Overview of path homology theory of digraphs II

Alexander Grigor’yan
University of Bielefeld

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University of Bielefeld

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1 Spaces of $\partial$-invariant paths

1.1 Paths and the boundary operator

Let us fix a finite set $V$ and a field $\mathbb{K}$. For any $p \geq 0$, an elementary $p$-path is any sequence $i_0, \ldots, i_p$ of $p + 1$ vertices of $V$; it will be denoted by $e_{i_0 \ldots i_p}$.

A $p$-path is any formal linear combinations of of elementary $p$-paths with coefficients in $\mathbb{K}$; that is, any $p$-path $u$ has a form

$$u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0i_1\ldots i_p} e_{i_0i_1\ldots i_p},$$

where $u^{i_0i_1\ldots i_p} \in \mathbb{K}$. The set of all $p$-paths is a $\mathbb{K}$-linear space denoted by $\Lambda_p = \Lambda_p(V, \mathbb{K})$.

For example, $\Lambda_0 = \langle e_i : i \in V \rangle$, $\Lambda_1 = \langle e_{ij} : i, j \in V \rangle$, $\Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle$.

Definition. Define for any $p \geq 1$ a linear boundary operator $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \hat{i}_q \ldots i_p},$$

(1.1)

where $\hat{\cdot}$ means omission of the index. For $p = 0$ set $\partial e_i = 0$ (and, hence, $\Lambda_{-1} = \{0\}$).
For example,
\[ \partial e_{ij} = e_j - e_i \quad \text{and} \quad \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}. \]
It is easy to show that \( \partial^2 = 0. \) Hence, we obtain a chain complex \( \Lambda_\ast (V): \)
\[
\begin{align*}
0 & \leftarrow \Lambda_0 \leftarrow \partial \Lambda_1 \leftarrow \ldots \leftarrow \partial \Lambda_{p-1} \leftarrow \partial \Lambda_p \leftarrow \ldots
\end{align*}
\]
An elementary \( p \)-path \( e_{i_0 \ldots i_p} \) is called \textit{regular} if \( i_k \neq i_{k+1} \) for all \( k = 0, \ldots, p-1, \) and \textit{irregular} otherwise. A \( p \)-path is called regular (resp. irregular) if it is a linear combination of regular (resp. irregular) elementary paths.
It is easy to show that if \( u \) is irregular then \( \partial u \) is also irregular. Denote by \( \mathcal{R}_p \) the space of all regular \( p \)-paths. Then \( \partial \) is well defined on the spaces \( \mathcal{R}_p \) if we identify all irregular paths with 0. For example, if \( i \neq j \) then \( e_{iji} \in \mathcal{R}_2 \) and
\[
\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij} \in \mathcal{R}_1,
\]
because \( e_{ii} = 0. \) Hence, we obtain a chain complex
\[
\begin{align*}
0 & \leftarrow \mathcal{R}_0 \leftarrow \partial \mathcal{R}_1 \leftarrow \ldots \leftarrow \partial \mathcal{R}_{p-1} \leftarrow \partial \mathcal{R}_p \leftarrow \ldots
\end{align*}
\]
1.2 Chain complex on digraphs

A digraph (directed graph) is a pair $G = (V, E)$ of a set $V$ of vertices and $E \subset \{V \times V \setminus \text{diag}\}$ is a set of arrows (directed edges). If $(i, j) \in E$ then we write $i \rightarrow j$.

**Definition.** Let $G = (V, E)$ be a digraph. An elementary $p$-path $e_{i_0 \ldots i_p}$ on $V$ is called allowed if $i_k \rightarrow i_{k+1}$ for any $k = 0, ..., p - 1$, and non-allowed otherwise. A $p$-path is called allowed if it is a linear combination of allowed elementary $p$-paths.

Let $\mathcal{A}_p = \mathcal{A}_p(G, \mathbb{K})$ be the space of all allowed $p$-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on spaces $\mathcal{A}_p$. However, in general $\partial$ does not act on the spaces $\mathcal{A}_p$. For example, in the digraph $\bullet \rightarrow \bullet \rightarrow \bullet$ we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because $e_{ac}$ is not allowed.

Consider the following subspace of $\mathcal{A}_p$:

$$\Omega_p \equiv \Omega_p(G, \mathbb{K}) := \{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \}.$$

We claim that $\partial \Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial (\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.
Definition. The elements of $\Omega_p$ are called $\partial$-invariant $p$-paths.

Hence, we obtain a chain complex $\Omega_\ast = \Omega_\ast (G, K)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \ldots$$  \hspace{1cm} (1.2)

that reflects the digraph structure of $G$. Homology groups of the chain complex (1.2) are called path homologies of $G$ and are denoted by $H_p(G)$.

By construction we have

$$\Omega_0 = A_0 = \langle e_i : i \in V \rangle \text{ and } \Omega_1 = A_1 = \{ e_{ij} : i \rightarrow j \}$$

while in general $\Omega_p \subset A_p$.

1.3 Examples of $\partial$-invariant paths

A triangle is a sequence of three distinct vertices $a, b, c$ such that $a \rightarrow b \rightarrow c, \ a \rightarrow c$.

It determines a 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in A_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in A_1$.

The path $e_{abc}$ is also referred to as a triangle.
A square is a sequence of four distinct vertices \( a, b, b', c \) such that \( a \to b \to c, a \to b' \to c \) while \( a \not\to c \).

It determines a 2-path \( u = e_{abc} - e_{ab'c} \in \Omega_2 \) because \( u \in A_2 \) and

\[
\partial u = (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) = e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in A_1.
\]

The path \( u \) is also referred to as a square.

An \( m \)-square is a sequence of \( m + 3 \) distinct vertices

\[ a, b_0, b_1, ..., b_m, c \]

such that \( a \to b_k \to c \; \forall k = 0, \ldots, m, \) while \( a \not\to c \).

Clearly, a square is an 1-square. Any \( m \)-square with \( m \geq 2 \) is also called a multisquare.

