Lectures on Path Chain Complexes on Digraphs

Alexander Grigor'yan
University of Bielefeld

Online minicourse at Nankai University

October-November 2024

Contents

References:

1. A. Grigor'yan, Yong Lin, Yu. Muranov, S.-T. Yau, Homologies of path complexes and digraphs, arxiv 1207.2834v4, 2013.

2. A. Grigor'yan, Yu. Muranov, S.-T. Yau, Homologies of digraphs and Künneth formulas, Comm. Anal. Geom. **25** (2017) 969–1018.

3. A. Grigor'yan, Advances in path homology theory of digraphs, Notices of the ICCM **10** (2022) 61–124.

4. A. Grigor'yan, Yong Lin, S.-T. Yau, Haohang Zhang, Eigenvalues of the Hodge Laplacian on digraphs, preprint, 2024

   . . .

All papers are available on my web page:

`https://www.math.uni-bielefeld.de/~grigor/pubs.htm`

1 A path chain complex

1.1 Paths in a finite set

Let V be a finite set. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V. Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Any element of Λ_p is called a p-path.

An elementary p-path $i_0, ..., i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \qquad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle$$

Any p-path u can be written in a form $u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p}$, where $u^{i_0 i_1 \ldots i_p} \in \mathbb{K}$.

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \ldots \widehat{i_q} \ldots i_p},$$

where $\widehat{\ }$ means omission of the index. For $p = 0$ set $\partial e_i = 0$.

For example, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 1.1 $\partial^2 = 0$.

Proof. Indeed, for any $p \geq 2$ we have

$$\begin{aligned} \partial^2 e_{i_0 \ldots i_p} &= \sum_{q=0}^{p} (-1)^q \, \partial e_{i_0 \ldots \widehat{i_q} \ldots i_p} \qquad (1.1) \\ &= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \, e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} \, e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p} \right) \\ &= \sum_{0 \leq r < q \leq p} (-1)^{q+r} \, e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} \, e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p}. \end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \ldots i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$. ∎

Hence, we obtain a chain complex $\Lambda_* (V)$:

$$0 \leftarrow \Lambda_0 \stackrel{\partial}{\leftarrow} \Lambda_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Lambda_{p-1} \stackrel{\partial}{\leftarrow} \Lambda_p \stackrel{\partial}{\leftarrow} \ldots$$

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and irregular otherwise.

Let I_p be the subspace of Λ_p spanned by irregular $e_{i_0...i_p}$. We claim that $\partial I_p \subset I_{p-1}$. Indeed, if $e_{i_0...i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\begin{aligned} \partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}. \end{aligned} \tag{1.2}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.2) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in I_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / I_p$, and we obtain the chain complex $\mathcal{R}_*(V)$:

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \dots$$

By setting all irregular p-paths to be equal to 0, we identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$. In what follows we always consider ∂ acting on $\mathcal{R}_p$.

1.2 Path chain complex and path homology of a digraph

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset \{V \times V \setminus \text{diag}\}$ of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0...i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p(G)$ be $\mathbb{K}$-linear space spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \langle e_{i_0...i_p} : i_0...i_p \text{ is allowed} \rangle.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\begin{tikzcd} \overset{a}{\bullet} \arrow[r] & \overset{b}{\bullet} \arrow[r] & \overset{c}{\bullet} \end{tikzcd}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Consider the following subspace of $\mathcal{A}_p$

$$\boxed{\Omega_p \equiv \Omega_p(G) := \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

Claim: $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

The elements of Ω_p are called *∂-invariant p-paths*. Hence, we obtain a *path chain complex* $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \Omega_{p+1} \overset{\partial}{\leftarrow} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Path homologies of G are defined as the homology groups of the path chain complex $\Omega_*(G)$:

$$H_p(G) = \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

The elements of $\ker \partial|_{\Omega_p}$ are called *closed paths* (or cycles), the elements of $\operatorname{Im} \partial|_{\Omega_{p+1}}$ are called *boundaries*. Hence, $H_p(G)$ is a linear space that consists of closed paths modulo boundaries.

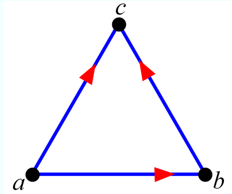
The Betti numbers of G are defined by $\beta_p(G) = \dim H_p(G)$.

It is easy to prove that $\beta_0(G) = \#$of (undirected) connected components of G.

1.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three vertices a, b, c such that $a \to b \to c$, $a \to c$.

