
0 & \rightarrow & 2
\end{array} \ast \{4, 5\} = \begin{array}{ccc}
3 & \leftarrow & 1 \\
0 & \rightarrow & 2
\end{array}
$$

Define the join $uv$ of $p$-path $u$ on $X$ and $q$-path $v$ on $Y$ as a $(p + q + 1)$-path on $X \ast Y$ as follows: first define it for elementary paths by

$$e_{i_0 \ldots i_p} e_{j_0 \ldots j_q} = e_{i_0 \ldots i_p j_0 \ldots j_q}$$

and then extend this definition by linearity to all $p$-paths $u$ on $X$ and $q$-paths $v$ on $Y$. 
If $u$ and $v$ are allowed on $X$ resp. $Y$ then $uv$ is allowed on $Z = X \ast Y$.

**Lemma 6.1** The join of paths satisfies the product rule

$$\partial (uv) = (\partial u) v + (-1)^{p+1} u \partial v.$$ 

If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $\partial u$ and $\partial v$ are allowed, which implies that $\partial (uv)$ is also allowed, that is, $uv \in \Omega_{p+q+1}(Z)$. The product rule implies also that the join $uv$ is well defined for homology classes $u \in \tilde{H}_p(X)$ and $v \in \tilde{H}_q(Y)$ so that $uv \in \tilde{H}_{p+q+1}(Z)$. 
Theorem 6.2 (Künneth formula) We have the following isomorphism: for any $r \geq -1$,

$$
\Omega_r (X \ast Y) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} (\Omega_p (X) \otimes \Omega_q (Y))
$$

(6.1)

that is given by the map $u \otimes v \mapsto uv$ with $u \in \Omega_p (X)$ and $v \in \Omega_q (Y)$, and, for any $r \geq 0$,

$$
\widetilde{H}_r (X \ast Y) \cong \bigoplus_{\{p,q \geq 0: p+q=r-1\}} \widetilde{H}_p (X) \otimes \widetilde{H}_q (Y)
$$

(6.2)

$$
\widetilde{\beta}_r (X \ast Y) \cong \sum_{\{p,q \geq 0: p+q=r-1\}} \widetilde{\beta}_p (X) \widetilde{\beta}_q (Y).
$$

(6.3)

The identity (6.1) means that any paths in $\Omega_r (Z)$ can be obtained as linear combination of joins $uv$ where $u \in \Omega_p (X)$ and $v \in \Omega_q (Y)$ with $p + q + 1 = r$, and (6.2) means the same for homology classes. Note that that the operation $\ast$ of digraphs is associative. For a sequence $X_1, \ldots, X_l$ of $l$ digraphs we obtain by induction from (6.1), (6.2) and (6.3) that

$$
\Omega_r (X_1 \ast X_2 \ast \ldots \ast X_l) \cong \bigoplus_{\{p_i \geq -1: p_1+p_2+\ldots+p_l=r-l+1\}} \Omega_{p_1} (X_1) \otimes \ldots \otimes \Omega_{p_l} (X_l)
$$

(6.4)

$$
\widetilde{H}_r (X_1 \ast X_2 \ast \ldots \ast X_l) \cong \bigoplus_{\{p_i \geq 0: p_1+p_2+\ldots+p_l=r-l+1\}} \widetilde{H}_{p_1} (X_1) \otimes \ldots \otimes \widetilde{H}_{p_l} (X_l)
$$

(6.5)

$$
\widetilde{\beta}_r (X_1 \ast X_2 \ast \ldots \ast X_l) = \sum_{\{p_i \geq 0: p_1+p_2+\ldots+p_l=r-l+1\}} \widetilde{\beta}_{p_1} (X_1) \ldots \widetilde{\beta}_{p_l} (X_l).
$$