The \( m \)-square determines \( \partial \)-invariant 2-paths (squares) as follows:

\[
u_{ij} = e_{ab_i c} - e_{ab_j c} \in \Omega_2 \quad \text{for all } i, j = 0, ..., m,
\]

and among them the following \( m \) squares are linearly independent:

\[
u_{0j} = e_{ab_0 c} - e_{ab_j c}, \quad j = 1, ..., m.
\]
A 3-cube is a sequence of 8 vertices 0, 1, 2, 3, 4, 5, 6, 7, connected by arrows as shown here:

A 3-cube determines a $\partial$-invariant 3-path

$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3,$$

also called a 3-cube. Indeed, we have $u \in A_3$ and

$$\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in A_2.$$

A trapezohedron of order $m \geq 2$ is a configuration of $2m + 2$ vertices: $a, b, i_0, \ldots, i_{m-1}, j_0, \ldots, j_{m-1}$ with $4m$ arrows: $a \to i_k, j_k \to b, i_k \to j_k, i_k \to j_{k+1}, \forall k = 0, \ldots, m - 1$, where $k + 1$ is understood mod $m$.

It determines the following $\partial$-invariant 3-path:

$$\tau_m = \sum_{k=0}^{m-1} (e_{a i_k j_k b} - e_{a i_k j_{k+1} b}) \hspace{1cm} (1.3)$$
that is called a *trapezohedral* path. Clearly, $\tau_{m}$ is allowed. Let us verify that $\partial \tau_{m} \in \mathcal{A}_{2}$. Indeed, we have

$$
\partial \tau_{m} = \sum_{k=0}^{m-1} \partial \left( e_{ai_{k}j_{k}b} - e_{ai_{k}j_{k+1}b} \right)
$$

$$
= \sum_{k=0}^{m-1} \left( e_{i_{k}j_{k}b} - e_{i_{k}j_{k+1}b} \right) - \sum_{k=0}^{m-1} \left( e_{ai_{k}j_{k}} - e_{ai_{k}j_{k+1}} \right)
$$

$$
- \sum_{k=0}^{m-1} \left( e_{aj_{k}b} - e_{aj_{k+1}b} \right) + \sum_{k=0}^{m-1} \left( e_{ai_{k}b} - e_{ai_{k}b} \right) \in \mathcal{A}_{2},
$$

(1.4)

(1.5)

because the both sums in (1.4) are allowed, while the both sums in (1.5) vanish.

For example, a trapezohedron of order $m = 2$ is shown here:

In this case we have

$$
\tau_{2} = e_{ai_{0}j_{0}b} - e_{ai_{0}j_{1}b} + e_{ai_{1}j_{1}b} - e_{ai_{1}j_{0}b}.
$$
Trapezohedra of order $m \geq 3$ can be realized as convex polyhedra in $\mathbb{R}^3$. For example, trapezohedron of order $m = 3$ coincides with a 3-cube:

In this case we have

$$\tau_3 = e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + e_{ai_2j_2b} - e_{ai_2j_0b},$$

and $\tau_3$ coincides (up to a sign) with the aforementioned $\partial$-invariant 3-path determined by a 3-cube (see p. 9).

Trapezohedron of order $m = 4$ can be realized in $\mathbb{R}^3$ as a tetragonal trapezohedron:

In this case we have

$$\tau_4 = e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + e_{ai_2j_2b} - e_{ai_2j_3b} + e_{ai_3j_3b} - e_{ai_3j_0b}.$$
Here are some pictures from Wikipedia of trapezohedra as convex polyhedra:

Tetragonal trapezohedron
\[ m = 4 \]

Pentagonal trapezohedron
\[ m = 5 \]

Heptagonal trapezohedron
\[ m = 7 \]

Decagonal trapezohedron
\[ m = 10 \]
1.4 Digraph morphisms

Let $X$ and $Y$ be two digraphs. For simplicity of notations, we denote the vertices of $X$ and $Y$ by the same letters $X$ resp. $Y$.

**Definition.** A mapping $f : X \to Y$ between the sets of vertices of $X$ and $Y$ called a *digraph map* (or *morphism*) if

$$a \to b \text{ on } X \Rightarrow f(a) \to f(b) \text{ or } f(a) = f(b) \text{ on } Y.$$ 

In other words, any arrow of $X$ under the mapping $f$ either goes to an arrow of $Y$ or collapses to a vertex of $Y$.

We say that a digraph $Y$ is a *subgraph* of a digraph $X$ if the sets of vertices and arrows of $Y$ are subset of the sets of vertices and arrows of $X$, respectively. In this case we have a natural inclusion $i : Y \to X$ that is clearly a digraph morphism.

To give another example of a morphism, let us split the vertex set of a digraph $X$ into a disjoint union of $n$ subsets $A_1, ..., A_n$, and construct a digraph $Y$ of $n$ vertices $a_1, ..., a_n$ that is obtained from $X$ by merging all the vertices from $A_i$ into a single vertex $a_i$ of $Y$. More precisely, we have an arrow $a_i \to a_j$ in $Y$ if and only if there are $x \in A_i$ and $y \in A_j$ such that $x \to y$ in $X$. 
An example of a merging map $\mu$

We have a natural merging map $\mu : X \to Y$ such that $\mu(x) = a_i$ for any $x \in A_i$. Clearly, a merging map is a digraph morphism that keeps any arrow $x \to y$ if $x$ and $y$ belong to different sets $A_i$ and collapses an arrow $x \to y$ into a vertex if $x, y$ belong to the same $A_i$.

Any mapping $f : X \to Y$ induces a mapping $f_* : \Lambda_n(X) \to \Lambda_n(Y)$ as follows: first set

$$f_*(e_{i_0...i_n}) = e_{f(i_0)...f(i_n)},$$

and then extend $f_*$ by linearity to all of $\Lambda_n(X)$.

**Proposition 1.1** Let $f : X \to Y$ be a digraph morphism. Then the induced mapping $f_* : \Lambda_n(X) \to \Lambda_n(Y)$ extends to a chain mapping $f_* : \Omega_n(X) \to \Omega_n(Y)$ and, hence, to homomorphism $f_* : H_n(X) \to H_n(Y)$. 
1.5 Structure of $\Omega_2$

As we know, $\Omega_0 = \langle e_i \rangle$ consists of all vertices and $\Omega_1 = \langle e_{ij} : i \to j \rangle$ consists of all arrows.

**Definition.** Let us call a *semi-arrow* any pairs $(x, y)$ of distinct vertices $x, y$ such that $x \not\to y$ but $x \to z \to y$ for some vertex $z$. We write in this case $x \xleftarrow{\cdot} y$

**Theorem 1.2**

(a) We have $\dim \Omega_2 = \dim A_2 - s$ where $s$ is the number of semi-arrows.