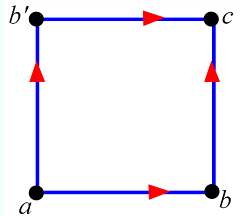
It determines 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.



A *square* is a sequence of four vertices a, b, b', c such that $a \to b$, $b \to c$, $a \to b'$, $b' \to c$.

It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

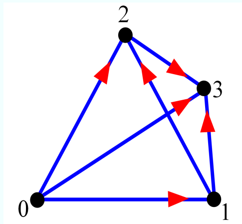
$$\partial u = \left(e_{bc} - \underline{e_{ac}} + e_{ab}\right) - \left(e_{b'c} - \underline{e_{ac}} + e_{ab'}\right)$$
$$= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1$$



A *p-simplex* (or p-clique) is a sequence of $p+1$ vertices, say, $0, 1, ..., p$, such that $i \to j$ for all $i < j$.

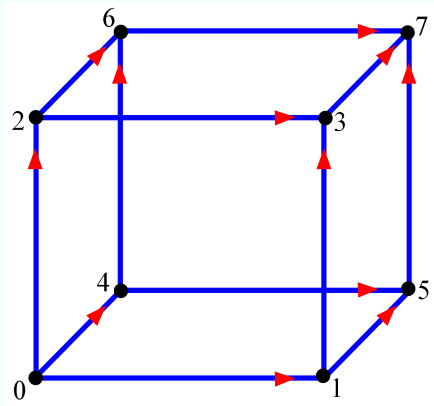
A p-simplex determines a p-path $e_{01\ldots p} \in \Omega_p$.

For example, on a 3-simplex $e_{0123} \in \Omega_3$.



A 3-simplex

A 3-*cube* is a sequence of 8 vertices $0, 1, 2, 3, 4, 5, 6, 7$, connected by arrows as here.



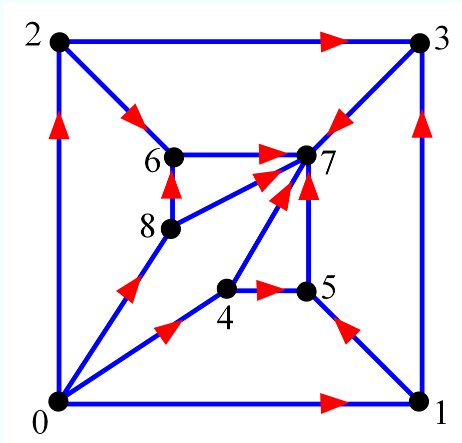
A 3-cube determines a ∂-invariant 3-path

$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$

because $u \in \mathcal{A}_3$ and

$$\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) \\ - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2$$

A *broken cube* consists of 9 vertices connected by arrows as here.



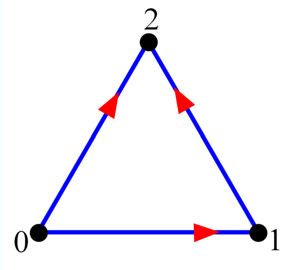
It determines a ∂-invariant 3-path

$v = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0867} - e_{0267} \in \Omega_3$

1.4 Examples of spaces Ω_p and H_p

For a vector space A over $\mathbb{K}$ we write $|A| = \dim_{\mathbb{K}} A$.

Consider a triangle as a digraph:



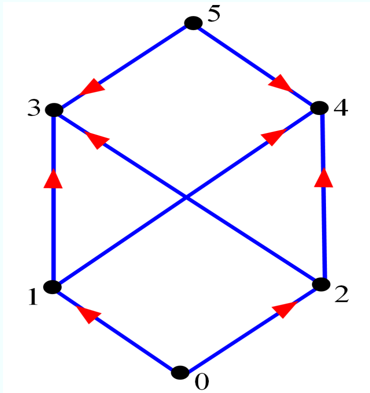
$\Omega_1 = \langle e_{01}, e_{02}, e_{12} \rangle$, $\Omega_2 = \langle e_{012} \rangle$, $\Omega_p = \{0\}$ for $p \geq 3$.

We have $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12} \rangle$ because
$\partial (\alpha e_{01} + \beta e_{02} + \gamma e_{12}) = 0 \quad \Leftrightarrow \quad \alpha = \gamma = -\beta.$

However, $e_{01} - e_{02} + e_{12} = \partial e_{012}$ so that $H_1 = \{0\}$.

Since $\partial e_{012} \neq 0$, we have $H_p = \{0\}$ for all $p \geq 2$.