(6.6)
**Example.** Consider an octahedron $Z = X_1 \ast X_2 \ast X_3$ where

$$X_1 = \{0, 1\}, \quad X_2 = \{2, 3\}, \quad X_3 = \{4, 5\}.$$  

(see p. 15). Then

$$\Omega_2 (Z) = \bigoplus_{\{p_i \geq -1: \ p_1 + p_2 + p_3 = 2 - 3 + 1\}} \Omega_{p_1} (X_1) \otimes \Omega_{p_2} (X_2) \otimes \Omega_{p_3} (X_3)$$

$$= \Omega_0 (X_1) \otimes \Omega_0 (X_2) \otimes \Omega_0 (X_3)$$

$$= \langle e_0, e_1 \rangle \otimes \langle e_2, e_3 \rangle \otimes \langle e_4, e_5 \rangle$$

$$= \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135} \rangle$$

and

$$H_2 (Z) = \tilde{H}_2 (Z) = \bigoplus_{\{p_i \geq 0: \ p_1 + p_2 + p_3 = 2 - 3 + 1\}} \tilde{H}_{p_1} (X_1) \otimes \tilde{H}_{p_2} (X_2) \otimes \tilde{H}_{p_3} (X_3)$$

$$= \tilde{H}_0 (X_1) \otimes \tilde{H}_0 (X_2) \otimes \tilde{H}_0 (X_3)$$

$$= \langle e_0 - e_1 \rangle \otimes \langle e_2 - e_3 \rangle \otimes \langle e_4 - e_5 \rangle$$

$$= \langle e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135} \rangle.$$  

(see p. 13).
A generalized join of digraphs

Given a digraph $G$ of $l$ vertices $\{1, 2, ..., l\}$ and a sequence $X_1, ..., X_l$ of $l$ digraphs, define their generalized join $(X_1 ... X_l)_G = X_G$ as follows: $X_G$ is obtained from the disjoint union $\bigcup_i X_i$ of digraphs $X_i$ by keeping all the arrows in each $X_i$ and by adding arrows $x \rightarrow y$ whenever $x \in X_i$, $y \in X_j$ and $i \rightarrow j$ in $G$.

Digraph $X_G$ is also referred to as a $G$-join of $X_1, ..., X_l$, and $G$ is called the base of $X_G$. 
The main problem to be discussed here is

how to compute the homology groups and Betti numbers of $X_G$.

Denote by $K_l$ a complete digraph with vertices $\{1, ..., l\}$ and arrows

$$i \to j \iff i < j$$

that is, $K_l$ is an $(l - 1)$-simplex. For example, $K_2 = \{1 \to 2\}$ and $K_3 = \{1 \to 2 \to 3, 1 \to 3\}$ is a triangle.

The digraph $X_{K_l}$ is called a complete join of $X_1, ..., X_l$. It is easy to see that

$$X_{K_l} = X_1 \ast X_2 \ast ... \ast X_l$$

It follows from (6.6) that, for any $r \geq 0$,

$$\tilde{\beta}_r (X_{K_l}) = \sum_{\{p_i \geq 0: p_1 + p_2 + ... + p_l = r - l + 1\}} \tilde{\beta}_{p_1} (X_1) \ast \tilde{\beta}_{p_l} (X_l). \quad (7.1)$$
A monotone linear join

Denote by $I_l$ a monotone linear digraph with the vertices $\{1, \ldots, l\}$ and arrows $i \rightarrow i + 1$:

$$I_l = \{1 \rightarrow 2 \rightarrow \ldots \rightarrow l\}.$$  \hfill (8.1)

If $G = I_l$ then we use the following simplified notation:

$$(X_1X_2\ldots X_l)_{I_l} = X_1X_2\ldots X_l$$

and refer to this digraph as a monotone linear join of $X_1, \ldots, X_l$.

Clearly, $X_1X_2\ldots X_n$ can be constructed as follows: take first a disjoint union $\bigsqcup_{i=1}^{l} X_i$ and then add arrows from any vertex of $X_i$ to any vertex of $X_{i+1}$ (see p. 19).