(b) Space $\Omega_2$ is spanned by all triangles $e_{abc}$, squares $e_{abc} - e_{ab'}c$ and double arrows $e_{aba}$:

Observe that all the triangles and double edges are linearly independent whereas the squares can be dependent as the example of multisquare shows.
Proof. (a) Recall that
\[ \mathcal{A}_2 = \text{span} \{ e_{abc} : a \rightarrow b \rightarrow c \} \]
and
\[ \Omega_2 = \{ v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1 \} = \{ v \in \mathcal{A}_2 : \partial v = 0 \text{ mod } \mathcal{A}_1 \} . \]
Since \( a \rightarrow b \) and \( b \rightarrow c \), we have
\[ \partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \text{ mod } \mathcal{A}_1 . \]
If \( a = c \) or \( a \rightarrow c \) then \( e_{ac} = 0 \text{ mod } \mathcal{A}_1 \). Otherwise we have a semi-arrow \( a \rightarrow c \), and in this case
\[ e_{ac} \neq 0 \text{ mod } \mathcal{A}_1 . \]
For any \( v \in \mathcal{A}_2 \), we have
\[ v = \sum_{\{ a \rightarrow b \rightarrow c \}} v^{abc} e_{abc} \]
whence it follows that
\[ \partial v = - \sum_{\{ a \rightarrow b \rightarrow c, a \rightarrow c \}} v^{abc} e_{ac} \text{ mod } \mathcal{A}_1 . \]
The condition \( \partial v = 0 \text{ mod } \mathcal{A}_1 \) is equivalent to
\[ \sum_{\{ a \rightarrow b \rightarrow c, a \rightarrow c \}} v^{abc} e_{ac} = 0 \text{ mod } \mathcal{A}_1 . \]  (1.6)
Fixing a semi-arrow \( a \rightarrow c \) and summing up in all possible \( b \), we obtain that (1.6) is equivalent to

\[
\sum_{\{b:a \rightarrow b \rightarrow c\}} v^{abc} = 0 \quad \text{for any semi-arrow } a \rightarrow c.
\]  
\[(1.7)\]

The number of the equations in (1.7) is exactly \( s \), and they all are linearly independent for different semi-arrows. Hence, \( \Omega_2 \) is obtained from \( \mathcal{A}_2 \) by imposing \( s \) linearly independent conditions on \( v^{abc} \), which implies \( \dim \Omega_2 = \dim \mathcal{A}_2 - s \).

(b) Let us prove that any \( \partial \)-invariant 2-path \( \omega \) is a linear combination of triangles, squares and double arrows. Since \( \omega \) is allowed, it is a linear combination of some elementary 2-paths \( e_{abc} \) with \( a \rightarrow b \rightarrow c \), with non-zero coefficients. If \( a = c \) then \( e_{abc} \) is a double arrow. If \( a \rightarrow c \) then \( e_{abc} \) is a triangle. Subtracting from \( \omega \) all double arrows and triangles, we can assume that \( \omega \) has no such terms any more.

Then, for any term \( e_{abc} \) in \( \omega \), we have \( a \neq c \) and \( a \not\rightarrow c \), that is, \( a \rightarrow c \). Fix such \( a, c \) and consider all vertices \( b \) with \( a \rightarrow b \rightarrow c \) so that we get a multisquare:
Denote by $\gamma_b$ the coefficient with which $e_{abc}$ enters $\omega$, and set
\[
\omega_{ac} = \sum_b \gamma_b e_{abc}.
\] (1.8)

Clearly, we have $\omega = \sum_{a \rightarrow c} \omega_{ac}$. Hence, it suffices to verify that each $\omega_{ac}$ is a linear combination of squares. We have
\[
\partial \omega_{ac} = \sum_b \gamma_b e_{ab} - \gamma_b e_{ac} + \gamma_b e_{bc} = - \sum_b \gamma_b e_{ac} \mod A_1.
\]

Since $\partial \omega$ is allowed but $e_{ac}$ is not allowed, the terms $\gamma_b e_{ac}$ should cancel out that is,
\[
\sum_b \gamma_b = 0.
\] (1.9)

Let us fix one of the vertices $b_0$ such that $a \rightarrow b_0 \rightarrow c$. It follows from (1.8) and (1.9) that
\[
\omega_{ac} = \sum_b \gamma_b e_{abc} = \sum_b \gamma_b (e_{abc} - e_{ab_0c}) = \sum_{b \neq b_0} \gamma_b (e_{abc} - e_{ab_0c}).
\]

Hence, $\omega_{ac}$ is a linear combination of the squares $e_{abc} - e_{ab_0c}$, which was to be proved. ■
Observe that a triangle $e_{abc}$ and a double arrow $e_{aba}$ are images of a square $e_{013} - e_{023}$ under some merging maps (cf. Section 1.4) as shown on these pictures:

- A merging map from a square onto a triangle:
  $$e_{013} - e_{023} \mapsto e_{abc} - e_{acc} = e_{abc}$$

- A merging map from a square onto a double arrow:
  $$e_{013} - e_{023} \mapsto e_{aba} - e_{aaa} = e_{aba}$$

Hence, we can rephrase Theorem 1.2 as follows: $\Omega_2$ is spanned by squares and their morphism images. Or: squares are basic shapes of $\Omega_2$. 
2 Trapezohedra and structure of $\Omega_3$

2.1 Spaces $\Omega_p$ for trapezohedron

For any integer $m \geq 2$, define a trapezohedron $T_m$ of order $m$ as the following digraph:

$T_m$ consists of $2m + 2$ vertices

$a, b, i_0, ..., i_{m-1}, j_0, j_1, ..., j_{m-1}$

and $4m$ arrows

$a \rightarrow i_k, \ j_k \rightarrow b, \ i_k \rightarrow j_k, \ i_k \rightarrow j_{k+1}$

for all $k = 0, \ldots, m - 1 \text{ mod } m$.

A fragment of $T_m$ is shown here:

It is clear that all allowed paths in $T_m$ have the length $\leq 3$, and, hence, $\Omega_p(T_m) = \{0\}$ $\forall p > 3$. 
Proposition 2.1 For the trapezohedron $T_m$ we have

$$\dim \Omega_2 = 2m, \quad \dim \Omega_3 = 1,$$

and $H_p = \{0\}$ for all $p \geq 1$.