Consider a hexagon with two diagonals:



We have $|\Omega_0| = 6$, $|\Omega_1| = 8$, $|\Omega_2| = 2$,
where Ω_2 is spanned by 2 squares:

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{014} - e_{024} \rangle,$$

and $\Omega_p = \{0\}$ for all $p \geq 3$.

For this digraph $H_1 = \langle e_{13} - e_{53} + e_{54} - e_{14} \rangle$,
so that $|H_1| = 1$, and $H_p = \{0\}$ for all $p \geq 2$.

Consider an octahedron, where $|\Omega_0| = 6, \quad |\Omega_1| = 12.$

The space Ω_2 is spanned by 8 triangles:

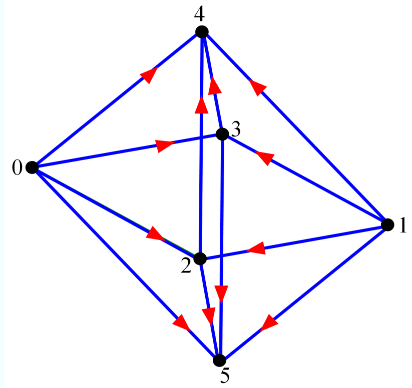
$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$

so that $|\Omega_2| = 8$, whereas $\Omega_p = \{0\}$ for all $p \geq 3$.

We have

$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \rangle$

so that $|H_2| = 1$, and $|H_p| = 0$ for $p = 1$ and $p \geq 3$



Consider an octahedron with a different orientation:

$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$

$\Omega_3 = \langle e_{0234} - e_{0134}, \; e_{0235} - e_{0135} \rangle$

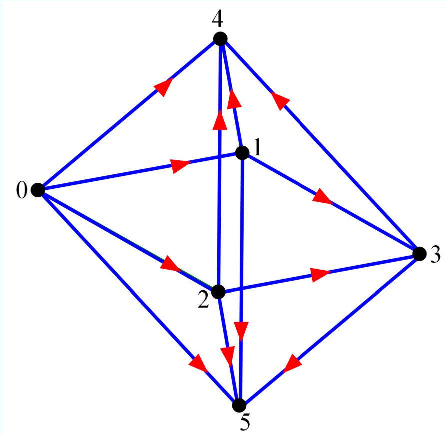
$|\Omega_2| = 9, \quad |\Omega_3| = 2$ and $\Omega_p = \{0\}$ for all $p \geq 4$.

We have $\ker \partial|_{\Omega_2} = \langle u, v \rangle$ where

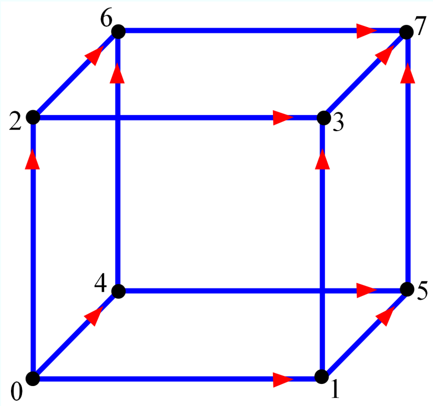
$$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$$
$$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$$

but $H_2 = \{0\}$ because

$$u = \partial\left(e_{0234} - e_{0134}\right) \quad \text{and} \quad v = \partial\left(e_{0235} - e_{0135}\right)$$



Consider a 3-cube:



Here $|\Omega_0| = 8, \quad |\Omega_1| = 12.$

Space Ω_2 is spanned by 6 squares:

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},\\ e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467} \rangle$$

hence, $|\Omega_2| = 6.$

Space Ω_3 is spanned by one 3-cube:

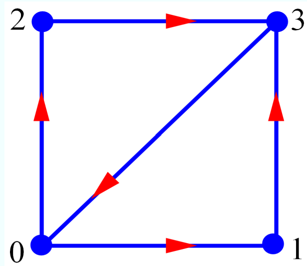
$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \rangle$$

hence, $|\Omega_3| = 1.$

$|\Omega_p| = 0$ for all $p \geq 4$ and $|H_p| = 0$ for all $p \geq 1$.