In the case $l = 2$ we obviously have $X_1X_2 = X_1 \ast X_2$ but in general $X_1X_2\ldots X_l$ is a subgraph of $X_1 \ast X_2 \ast \ldots \ast X_l$. For example, we have

\[
\begin{align*}
\{0\} \{1, 2\} \{3\} &= \uparrow \quad \uparrow \\
0 \rightarrow 2 &\quad \text{while} \quad \{0\} \ast \{1, 2\} \ast \{3\} = \uparrow \quad \nearrow \quad \uparrow \\
1 \rightarrow 3 &\quad 0 \rightarrow 2
\end{align*}
\]  \hfill (8.2)
Theorem 8.1 We have

$$\tilde{H}_r (X_1 X_2 \ldots X_l) \cong \bigoplus_{\{p_i \geq 0: \ p_1 + p_2 + \ldots + p_l = r - l + 1\}} \tilde{H}_{p_1} (X_1) \otimes \ldots \otimes \tilde{H}_{p_l} (X_l) \quad (8.3)$$

and

$$\tilde{\beta}_r (X_1 X_2 \ldots X_l) = \sum_{\{p_i \geq 0: \ p_1 + p_2 + \ldots + p_l = r - l + 1\}} \tilde{\beta}_{p_1} (X_1) \ldots \tilde{\beta}_{p_l} (X_l). \quad (8.4)$$

By (6.5) and (8.3), $X_1 X_2 \ldots X_l$ and $X_1 \ast X_2 \ast \ldots \ast X_l$ are homologically equivalent.

Example. Let the base $G$ be a square:

We have $G = \{1\} \{2, 3\} \{4\}$ which implies that $X_G = X_1 (X_2 \sqcup X_3) X_4$. Hence, by Theorem 8.1,

$$\tilde{\beta}_r (X_G) = \sum_{\{p_i \geq 0: \ p_1 + p_2 + p_3 = r - 2\}} \tilde{\beta}_{p_1} (X_1) \tilde{\beta}_{p_2} (X_2 \sqcup X_3) \tilde{\beta}_{p_3} (X_4)$$

$$= \sum_{\{p_i \geq 0: \ p_1 + p_2 + p_3 = r - 2\}} \tilde{\beta}_{p_1} (X_1) \left( \tilde{\beta}_{p_2} (X_2) + \tilde{\beta}_{p_2} (X_3) + 1_{\{p_2 = 0\}} \right) \tilde{\beta}_{p_3} (X_4)$$

$$= \tilde{\beta}_r (X_1 X_2 X_4) + \tilde{\beta}_r (X_1 X_3 X_4) + \tilde{\beta}_{r-1} (X_1 X_4). \quad (8.5)$$
For a general base $G$, if $i_1...i_k$ is an arbitrary sequence of vertices in $G$ then denote

$$X_{i_1...i_k} = X_{i_1}X_{i_2}...X_{i_k}.$$ 

Note that by (8.4)

$$\tilde{\beta}_r (X_{i_1...i_k}) = \sum_{p_1+...+p_k=r-(k-1)} \tilde{\beta}_{p_1} (X_{i_1}) ... \tilde{\beta}_{p_k} (X_{i_k}),$$

and we consider the numbers $\tilde{\beta}_r (X_{i_1...i_k})$ as known.

Using this notation, we can rewrite (8.5) as follows: if $G$ is a square then

$$\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{124}) + \tilde{\beta}_r (X_{134}) + \tilde{\beta}_{r-1} (X_{14}).$$
**Example.** Let $G$ be an octahedron:

We have $G = \{1, 2\} \ast \{3, 4\} \ast \{5, 6\}$ whence

$$X_G = (X_1 \sqcup X_2) \ast (X_3 \sqcup X_4) \ast (X_5 \sqcup X_6)$$

By (7.1) we obtain

$$\tilde{\beta}_r(X_G) = \sum_{\{p_i \geq 0: p_1 + p_2 + p_3 = r-2\}} \tilde{\beta}_{p_1}(X_1 \sqcup X_2)\tilde{\beta}_{p_2}(X_3 \sqcup X_4)\tilde{\beta}_{p_3}(X_5 \sqcup X_6)$$

$$= \sum_{\{p_i \geq 0: p_1 + p_2 + p_3 = r-2\}} (\tilde{\beta}_{p_1}(X_1) + \tilde{\beta}_{p_1}(X_2) + 1_{\{p_1=0\}}) (\tilde{\beta}_{p_2}(X_3) + \tilde{\beta}_{p_2}(X_4) + 1_{\{p_2=0\}}) \times (\tilde{\beta}_{p_3}(X_5) \sqcup \tilde{\beta}_{p_3}(X_6) + 1_{\{p_3=0\}})$$