Proof. It is easy to detect all the squares in $T_m$:

$$e_{ai_{k-1}jk} - e_{ai_kjk} \quad \text{and} \quad e_{i_kjkb} - e_{i_kjk+1b},$$

where $k = 0, \ldots, m-1$. Hence, $T_m$ contains $2m$ squares, and they are linearly independent. Since there are neither triangles no double arrows in $T_m$, we conclude by Theorem 1.2 that $\dim \Omega_2 = 2m$.

All allowed 3-paths in $T_m$ are as follows:

$$e_{ai_kjk}b \quad \text{and} \quad e_{ai_kjk+1}b,$$

for all $k = 0, \ldots, m - 1$. 

![Diagram of trapezohedron](image)
Let us find all linear combinations of these paths that are $\partial$-invariant. Consider such a linear combination

$$\omega = \sum_{k=0}^{m-1} (\alpha_k e_{a_{i_k j_k}b} + \beta_k e_{a_{i_k j_k+1}b})$$

with coefficients $\alpha_k, \beta_k$. We have

$$\partial \omega = \sum_{k=0}^{m-1} \partial (\alpha_k e_{a_{i_k j_k}b} + \beta_k e_{a_{i_k j_k+1}b})$$

$$= \sum_{k=0}^{m-1} (\alpha_k e_{i_k j_k b} + \beta_k e_{i_k j_k+1 b}) - \sum_{k=0}^{m-1} (\alpha_k e_{a_{i_k j_k}b} + \beta_k e_{a_{i_k j_k+1}b})$$

$$- \sum_{k=0}^{m-1} (\alpha_k e_{a_{j_k}b} + \beta_k e_{a_{j_k+1} b}) + \sum_{k=0}^{m-1} (\alpha_k e_{a_{i_k}b} + \beta_k e_{a_{i_k}b}).$$

(2.2)

(2.3)

The both sums in (2.2) consist of allowed paths. In the rightmost sum in (2.3), the path $e_{a_{i_k}b}$ is not allowed and, hence, must cancel out, which yields

$$\alpha_k = -\beta_k.$$ 

The leftmost sum in (2.3) is then equal to

$$\sum_{k=0}^{m-1} (\alpha_k e_{a_{j_k}b} - \alpha_k e_{a_{j_k+1}b}) = \sum_{k=0}^{m-1} (\alpha_k - \alpha_{k-1}) e_{a_{j_k}b},$$

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and it must vanish as $e_{a_{jk}b}$ is not allowed, whence

$$\alpha_k = \alpha_{k-1}.$$  

Setting $\alpha_k \equiv \alpha$ and, hence, $\beta_k \equiv -\alpha$, we obtain that

$$\omega = \alpha \sum_{k=0}^{m-1} (e_{a_{i_{k}j_{k}b}} - e_{a_{i_{k}j_{k+1}b}}) = \alpha \tau_m,$$

where $\tau_m$ is a trapezohedral path that was defined by (1.3). It follows that $\Omega_3 = \langle \tau_m \rangle$ and, hence, $\dim \Omega_3 = 1$.

It follows from (2.2)-(2.3) that

$$\partial \tau_m = \sum_{k=0}^{m-1} (e_{i_{k}j_{k}b} - e_{i_{k}j_{k+1}b}) - \sum_{k=0}^{m-1} (e_{a_{i_{k}j_{k}} - e_{a_{i_{k}j_{k+1}}}) \neq 0.$$

Hence, ker $\partial|_{\Omega_3} = 0$ whence $H_3 = \{0\}$. Let us show that $H_2 = \{0\}$. Since dim Im $\partial|_{\Omega_3} = 1$, it suffices to show that

$$\dim \ker \partial|_{\Omega_2} = 1.$$  \hspace{1cm} (2.4)

Consider the following general element of $\Omega_2$:

$$u = \sum_{k=0}^{m-1} \alpha_k (e_{a_{i_{k-1}j_{k}}} - e_{a_{i_{k}j_{k}}}) + \beta_k (e_{i_{k}j_{k}b} - e_{i_{k}j_{k+1}b})$$
with arbitrary coefficients $\alpha_k, \beta_k$. We have

$$
\partial u = \sum_{k=0}^{m-1} \alpha_k \left(e_{ai_{k-1}} + e_{i_{k-1}j_k} - e_{ai_k} - e_{i_kj_k}\right) + \beta_k \left(e_{j_kb} + e_{i_kj_k} - e_{j_{k+1}b} - e_{i_kj_{k+1}}\right)
$$

$$
= \sum_{k=0}^{m-1} \left(\alpha_{k+1} - \alpha_k\right) e_{ai_k} + \sum_{k=0}^{m-1} \left(\beta_k - \beta_{k-1}\right) e_{j_kb}
$$

$$
+ \sum_{k=0}^{m-1} \left(\beta_k - \alpha_k\right) e_{i_kj_k} + \sum_{k=0}^{m-1} \left(\alpha_{k+1} - \beta_k\right) e_{i_kj_{k+1}}.
$$

The condition $\partial u = 0$ is equivalent to

$$
\alpha_{k+1} = \alpha_k = \beta_k = \beta_{k-1} \text{ for all } k = 0, \ldots, m - 1
$$

which implies (2.4).

Finally, we determine $\dim H_1$ by means of the Euler characteristic

$$
\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 - \dim \Omega_3 = (2m + 2) - 4m + 2m - 1 = 1.
$$

Hence, we obtain

$$
\dim H_0 - \dim H_1 + \dim H_2 - \dim H_3 = 1,
$$

which yields $\dim H_1 = 0$. ■
2.2 A cluster basis in $\Omega_p$

We start with the following definition.

**Definition.** A $p$-path $v = \sum v^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$ is called an $(a, b)$-cluster if all the elementary paths $e_{i_0 \ldots i_p}$ with non-zero values of $v^{i_0 \ldots i_p}$ have $i_0 = a$ and $i_p = b$. A path $v$ is called a cluster if it is an $(a, b)$-cluster for some $a, b$.

**Lemma 2.2** Any $\partial$-invariant $p$-path is a sum of $\partial$-invariant clusters.

**Proof.** Let $v \in \Omega_p$. For any points $a, b \in V$, denote by $v_{a,b}$ the sum of all terms $v^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$ with $i_0 = a$ and $i_p = b$.

Then $v_{a,b}$ is a cluster and $v = \sum_{a,b \in V} v_{a,b}$, that is, $v$ is a sum of clusters. Let us prove that each non-zero cluster $v_{a,b}$ is $\partial$-invariant.