1.5 An example of computation of Ω_p and H_p

Consider the following digraph with 4 vertices and 5 arrows (a square with a diagonal):



$$\Omega_0 = \mathcal{A}_0 = \langle e_0, e_1, e_2, e_3 \rangle, \quad |\Omega_0| = 4,$$
$$\Omega_1 = \mathcal{A}_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}, e_{30} \rangle, \quad |\Omega_1| = 5,$$
$$\mathcal{A}_2 = \langle e_{013}, e_{023}, e_{130}, e_{230}, e_{301}, e_{302} \rangle \quad |\mathcal{A}_2| = 6.$$

In order to determine $\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\}$,
we first compute $\partial|_{\mathcal{A}_2} \operatorname{mod} \mathcal{A}_1$:

$$\begin{aligned}
\partial e_{013} &= e_{13} - e_{03} + e_{01} = -e_{03} \mod \mathcal{A}_1 \\
\partial e_{023} &= e_{23} - e_{03} + e_{02} = -e_{03} \mod \mathcal{A}_1 \\
\partial e_{130} &= e_{30} - e_{10} + e_{13} = -e_{10} \mod \mathcal{A}_1 \\
\partial e_{230} &= e_{30} - e_{20} + e_{23} = -e_{20} \mod \mathcal{A}_1 \\
\partial e_{301} &= e_{01} - e_{31} + e_{30} = -e_{31} \mod \mathcal{A}_1 \\
\partial e_{302} &= e_{02} - e_{32} + e_{30} = -e_{32} \mod \mathcal{A}_1
\end{aligned}$$

Hence,

$$\text{matrix of } \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1 = \begin{array}{c} \\ e_{03} \\ e_{10} \\ e_{20} \\ e_{31} \\ e_{32} \end{array} \begin{array}{cccccc} e_{013} & e_{023} & e_{130} & e_{230} & e_{301} & e_{302} \end{array} \hspace{-20em} \left(\begin{array}{cccccc} \phantom{e_{013}} & & & & & \\ -1 & -1 & & & & 0 \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ 0 & & & & & -1 \end{array} \right) := D$$

$$\Omega_2 = \ker \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1 = \text{nullspace}\, D = \langle e_{013} - e_{023} \rangle.$$

One can show that $|\Omega_p| = 0$ for all $p \geq 3$ and, hence, $|H_p| = 0$ for all $p \geq 3$.

Let us compute H_1 and H_2. We have for the basis in Ω_1:

$$\begin{aligned} \partial e_{01} &= -e_0 + e_1 \\ \partial e_{02} &= -e_0 + e_2 \\ \partial e_{13} &= -e_1 + e_3 \\ \partial e_{23} &= -e_2 + e_3 \\ \partial e_{30} &= e_0 - e_3 \end{aligned}$$

Hence,

$$\text{matrix of } \partial|_{\Omega_1} = \begin{array}{c} \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{pmatrix} e_{01} & e_{02} & e_{13} & e_{23} & e_{30} \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} =: D$$

and

$$\ker \partial|_{\Omega_1} = \operatorname{nullspace} D = \langle e_{01} + e_{13} - e_{02} - e_{23},\ e_{01} + e_{13} + e_{30} \rangle .$$

Similarly, for the basis in Ω_2 we have

$$\partial \left(e_{013} - e_{023} \right) = \left(e_{13} - e_{03} + e_{01} \right) - \left(e_{23} - e_{03} + e_{02} \right) = e_{01} + e_{13} - e_{02} - e_{23}$$

whence

$$\operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} - e_{02} - e_{23} \rangle \quad \text{and} \quad \ker \partial|_{\Omega_2} = \{0\}\,.$$

It follows that $H_2 = \{0\}$ and

$$H_1 = \ker \partial|_{\Omega_1} / \operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} + e_{30} \rangle .$$

As we have seen, computation of the spaces $\Omega_p \left(G \right)$ and $H_p \left(G \right)$ amounts to computing ranks and null-spaces of large matrices. We currently use for numerical computation of $H_p \left(G, \mathbb{F}_2 \right)$ a C++ program written by Chao Chen in 2012.

1.6 Structure of Ω_2

As we know, $\Omega_0 = \langle e_i \rangle$ consists of all vertices and $\Omega_1 = \{e_{ij} : i \to j\}$ consists of all arrows.

Proposition 1.2 (a) *The space* Ω_2 *is spanned by all triangles* e_{abc}*, squares* $e_{abc} - e_{ab'c}$ *and double arrows* e_{aba}*.*

(b) $|\Omega_2| = |\mathcal{A}_2| - s$ *where* s *is the number of semi-arrows, that is, pairs of vertices* (x, y) *such that* $x \not\to y$ *but* $x \to z \to y$ *for some vertex* z*.*

The triangles and double arrows are always linearly independent but the squares can be dependent.