$$= \tilde{\beta}_r(X_{135}) + \tilde{\beta}_r(X_{145}) + \tilde{\beta}_r(X_{235}) + \tilde{\beta}_r(X_{245}) + \tilde{\beta}_r(X_{136}) + \tilde{\beta}_r(X_{146}) + \tilde{\beta}_r(X_{236}) + \tilde{\beta}_r(X_{246})$$

$$+ \tilde{\beta}_{r-1}(X_{13}) + \tilde{\beta}_{r-1}(X_{23}) + \tilde{\beta}_{r-1}(X_{14}) + \tilde{\beta}_{r-1}(X_{24}) + \tilde{\beta}_{r-1}(X_{15}) + \tilde{\beta}_{r-1}(X_{25})$$

$$+ \tilde{\beta}_{r-1}(X_{35}) + \tilde{\beta}_{r-1}(X_{45}) + \tilde{\beta}_{r-1}(X_{16}) + \tilde{\beta}_{r-1}(X_{16}) + \tilde{\beta}_{r-1}(X_{36}) + \tilde{\beta}_{r-1}(X_{46})$$

$$+ \tilde{\beta}_{r-2}(X_1) + \tilde{\beta}_{r-2}(X_2) + \tilde{\beta}_{r-2}(X_3) + \tilde{\beta}_{r-2}(X_4) + \tilde{\beta}_{r-2}(X_5) + \tilde{\beta}_{r-2}(X_6) + 1_{\{r=2\}}.$$
9 An arbitrary linear join

Let now $G$ be a linear digraph but not necessarily monotone. That is, the vertex set of $G$ is $\{1, ..., l\}$ and, for any pair $(i, i + 1)$ of consecutive numbers there is exactly one arrow: either $i \to i + 1$ or $i \leftarrow i + 1$.

**Definition.** We say that a vertex $v$ of $G$ is a *turning point* if $v$ has either two incoming arrows or two outcoming arrows. Denote by $T$ the set of all turning points.

An allowed path in $G$ is called *maximal* if it is not a proper subset (as a set of vertices) of another allowed path. Denote by $A_{\max}$ the family of all maximal allowed paths in $G$.

Clearly, the end vertices of a maximal path are either turning points or the vertices $1, l$. 
Theorem 9.1 If $G$ is an arbitrary linear digraph then

$$
\tilde{\beta}_r (X_G) = \sum_{u \in \mathcal{A}_{\text{max}}} \tilde{\beta}_r (X_u) + \sum_{v \in \mathcal{T}} \tilde{\beta}_{r-1} (X_v).
$$

In other words, $\tilde{\beta}_r (X_G)$ is the sum of all $\tilde{\beta}_r$ of the linear joins of $X_i$ along all maximal allowed paths in $G$ plus the sum of $\tilde{\beta}_{r-1}$ of all $X_v$ sitting at the turning points $v$.

**Example.** Consider the base

$$
G = \{1 \to 2 \leftarrow 3 \leftarrow 4 \to 5\}.
$$

Then $\mathcal{T} = \{2, 4\}$, while maximal paths of $L$ are

$$
\mathcal{A}_{\text{max}} = \{1 \to 2, \ 4 \to 3 \to 2, \ 4 \to 5\}.
$$

Hence, by Theorem 9.1,

$$
\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{12}) + \tilde{\beta}_r (X_{432}) + \tilde{\beta}_r (X_{45}) + \tilde{\beta}_{r-1} (X_2) + \tilde{\beta}_{r-1} (X_4).
$$
Example. Consider the following base:

\[
\begin{array}{cccc}
2 & \downarrow & \leftarrow & 5 \\
\downarrow & 1 & \leftarrow & 3 \\
4 & \leftarrow & \downarrow & 6 \\
\downarrow & 7
\end{array}
\]

\[
G = \{1\} \{2, 4\} \{3\} \{5, 7\} \{6\}
\]