Since $v$ is allowed, also all non-zero terms $v^{i_0 \ldots i_p} e_{i_0 \ldots i_p}$ are allowed, whence $v_{a,b}$ is also allowed. Let us prove that $\partial v_{a,b}$ is allowed, which will yield the $\partial$-invariance of $v_{a,b}$. The
path \( v_{a,b} \) is a linear combination of allowed paths of the form \( e_{ai_1...i_{p-1}b} \). We have

\[
\partial e_{ai_1...i_{p-1}b} = e_{i_1...i_{p-1}b} + (-1)^p e_{ai_1...i_{p-1}} + \sum_{k=1}^{p-1} (-1)^k e_{ai_1...\hat{i}_k...i_{p-1}b}.
\]

The terms \( e_{i_1...i_{p-1}b} \) and \( e_{ai_1...i_{p-1}} \) are clearly allowed, while among the terms \( e_{ai_1...\hat{i}_k...i_{p-1}b} \) there may be non-allowed. In the full expansion of

\[
\partial v = \sum_{a,b \in V} \partial v_{a,b}
\]

all non-allowed terms must cancel out. Since all the terms \( e_{ai_1...\hat{i}_k...i_{p-1}b} \) form a \((a,b)\)-cluster, they cannot cancel with terms containing different values of \( a \) or \( b \). Therefore, they have to cancel already within \( \partial v_{a,b} \), which implies that \( \partial v_{a,b} \) is allowed. ■

**Definition.** For any \( p \)-path \( v = \sum v^{i_0...i_p} e_{i_0...i_p} \) define its *width* \( \|v\| \) as the number of non-zero coefficients \( v^{i_0...i_p} \).

**Definition.** A \( \partial \)-invariant path \( \omega \) is called *minimal* if \( \omega \) cannot be represented as a sum of other \( \partial \)-invariant paths with smaller widths.

**Example.** A square \( \omega = e_{abc} - e_{ab'c} \) has width 2 and is minimal because \( e_{abc} \) and \( e_{ab'c} \) having width 1 are not \( \partial \)-invariant.
Let $a, \{b_0, b_1, b_2\}, c$ be a 2-square. The following path

$$\omega = e_{ab_1c} + e_{ab_2c} - 2e_{ab_0c}$$

is then $\partial$-invariant, has width 3 but is not minimal because it can be represented as a sum of two squares:

$$\omega = (e_{ab_1c} - e_{ab_0c}) + (e_{ab_2c} - e_{ab_0c}),$$

where each square has width 2.

**Lemma 2.3** Every $\partial$-invariant cluster is a sum of minimal $\partial$-invariant clusters.

**Proof.** Let $\omega$ be a $\partial$-invariant cluster that is not minimal. Then we have

$$\omega = \sum_{k=1}^{n} \omega^{(k)}, \quad (2.5)$$

where each $\omega^{(k)}$ is a $\partial$-invariant path with $\|\omega^{(k)}\| < \|\omega\|$. By Lemma 2.2, each $\omega^{(k)}$ is a sum of clusters $\omega^{(k)}_{a,b}$, and it is clear from the definition of $\omega^{(k)}_{a,b}$ that

$$\|\omega^{(k)}_{a,b}\| \leq \|\omega^{(k)}\|.$$
Hence, we can replace in (2.5) each $\omega^{(k)}$ by $\sum_{a,b} \omega_{a,b}^{(k)}$ and, hence, assume without loss of generality that all terms $\omega^{(k)}$ in (2.5) are $\partial$-invariant clusters.

If some $\omega^{(k)}$ in this sum is not minimal then we replace it further with sum of $\partial$-invariant clusters with smaller widths. Continuing this procedure we obtain in the end a representation $\omega$ as a sum of minimal $\partial$-invariant clusters.

**Proposition 2.4** The space $\Omega_p$ has a basis that consists of minimal $\partial$-invariant clusters.

**Proof.** Indeed, let $\mathcal{M}$ denote the set of all minimal $\partial$-invariant clusters in $\Omega_p$. By Lemmas 2.2, 2.3, every element of $\Omega_p$ is a sum of some elements of $\mathcal{M}$. Choosing in $\mathcal{M}$ a maximal linearly independent subset, we obtain a basis in $\Omega_p$. ■

### 2.3 Structure of $\Omega_3$

We use here the trapezohedra $T_m$ and associated trapezohedral paths $\tau_m$ that are $\partial$-invariant 3-paths for all $m \geq 2$ (see (1.3) and Section 2.1). We prove here that, under an additional mild hypothesis, $\Omega_3(G)$ has a basis that consists of trapezohedral paths and their morphism images.
We start with some examples of morphism images of $\tau_m$.

**Example.** Here is a merging map from $T_2$ onto a 3-snake:

The trapezohedral path $\tau_2$ is given by

$$\tau_2 = e_{0123} - e_{0153} + e_{0453} - e_{0423},$$

and its merging image is the 3-path

$$v = e_{0123} - e_{0133} + e_{0233} - e_{0223} = e_{0123},$$

that is, the $\partial$-invariant 3-path $e_{0123}$ associated with a 3-snake.
Example. Here is a merging morphism of $T_3$ (a 3-cube) onto a pyramid:

![Diagram](image)

The cubical 3-path is given by

$$
\tau_3 = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}
$$

and its merging image of $\tau_3$ is the following $\partial$-invariant 3-path in a pyramid:

$$
v = e_{0234} - e_{0134} + e_{0144} - e_{0444} + e_{0444} - e_{0244} = e_{0234} - e_{0134}.
$$
Example. Consider another merging morphism of $T_3$ onto a prism:

The merging image of the cubical 3-path

$$\tau_3 = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}$$

is the following $\partial$-invariant 3-path of the prism:

$$u = e_{0233} - e_{0133} + e_{0153} - e_{0453} + e_{0423} - e_{0223}$$

$$= e_{0153} - e_{0453} + e_{0423}.$$
Example. Here is a merging morphism $\mu : T_4 \to G$ where the digraph $G$ is a broken cube:

The path $\tau_4$ in the present notation is given by

$$\tau_4 = e_{0159} - e_{0169} + e_{0269} - e_{0279} + e_{0379} - e_{0389} + e_{0489} - e_{0459},$$

and the merging image of $\tau_4$ is the following $\partial$-invariant 3-path on the broken cube:

$$w = e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0388} + e_{0488} - e_{0458}$$

$$= e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0388} - e_{0458}.$$
The next theorem describes the structure of $\Omega_3(G')$ for a digraph $G$ under the following hypothesis:

$$G \text{ contains neither multisquares (see p.8) nor double arrows.} \quad (N)$$

Under the hypothesis $(N)$, $\Omega_2(G')$ has a basis that consists of triangles and squares. The condition $(N)$ implies that if $a \rightarrow b \rightarrow c$ and $a \not\rightarrow c$ then there is at most one $b' \neq b$ such that $a \rightarrow b' \rightarrow c$.