For example, on this digraph we have three squares:

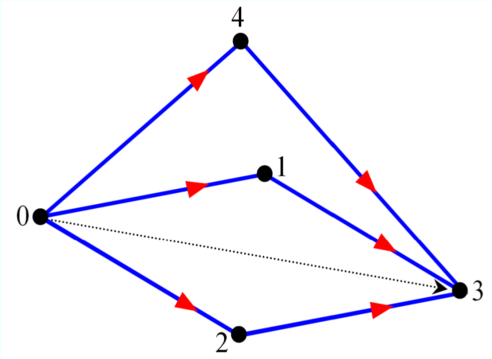
$e_{013} - e_{023}, \quad e_{043} - e_{013}, \quad e_{023} - e_{043}$

but they are linearly dependent as their sum is 0.

Since $\mathcal{A}_2 = \{e_{013}, e_{023}, e_{043}\}$ and there is only one semi-arrow $(0, 3)$, we obtain

$$|\Omega_2| = 2 = |\mathcal{A}_2| - s = 3 - 1 = 2.$$

Clearly, $\Omega_2 = \langle e_{013} - e_{023}, e_{043} - e_{013} \rangle$.

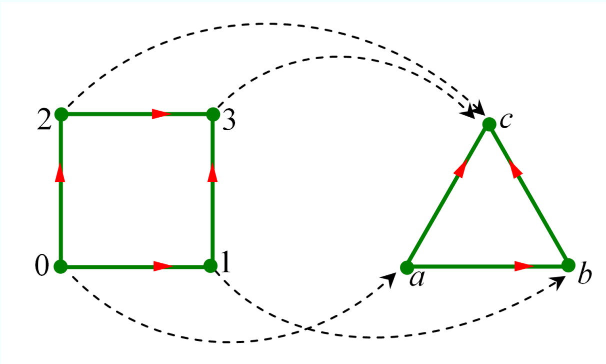


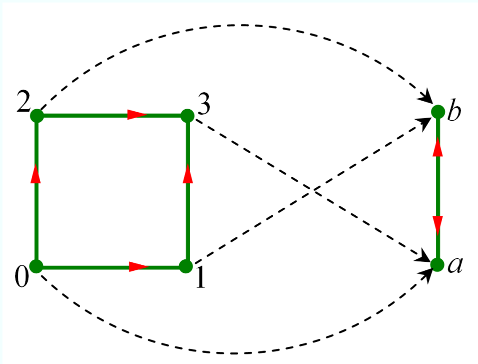
Let X, Y be two digraphs. A map $f : X \to Y$ is called a *morphism of digraphs* if for any arrow $a \to b$ in X we have either $f(a) \to f(b)$ or $f(a) = f(b)$ (that is, the image of an arrow is either an arrow or a vertex). Define images of paths by

$$f\left(e_{i_0 \ldots i_p}\right) = e_{f(i_0)\ldots f(i_p)}$$

so that the image of an allowed path is either allowed or zero (that is also allowed). It is easy to see that $f \circ \partial = \partial \circ f$ so that the morphism images of ∂-invariant paths are again ∂-invariant.

A triangle e_{abc} and a double arrow e_{aba} are morphism images of a square $e_{013} - e_{023}$ as on these pictures:





Hence, we can rephrase Proposition 1.2 as follows: Ω_2 is spanned by squares and their morphism images. Or: squares are *basic shapes* of Ω_2.

Problem 1.3 *Describe all basic shapes in* Ω_3 *(as well as in* Ω_p *for* $p > 3$*).*

1.7 Triangulation as a closed path

Let T be a triangulation of closed oriented n-manifold M, that is, a partition of M into n-dimensional simplexes. Denote by $V = \{0, 1, \dots\}$ the set of all vertices of the simplexes of T and by E – the set of all edges, so that (V, E) is a graph embedded on M.

Let us make each edge $(i, j) \in E$ into an arrow $i \to j$ if $i < j$. Then each simplex from T becomes a digraph-simplex. Denote by $\overrightarrow{T}$ the set of all digraph simplexes constructed in this way. That is, $i_0 \dots i_n \in \overrightarrow{T}$ if $i_0 \dots i_n$ is a monotone increasing sequence that determines a simplex from T. Clearly, any such path $i_0 \dots i_p$ is allowed in the digraph $G = (V, E)$.