It is easy to see that \( G \) itself is the following linear join:

\[
G = (\{1\} \{2, 4\} \{3\} \{5, 7\} \{6\})_L
\]

where \( L = \{\alpha \rightarrow \beta \leftarrow \gamma \leftarrow \delta \rightarrow \varepsilon\} \). Here the turning points of \( L \) are \( T = \{\beta, \delta\} \), while maximal paths of \( L \) are

\[
A_{\max} = \{\alpha \rightarrow \beta, \ \delta \rightarrow \gamma \rightarrow \beta, \ \delta \rightarrow \varepsilon\}.
\]

For \( L \)-join we have as above

\[
\tilde{\beta}_r (Y_L) = \tilde{\beta}_r (Y_{\alpha \beta}) + \tilde{\beta}_r (Y_{\delta \gamma \beta}) + \tilde{\beta}_r (Y_{\delta \varepsilon}) + \tilde{\beta}_{r-1} (Y_{\beta}) + \tilde{\beta}_{r-1} (Y_{\delta}).
\]

Setting \( Y_\alpha = X_1 \), \( Y_\beta = X_2 \sqcup X_3 \), \( Y_\gamma = X_3 \), \( Y_\delta = X_5 \sqcup X_7 \) and \( Y_\varepsilon = X_6 \) we obtain

\[
\tilde{\beta}_r (X_G) = \tilde{\beta}_r ((X_1 (X_2 \sqcup X_3) X_3 (X_5 \sqcup X_7) X_6)_L)
\]
\[
\begin{align*}
\tilde{\beta}_r(X_1(X_2 \sqcup X_4)) &+ \tilde{\beta}_r((X_5 \sqcup X_7)X_3(X_2 \sqcup X_4)) + \tilde{\beta}_r((X_5 \sqcup X_7)X_6) \\
&+ \tilde{\beta}_{r-1}(X_2 \sqcup X_4) + \tilde{\beta}_{r-1}(X_5 \sqcup X_7) \\
= &\tilde{\beta}_r(X_{12}) + \tilde{\beta}_r(X_{14}) + \tilde{\beta}_{r-1}(X_1) \\
&+ \tilde{\beta}_r(X_{532}) + \tilde{\beta}_r(X_{534}) + \tilde{\beta}_r(X_{732}) + \tilde{\beta}_r(X_{734}) \\
&\quad + \tilde{\beta}_{r-1}(X_{32}) + \tilde{\beta}_{r-1}(X_{34}) + \tilde{\beta}_{r-1}(X_{53}) + \tilde{\beta}_{r-1}(X_{73}) + \tilde{\beta}_{r-2}(X_3) \\
&+ \tilde{\beta}_r(X_{56}) + \tilde{\beta}_r(X_{76}) + \tilde{\beta}_{r-1}(X_6) \\
&+ \tilde{\beta}_{r-1}(X_2) + \tilde{\beta}_{r-1}(X_4) + 1_{\{r=1\}} + \tilde{\beta}_{r-1}(X_5) + \tilde{\beta}_{r-1}(X_7) + 1_{\{r=1\}}.
\end{align*}
\]

\[
\begin{align*}
\tilde{\beta}_r(X_G) &\quad = \quad \tilde{\beta}_r(X_{534}) + \tilde{\beta}_r(X_{532}) + \tilde{\beta}_r(X_{734}) + \tilde{\beta}_r(X_{732}) \\
&\quad + \tilde{\beta}_r(X_{12}) + \tilde{\beta}_r(X_{14}) + \tilde{\beta}_r(X_{56}) + \tilde{\beta}_r(X_{76}) \\
&\quad + \tilde{\beta}_{r-1}(X_{73}) + \tilde{\beta}_{r-1}(X_{53}) + \tilde{\beta}_{r-1}(X_{32}) + \tilde{\beta}_{r-1}(X_{34}) \\
&\quad + \tilde{\beta}_{r-1}(X_1) + \tilde{\beta}_{r-1}(X_2) + \tilde{\beta}_{r-1}(X_4) + \tilde{\beta}_{r-1}(X_5) + \tilde{\beta}_{r-1}(X_6) + \tilde{\beta}_{r-1}(X_7) \\
&\quad + \tilde{\beta}_{r-2}(X_3) + 2_{\{r=1\}}.
\end{align*}
\]
10 A cyclic join

A digraph $G$ is called *cyclic* if it is connected and each vertex has the undirected degree 2. Let $G$ be a cyclic digraph with the set of vertices $V = \{1, 2, \ldots, l\}$. We assume that the vertices are ordered so that every vertex $i \in V$ is connected by arrows to $i - 1$ and $i + 1$ (where $l$ is identified with 0). In the same way as above we define the set $\mathcal{A}_{\text{max}}$ and $\mathcal{T}$.