**Theorem 2.5** Under the hypothesis $(N)$, there is a basis in $\Omega_3(G')$ that consists of trapezohedral paths $\tau_m$ with $m \geq 2$ and their merging images.

In other words, trapezohedra are basic shapes for $\Omega_3$.

**Proof.** By Proposition 2.4, $\Omega_3$ has a basis that consists of minimal $\partial$-invariant clusters. Let a 3-path $\omega$ be a minimal $\partial$-invariant $(a, b)$-cluster. It suffices to prove that $\omega$ is a merging image of one of the trapezohedral paths $\tau_m$ up to a constant factor.

Denote by $Q$ the set of all elementary terms $e_{aijb}$ of $\omega$. 

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Clearly, the number $|Q|$ of elements in $Q$ is equal to $\|\omega\|$. We claim that, for any $e_{aijb} \in Q$, either $a \to j$ or $a \nearrow j$

where the notation $a \nearrow j$ means that $a$ and $j$ form a diagonal of a square.

Indeed, if $a \not\to j$ then the term $e_{aijb}$ appearing in $\partial e_{aijb}$ is non-allowed and must be cancelled out in $\partial \omega$ by the boundary of another elementary 3-path from $Q$ that can only be of the form $e_{ai'jb}$ with $a \to i' \to j$.

Hence, $a$ and $j$ form diagonal of a square $a, i, i', j$.

By hypothesis (N), the vertex $i'$ with these properties is unique. Hence, in this case we have

$$\omega = ce_{aijb} - ce_{ai'jb} + \ldots \quad (2.6)$$

for some scalar $c \neq 0$. In the same way, we have

either $i \to b$ or $i \nearrow b$,

and, for some $e_{aijb} \in Q$ and $c \neq 0$,

$$\omega = ce_{aijb} - ce_{ai'jb} + \ldots . \quad (2.7)$$

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If, for some path $e_{aijb} \in Q$, we have both conditions

$$a \rightarrow j \text{ and } i \rightarrow b,$$

then $e_{aijb}$ is $\partial$-invariant and, by the minimality of $\omega$,

$$\omega = \text{const } e_{aijb}.$$

Since $e_{aijb}$ is in this case a 3-snake, the path $\omega$ is a merging image of $\tau_2$ (see Example on p. 29).

Next, we can assume that, for any path $e_{aijb} \in Q$, we have $a \not\rightarrow j$ or $i \not\rightarrow b$, that is,

$$a \nearrow j \text{ or } i \nearrow b.$$  \hfill (2.8)

Define a graph structure on $Q$ with edges of two types (i) and (ii) as follows: for two distinct elements $e_{aijb}$ and $e_{ai'j'b}$ of $Q$ set

$$e_{aijb} \sim_{(i)} e_{ai'j'b} \text{ if } a \nearrow j = j'$$

and

$$e_{aijb} \sim_{(ii)} e_{ai'j'b} \text{ if } i' = i \nearrow b.$$

Both relations $\sim_{(i)}$ and $\sim_{(ii)}$ are symmetric and, hence, can be considered as edges.
Before continuing the proof, consider some examples of graphs $Q$.

**Example A.** Let $\omega$ be the trapezohedral path of $T_2$, that is,

$$\omega = \tau_2 = e_{0123} - e_{0153} + e_{0453} - e_{0423}.$$ 

This path is an $(a, b)$-cluster with $a = 0$ and $b = 3$. In this case the graph $Q$ consists of 4 vertices as follows:
Example B. Let $\omega$ be the $\partial$-invariant 3-path of the broken cube (see Example on p. 32), that is,

$$\omega = e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0458}.$$ 

This path is a $(a, b)$-cluster with $a = 0$ and $b = 8$. The graph $Q$ consists of 6 vertices as follows:
By the hypothesis (N), for any $e_{aijb} \in Q$, there is at most one edge of type (i) and at most one edge of type (ii).

In particular, the degree of any vertex of the graph $(Q, \sim)$ is at most 2.

Fix a path $e_{aijb} \in Q$. By (2.8) we have

$$a \nearrow j \text{ or } i \nearrow b.$$ 

By the above argument, if $a \nearrow j$ then there exists $e_{ai'jb} \in Q$ such that $e_{aijb} \sim e_{ai'jb}$ and

$$\omega = ce_{aijb} - ce_{ai'jb} + \ldots$$ \hspace{1cm} (2.9)

(cf. (2.6)). Similarly, if $i \nearrow b$ then there exists $e_{aijb} \in Q$ such that $e_{aijb} \sim e_{aijb}$ and

$$\omega = ce_{aijb} - ce_{aijb} + \ldots$$ \hspace{1cm} (2.10)

(cf. (2.7)). In particular, the degree of any vertex of the graph $Q$ is at least 1.

Let us prove that the graph $(Q, \sim)$ is connected. Assume from the contrary that $Q$ is disconnected, then $Q$ is a disjoint union of its connected components $\{Q_k\}_{k=1}^n$ with $n > 1$. 

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Denote by \( \omega^{(k)} \) the sum of all elementary terms of \( \omega \) lying in \( Q_k \), with the same coefficients as in \( \omega \), so that

\[
\omega = \sum_{k=1}^{n} \omega^{(k)}. \tag{2.11}
\]

Let us prove that each \( \omega^{(k)} \) is \( \partial \)-invariant. Clearly, \( \omega^{(k)} \) is allowed, and we need to verify that \( \partial \omega^{(k)} \) is also allowed. Indeed, assume that \( \partial \omega^{(k)} \) contains a non-allowed term. Then this term comes from the boundary \( \partial e_{aijb} \) of some term \( e_{aijb} \) of path \( \omega^{(k)} \). The non-allowed term of \( \partial e_{aijb} \) is either \( e_{aib} \) or \( e_{ajb} \); let it be \( e_{aib} \), that is, let \( i \not\rightarrow b \). Then the term \( e_{aib} \) cancels out in \( \partial \omega \), which can only happen when \( \omega \) contains another term of the form \( e_{aij'b} \). However, then \( e_{aijb} \) and \( e_{aij'b} \) are connected by an edge in \( Q \):

\[
e_{aijb} \sim^{(ii)} e_{aij'b}.
\]

Therefore, \( e_{aij'b} \) and \( e_{aijb} \) belong to the same connected component of \( Q \), that is, to \( Q_k \). Hence, \( e_{aij'b} \) is also an elementary term of \( \omega^{(k)} \), and \( e_{aib} \) cancels out also in \( \partial \omega^{(k)} \). This proves that \( \partial \omega^{(k)} \) is allowed and, hence, \( \omega^{(k)} \) is \( \partial \)-invariant.
As the number $n$ of components is $> 1$, we have $|Q_k| < |Q|$, whence $||\omega^{(k)}|| < ||\omega||$. But then (2.11) is impossible by the minimality of $\omega$. Hence, $n = 1$ and $Q$ is connected.