For any simplex from T with the vertices $i_0 \dots i_n$ define the quantity $\sigma^{i_0 \dots i_n}$ to be equal to 1 if the orientation of the simplex $i_0 \dots i_n$ matches the orientation of the manifold M, and -1 otherwise. Then consider the following allowed n-path on G:

$$\sigma = \sum_{i_0 \dots i_n \in \overrightarrow{T}} \sigma^{i_0 \dots i_n} e_{i_0 \dots i_n}. \tag{1.3}$$

Lemma 1.4 *The path σ is closed, that is, $\partial \sigma = 0$, which, in particular, implies that σ is ∂-invariant.*

Proof. Observe that $\partial \sigma$ is the a linear combination with coefficients ± 1 of the terms

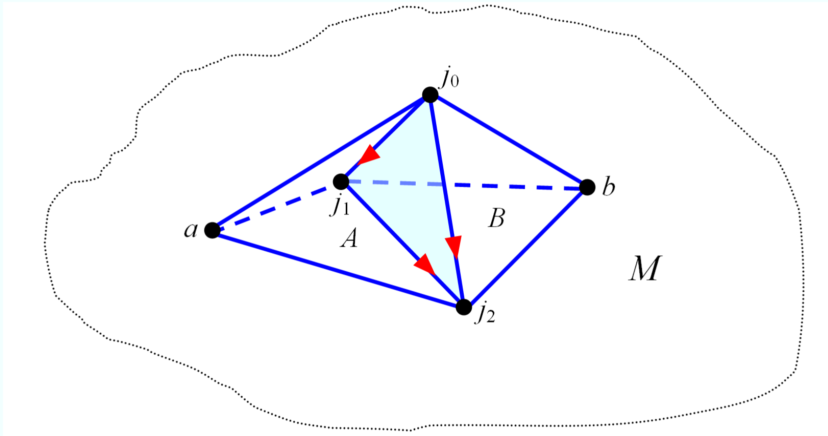
$e_{j_0 \dots j_{n-1}}$ where the sequence $j_0, \dots, j_{n-1}$ is monotone increasing and forms an $(n-1)$-dimensional face of one of the n-simplexes from T. In fact, every $(n-1)$-face arises from *two* n-simplexes, say

$A = j_0 \dots j_{k-1} a j_k \dots j_{n-1}$

and

$B = j_0 \dots j_{l-1} b j_l \dots j_{n-1}$

that is, two n-simplexes A, B have a common $(n-1)$-dimensional face $j_0 \dots j_{n-1}$.



3-simplexes A and B

We have

$$\partial e_A = \partial e_{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = \dots + (-1)^k \, e_{j_0 \dots j_{k-1} j_k \dots j_{n-1}} + \dots \, .$$

Since interchanging the order of two neighboring vertices in an n-simplex changes its orientation, we have

$$\sigma^A = \sigma^{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = (-1)^k \, \sigma^{a j_0 \dots j_{k-1} j_k \dots j_{n-1}}.$$

Multiplying the above lines, we obtain

$$\partial\left(\sigma^{A}e_{A}\right)=...+\sigma^{aj_0...j_{n-1}}e_{j_0...j_{n-1}}+...\ ,$$

and in the same way

$$\partial\left(\sigma^{B}e_{B}\right)=...+\sigma^{bj_0...j_{n-1}}e_{j_0...j_{n-1}}+...$$

However, the vertices a and b are located on the opposite sides of the face $j_0...j_{n-1}$, which implies that the simplexes $aj_0...j_{n-1}$ and $bj_0...j_{n-1}$ have the opposite orientations relative to that of M. Hence,

$$\sigma^{aj_0...j_{n-1}}+\sigma^{bj_0...j_{n-1}}=0,$$

which means that the term $e_{j_0...j_{n-1}}$ cancels out in the sum $\partial\left(\sigma^{A}e_{A}+\sigma^{B}e_{B}\right)$ and, hence, in $\partial\sigma$. This proves that $\partial\sigma=0$. $\blacksquare$

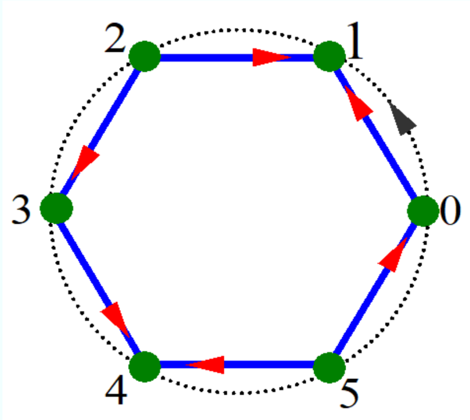
The closed path σ defined by (1.3) is called a *surface path* on M.

There is a number of examples when a surface path σ happens to be exact, that is, $\sigma=\partial v$ for some $(n+1)$-path v. In this case v is called a *solid path* on M because v represents a "solid" shape whose boundary is given by a surface path. If σ is not exact then σ determines a non-trivial homology class from $H_n\left(G\right)$ and, hence, represents a "cavity" of triangulation T.