For example, consider the following hexagon:

Here $\mathcal{T} = \{1, 4\}$ and
$\mathcal{A}_{\text{max}} = \{4 \rightarrow 3 \rightarrow 2 \rightarrow 1, \ 4 \rightarrow 5 \rightarrow 6 \rightarrow 1\}$

**Theorem 10.1** Let $G$ be a cyclic digraph that is neither triangle nor square nor double arrow. Then

$$
\tilde{\beta}_r (X_G) = \sum_{u \in \mathcal{A}_{\text{max}}} \tilde{\beta}_r (X_u) + \sum_{v \in \mathcal{T}} \tilde{\beta}_{r-1} (X_v) + \tilde{\beta}_r (G).
$$

(10.1)

Note that in this case $\tilde{\beta}_r (G) = 1_{\{r=1\}}$. If $G$ is a triangle or square or double arrow then (10.1) is wrong, which is shown in Examples below.
**Example.** If $G$ is the above hexagon then we obtain

$$\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{4321}) + \tilde{\beta}_r (X_{4561}) + \tilde{\beta}_{r-1} (X_1) + \tilde{\beta}_{r-1} (X_4) + 1_{\{r=1\}}.$$ 

**Example.** Consider the following 4-cyclic base:

$$G = \begin{array}{c}
2 \rightarrow 3 \\
\uparrow & \downarrow \\
1 \rightarrow 4
\end{array}$$

Since $\mathcal{T} = \{1, 4\}$ and $\mathcal{A}_{\text{max}} = \{1 \rightarrow 2 \rightarrow 3 \rightarrow 4, 1 \rightarrow 4\}$, we obtain

$$\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{1234}) + \tilde{\beta}_r (X_{14}) + \tilde{\beta}_{r-1} (X_1) + \tilde{\beta}_{r-1} (X_4) + 1_{\{r=1\}}. \quad (10.2)$$

**Example.** Consider the following 3-cyclic base: $G = \begin{array}{c}
\begin{array}{c}
2 \end{array} \\
\begin{array}{c}
\downarrow \downarrow \downarrow \\
1 \leftarrow 3
\end{array}
\end{array}.$

Then $\mathcal{A}_{\text{max}}$ and $\mathcal{T}$ are empty, and we obtain $\tilde{\beta}_r (X_G) = 1_{\{r=1\}} = \tilde{\beta}_r (G)$.
Example. Consider the following tetrahedron as a base $G$:

We have $G = C \ast \{4\}$ where $C = \{1 \rightarrow 2 \rightarrow 3 \rightarrow 1\}$

It follows that $X_G = X_C \ast X_4$

and

$$\tilde{\beta}_r (X_G) = \sum_{p+q=r-1} \tilde{\beta}_p (X_C) \tilde{\beta}_q (X_4) = \sum_{p+q=r-1} 1_{\{p=1\}} \tilde{\beta}_q (X_4) = \tilde{\beta}_{r-2} (X_4).$$

Hence, $\tilde{\beta}_r (X_G) = \tilde{\beta}_{r-2} (X_4)$.