Since each vertex of $Q$ has at most two adjacent edges, there are only two possibilities:

(A): $Q$ is a simple closed polygon; 

(B): $Q$ is a linear graph.

Consider first the case (A). In this case every vertex of $Q$ has two edges: exactly one edge of each type (i), (ii). Hence, the number of edges is even, let $2m$, and $Q$ has necessarily the following form:

$$e_{ai_0j_0b} \sim e_{ai_0j_1b} \sim e_{ai_1j_1b} \sim \ldots \sim e_{ai_{m-1}j_{m-1}b} \sim e_{ai_{m-1}j_0b} \sim e_{ai_0j_0b} \quad (2.12)$$

for some vertices $i_0, \ldots, i_{m-1}$ and $j_0, \ldots, j_{m-1}$ of $G$. Note that $m \geq 2$ because if $m = 1$ then (2.12) becomes

$$e_{ai_0j_0b} \sim e_{ai_0j_1b} \sim e_{ai_0j_0b},$$
which is impossible as edges of different types between the same vertices of $Q$ do not exist.

Since all the terms in (2.12) enter $\omega$ with the same coefficients $\pm c$ (cf. (2.9) and (2.10)), we see that

$$\omega = c(e_{ai_0j_0b} - e_{ai_1j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \ldots + e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_0b}).$$  \hspace{1cm} (2.13)

Suppose that all the vertices $a, i_0, \ldots, i_{m-1}, j_0, \ldots, j_{m-1}, b$ are distinct. It follows from (2.12) that these vertices form a trapezohedron $T_m$ as on the next picture:

By (1.3), the trapezohedral path of $T_m$ is

$$\tau_m = (e_{ai_0j_0b} - e_{ai_0j_1b}) + (e_{ai_1j_1b} - e_{ai_1j_2b})$$
$$\ldots + (e_{ai_{m-2}j_{m-2}b} - e_{ai_{m-2}j_{m-1}b})$$
$$+ (e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_0b}).$$

Comparison with (2.13) shows that $\omega = c\tau_m$.

If some of these vertices coincide then the configuration (2.12) in $G$ is a merging image of $T_m$, and $\omega$ is a merging image of $c\tau_m$. 

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Consider now the case (B). In this case the linear graph $Q$ has two end vertices of degree 1, while all other vertices have degree 2. There are two essentially different subcases:

(B$_1$) the end edges of $Q$ are of different types:

\[ e_{ai_0j_0b} \sim e_{ai_1j_1b} \sim e_{ai_2j_2b} \sim \ldots \sim e_{ai_{m-1}j_{m-1}b} \sim e_{ai_mb}. \] (2.14)

Consequently, we have

\[ \omega = c(e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \ldots - e_{ai_{m-1}j_{m-1}b} + e_{ai_mb}). \] (2.15)

Computation of $\partial \omega$ gives

\[ \partial \omega = c(-e_{aj_0b} + e_{ai_mb}) \mod A_2. \]
Since $\partial \omega = 0 \mod A_2$, we must have either $e_{aj_0b} = e_{aimb}$ or the both $e_{aj_0b}$ and $e_{aimb}$ are allowed, that is,

$$a \to j_0 \text{ and } i_m \to b.$$  \hfill (2.16)

In the case $e_{aj_0b} = e_{aimb}$ we have $j_0 = i_m$ whence (2.16) follows again so that (2.16) is satisfied in the both cases.

We claim that in the case $(B_1)$ the configuration (2.14) is a merging image of $T_{m+2}$. Indeed, denote the vertices of $T_{m+2}$ by

$$a, i_0, ..., i_m, i_{m+1}, j_0, ..., j_m, j_{m+1}, b,$$

and map all the vertices of $T_{m+2}$, except for $i_{m+1}, j_{m+1}$, to the vertices of $G$ with the same names; then merge: $i_{m+1} \mapsto j_0$ and $j_{m+1} \mapsto b$.

The following arrows in $T_{m+2}$

$$a \to i_{m+1}, i_m \to j_{m+1}, i_{m+1} \to j_{m+1}$$

are mapped to the arrows in $G$:

$$a \to j_0, \quad i_m \to b, \quad j_0 \to b$$

(cf. (2.16)), while the arrows

$$i_{m+1} \to j_0 \text{ and } j_{m+1} \to b$$

go to vertices.
It follows that this mapping of $T_{m+2}$ into $G$ is a digraph morphism. Since by (1.3)

$$
\tau_{m+2} = (e_{a_0 j_0 b} - e_{a_0 j_1 b}) + (e_{a_1 j_1 b} - e_{a_1 j_2 b}) + \ldots + (e_{a_m j_m b} - e_{a_m j_{m+1} b}) + (e_{a_{m+1} j_{m+1} b} - e_{a_{m+1} j_0 b}),
$$

the image of $\tau_{m+2}$ is the following path, where we replace $i_{m+1}$ by $j_0$ and $j_{m+1}$ by $b$:

$$
u = (e_{a_0 j_0 b} - e_{a_0 j_1 b}) + (e_{a_1 j_1 b} - e_{a_1 j_2 b}) + \ldots + (e_{a_m j_m b} - e_{a_m b b}) + (e_{a_0 b b} - e_{a_0 j_0 b})
= e_{a_0 j_0 b} - e_{a_0 j_1 b} + e_{a_1 j_1 b} - e_{a_1 j_2 b} + \ldots - e_{a_{m-1} j_m b} + e_{a_m j_m b}.
$$

Comparison with (2.15) shows that $\omega = c\nu$, that is, $\omega$ is a merging image of $c\tau_{m+2}$.