Example. $M = \mathbb{S}^1$.

A triangulation of $\mathbb{S}^1$ is a polygon, and the corresponding digraph G is *cyclic*.

On each edge (i, j) of a polygon we choose an arrow $i \to j$ arbitrary (not necessarily if $i < j$).



We have

$$\sigma = \sum_{i \to j} \sigma^{ij} e_{ij}$$

where $\sigma^{ij} = 1$ if the arrow $i \to j$ goes counterclockwise, and $\sigma^{ij} = -1$ otherwise.

On the digraph on the picture we have

$$\sigma = e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50}$$

Proposition 1.5 *(a) If a polygon G is neither triangle nor square then $\Omega_p = \{0\}$ for all $p \geq 2$, $H_1 = \langle \sigma \rangle$ and $H_p = \{0\}$ for all $p \geq 2$.*

(b) If G is either triangle or square then $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for all $p \geq 1$.

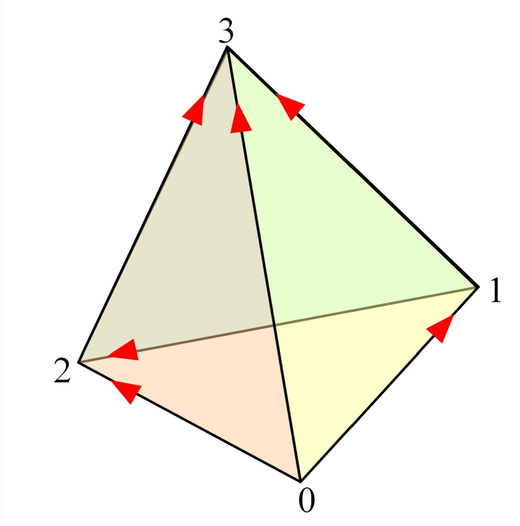
Example. Let $M = \mathbb{S}^n$ and let triangulation of $\mathbb{S}^n$ be given by an $(n+1)$-simplex.

Then G is a $(n+1)$-simplex digraph.

On this picture $n = 2$,

$\sigma = e_{123} - e_{023} + e_{013} - e_{012} = \partial e_{0123}$

so that e_{0123} is a solid path representing a tetrahedron.



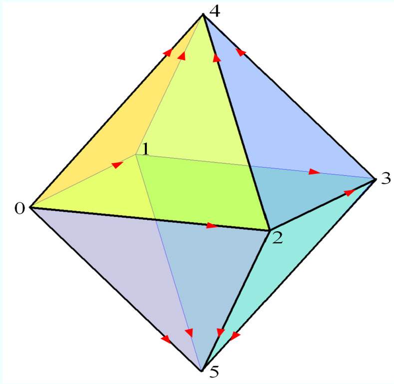
In general we also have

$$\sigma = \partial e_{0\ldots n+1}$$

so that $e_{0\ldots n+1}$ is a solid path representing a $(n+1)$-simplex.

Example. $M = \mathbb{S}^2$, octahedron.

Here is a triangulation of $\mathbb{S}^2$ by an *octahedron* with two ways of numbering.



A

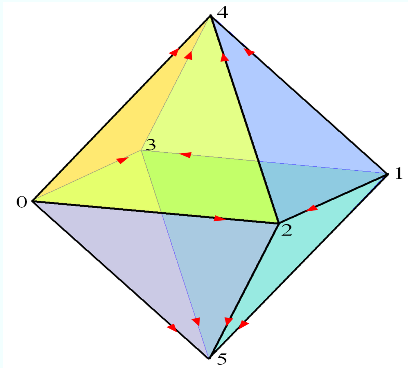
Case A: $\quad H_2 = \{0\}$

$$\sigma = e_{024} - e_{025} - e_{014} + e_{015} - e_{234} + e_{235} + e_{134} - e_{135}$$
$$= \partial\left(e_{0134} - e_{0234} + e_{0135} - e_{0235}\right)$$

Hence,

$$v = e_{0134} - e_{0234} + e_{0135} - e_{0235}$$

is a solid path, and the octahedron represents a solid shape.



B

Case B: $\quad H_2 = \langle \sigma \rangle$

$$\sigma = e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135}$$

and the octahedron represents a cavity.

Example. $M = \mathbb{S}^2$, icosahedron.