Example. Let $G$ be a triangle: $G = \xymatrix{ 1 \ar[r] & 2 \ar[r] & 3 }$. Then $X_G = X_1 \ast X_2 \ast X_3$ and we know that

$$\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{123}).$$

However, the right hand side of (10.1) is in this case

$$\tilde{\beta}_r (X_{123}) + \tilde{\beta}_{r-1} (X_1) + \tilde{\beta}_{r-1} (X_3) \neq \tilde{\beta}_r (X_G).$$
**Example.** Let $G$ be a square:

\[
\begin{array}{c@{}c@{}c@{}c@{}c@{}c@{}c@{}c}
 & 2 & \rightarrow & 4 \\
G = & \uparrow & & \uparrow \\
 & 1 & \rightarrow & 3 \\
\end{array}
\]

Then we that by (8.5)

\[
\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{124}) + \tilde{\beta}_r (X_{134}) + \tilde{\beta}_{r-1} (X_{14}),
\]

while the right hand side of (10.1) is in this case

\[
\tilde{\beta}_r (X_{124}) + \tilde{\beta}_r (X_{134}) + \tilde{\beta}_{r-1} (X_1) + \tilde{\beta}_{r-1} (X_4).
\]

**Example.** Let $G$ be a double arrow: $G = \{1 \leftrightarrows 2\}$. Then

\[
X_G = X_1 \ast X_2 \ast X_1
\]

whence $\tilde{\beta}_r (X_G) = \tilde{\beta}_r (X_{121})$. However, in this case $A_{\text{max}}$ and $T$ are empty, so that the right hand side of (10.1) is $\tilde{\beta}_r (G) = 0$. 

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**Example.** Let $G$ be as here:

We have

$G = \{1, 2, 3, 4\} \{5, 6\} \{7 \to 8 \to 9 \to 7\}$

so that

$X_G = (X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4) (X_5 \sqcup X_6) X_{\{7 \to 8 \to 9 \to 7\}}$

It follows that

$$\tilde{\beta}_r (X_G) = \sum_{p+q+s=r-2} \left( \tilde{\beta}_p (X_1) + \tilde{\beta}_p (X_2) + \tilde{\beta}_p (X_3) + \tilde{\beta}_p (X_4) + 3\{p=0\} \right) \times \left( \tilde{\beta}_q (X_5) + \tilde{\beta}_q (X_6) + 1\{q=0\} \right) 1\{s=1\}$$

which yields after computation

$$\tilde{\beta}_r (X_G) = \tilde{\beta}_{r-2}(X_{15}) + \tilde{\beta}_{r-2}(X_{16}) + \tilde{\beta}_{r-2}(X_{25}) + \tilde{\beta}_{r-2}(X_{26}) + \tilde{\beta}_{r-2}(X_{35}) + \tilde{\beta}_{r-2}(X_{36}) + \tilde{\beta}_{r-2}(X_{45}) + \tilde{\beta}_{r-2}(X_{46}) + \tilde{\beta}_{r-3}(X_1) + \tilde{\beta}_{r-3}(X_2) + \tilde{\beta}_{r-3}(X_3) + \tilde{\beta}_{r-3}(X_4) + 3\tilde{\beta}_{r-3}(X_5) + 3\tilde{\beta}_{r-3}(X_6) + 3\{r=3\}.$$
11 Homology of a generalized join

Theorem 11.1 There exists a finite sequence of paths $\{u_k\}$ in $G$ and a sequence $\{s_k\}$ of non-negative integers such that, for any sequence $\{X_i\}$ of digraphs and any $r \geq 0$,

$$
\tilde{\beta}_r(X_G) = \sum_k \tilde{\beta}_{r-s_k}(X_{u_k}) + \tilde{\beta}_r(G).
$$

(11.1)

Besides, the sequence $\{u_k\}$ contains all maximal allowed paths, and $u_k \in \mathcal{A}_{\text{max}} \iff s_k = 0$.

Example. Let the base $G$ be a cube.

Use description of paths $u_k$ from the proof of Theorem 11.1, we obtain

$$
\tilde{\beta}_r(X_G) = \tilde{\beta}_r(X_{1248}) + \tilde{\beta}_r(X_{1268}) + \tilde{\beta}_r(X_{1348}) + \tilde{\beta}_r(X_{1378}) + \beta_r(X_{1568}) + \beta_r(X_{1578}) + \beta_{r-1}(X_{178}) + \beta_{r-1}(X_{168}) + \beta_{r-1}(X_{148}) + \beta_{r-1}(X_{128}) + \beta_{r-1}(X_{138}) + \beta_{r-1}(X_{158}) + \beta_{r-2}(X_{18})
$$