In the case $m = 1$, this merging morphism of $T_3$ is shown here (cf. Example on p. 31):
Consider now the case \((B_2)\) when the graph \(Q\) has the form

\[ e_{ai_0j_0b} \sim e_{ai_0j_1b} \sim e_{ai_1j_1b} \sim e_{ai_1j_2b} \sim \ldots \sim e_{ai_{m-1}j_{m-1}b} \sim e_{ai_{m-1}j_mb} \]  

(2.17)

so that

\[ \omega = c(e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \ldots + e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_mb}). \]  

(2.18)

Since

\[ \partial \omega = c(-e_{aj_0b} + e_{aj_mb}) \mod A_2, \]

it follows that either \(j_0 = j_m\) or the both paths \(e_{aj_0b}\) and \(e_{aj_mb}\) are allowed, that is,

\[ a \to j_0 \quad \text{and} \quad a \to j_m. \]

(2.19)

However, \(j_0 = j_m\) is not possible because it would imply that

\[ e_{ai_0j_0b} \sim e_{ai_{m-1}j_0b} \]

and the line graph \(Q\) would close into a polygon, which gives the case \((A)\). Hence, (2.19) is satisfied. We claim that the configuration (2.17) is then a merging image of \(T_{m+1}\). Indeed, denote the vertices of \(T_{m+1}\) by

\[ a, i_0, \ldots, i_m, j_0, \ldots j_m, b. \]
Then we map all the vertices of $T_{m+1}$, except for $i_m$, to the vertices of $G$ with the same names; then map $i_m \mapsto a$.

Clearly, the following arrows in $T_{m+1}$

$i_m \to j_0$ and $i_m \to j_m$

are mapped to the arrows in $G$:

$a \to j_0$ and $a \to j_m$ (cf. (2.19)),

and the arrow $a \to i_m$ goes to a vertex.

Hence, we obtain a merging morphism of $T_{m+1}$ into $G$. Since by (1.3)

$$\tau_{m+1} = (e_{ai_0j_0b} - e_{ai_0j_1b}) + (e_{ai_1j_1b} - e_{ai_1j_2b}) + \ldots + (e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_mb}) + (e_{ai_mj_mb} - e_{ai_mj_0b}),$$

the image of $\tau_{m+1}$ is the following path, where we replace $i_m$ by $a$:

$$v = (e_{ai_0j_0b} - e_{ai_0j_1b}) + (e_{ai_1j_1b} - e_{ai_1j_2b}) + \ldots + (e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_mb}) + (e_{aa_j_mb} - e_{aa_j_0b})$$

$$= e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \ldots + e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_mb}.$$

Comparison with (2.18) shows that $\omega = cv$ so that $\omega$ is a merging image of $c\tau_{m+1}$. ■
2.4 Examples and problems

For example, in the case $m = 2$ the above morphism gives the following merging image of $T_3$: ($T_3=$3-cube)

In the case $m = 3$, the above morphism gives the merging image of $T_4$ as broken cube: (cf. Example on p. 32)
**Problem 2.6** Prove Theorem 2.5 in the general case without the hypothesis (N).

Perhaps, one can prove the absence of multisquares inside each minimal cluster \( \omega \) using the minimality of \( \omega \). Then the rest of the proof remains unchanged.

**Problem 2.7** Devise an algorithm for computing a basis in \( \Omega_3 \) based on Theorem 2.5.

Denote by \( \mathcal{Q} \) the set of all elementary allowed 3-paths. For each \( e_{aijb} \in \mathcal{Q} \), we have

\[
\partial e_{aijb} = -e_{ajb} + e_{aib} \mod \mathcal{A}_2.
\]

We say that \( e_{ajb} \) is a bond of type (i) if \( a \not\to j \); and \( e_{aib} \) is a bond of type (ii), if \( i \not\to b \).

Define edges between elements \( q_1, q_2 \in \mathcal{Q} \) as follows:

\[ q_1 \overset{\text{(i)}}{\sim} q_2 \quad \text{if} \quad q_1, q_2 \text{ have a common bond of the type (i)}; \]

\[ q_1 \overset{\text{(ii)}}{\sim} q_2 \quad \text{if} \quad q_1, q_2 \text{ have a common bond of the type (ii)}. \]

Some bonds may be attached to only one vertex of \( \mathcal{Q} \), so that we allow in \( \mathcal{Q} \) edges with only one vertex. Then the minimal \( \partial \)-invariant clusters in \( G \) are determined by the maximal paths in graph \( \mathcal{Q} \) that go along the edges with alternating types.
For example, consider the following digraph: and try to determine $\Omega_3$. For that first find all elementary allowed 3-paths with all their bonds as shown in the following table:

<table>
<thead>
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<th>$\mathcal{Q}$ \ bonds</th>
<th>054</th>
<th>034</th>
<th>154</th>
<th>012</th>
<th>123</th>
<th>124</th>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>0152</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(ii)</td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>0234</td>
<td></td>
<td>(i)</td>
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</tbody>
</table>
This table determines a (hyper)graph structure in $\mathcal{Q}$ is as follows:

| 0134 \(\begin{array}{c}
(i) \\
\end{array}\) | 0534 \(\begin{array}{c}
(ii) \\
\end{array}\) \(-\) 0524 | 1524 \(\begin{array}{c}
(ii) \\
\end{array}\) | 0152 | 1523 | 0523 |
|---|---|---|---|---|---|
| 0234 \(\begin{array}{c}
(i) \\
\end{array}\) | 1534 | 0153 | 5234 |

The maximal alternating paths in this graph are

$$
\begin{align*}
0134 \sim 0234, & \quad 0134 \sim 0534 \sim 0524, \\
0153, & \quad 0523, \quad 5234,
\end{align*}
$$

which yields five minimal $\partial$-invariant clusters

$$
e_{0134} - e_{0234}, \quad e_{0134} - e_{0534} + e_{0524}, \quad e_{0153}, \quad e_{0523}, \quad e_{5234},$$

that form a basis in $\Omega_3$. In particular, $\dim \Omega_3 = 5$.

**Problem 2.8** State and prove similar results for $\Omega_4$. Are the basic shapes in $\Omega_4$ given by polyhedra in $\mathbb{R}^4$? Devise an algorithm for computing a basis in $\Omega_4$. The same questions for $\Omega_p$ with $p > 4$.

... to be continued ...