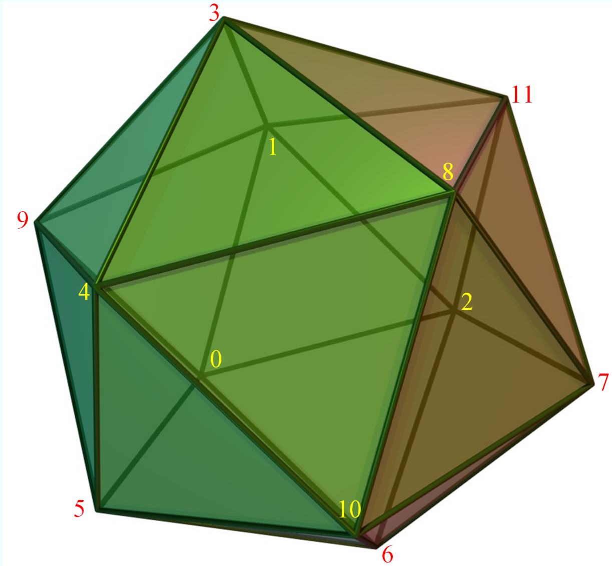
Consider an *icosahedron* as a triangulation of $\mathbb{S}^2$:
(here $i \to j$ if $i < j$).

We have $|V| = 12$, $|E| = 30$, $H_1 = \{0\}$,

and $H_2 = \langle \sigma \rangle$ where

$$\sigma = -e_{0\,1\,9} + e_{0\,1\,2} - e_{1\,2\,11} + e_{0\,2\,6} + e_{0\,5\,9} \\ -e_{0\,5\,6} + e_{5\,6\,10} - e_{1\,3\,9} + e_{1\,3\,11} - e_{2\,6\,7} \\ +e_{6\,7\,10} - e_{2\,7\,11} - e_{3\,4\,9} + e_{3\,4\,8} - e_{4\,8\,10} \\ +e_{3\,8\,11} - e_{4\,5\,9} + e_{4\,5\,10} + e_{7\,8\,10} - e_{7\,8\,11}.$$

Hence, the icosahedron represents a cavity.



Conjecture 1.6 *For icosahedron* $\dim H_2(G) = 1$ *for any numbering of the vertices.*

Conjecture 1.7 *For a general triangulation of* $\mathbb{S}^n$, *the homology group* $H_n(G)$ *is either trivial or is generated by* σ. *All other homology groups* $H_p(G)$ *are trivial.*

1.8 Computational challenge

An interesting paper:

frontiers in Computational Neuroscience

ORIGINAL RESEARCH
published: 12 June 2017
doi: 10.3389/fncom.2017.00048

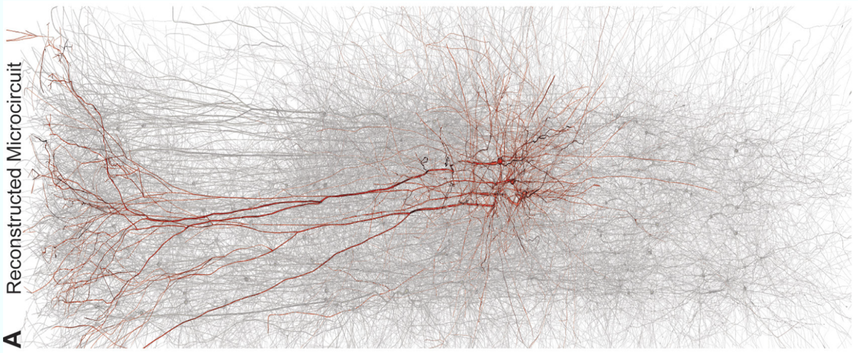
Check for updates

Cliques of Neurons Bound into Cavities Provide a Missing Link between Structure and Function

Michael W. Reimann[1†], Max Nolte[1†], Martina Scolamiero[2], Katharine Turner[2], Rodrigo Perin[3], Giuseppe Chindemi[1], Paweł Dłotko[4‡], Ran Levi[5‡], Kathryn Hess[2‡] and Henry Markram[1,3*‡]*

[1] Blue Brain Project, École Polytechnique Fédérale de Lausanne, Geneva, Switzerland, [2] Laboratory for Topology and Neuroscience, Brain Mind Institute, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland, [3] Laboratory of Neural Microcircuitry, Brain Mind Institute, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland, [4] DataShape, INRIA Saclay, Palaiseau, France, [5] Institute of Mathematics, University of Aberdeen, Aberdeen, United Kingdom

They reconstruct a microcircuit from a rat brain as a graph (neurons and connections between them). The size of the graph is $|V| \sim 31,000$ and $|E| \sim 8,000,000$.



Then they detect cliques in this graph, form out of the cliques a simplicial complex, and compute its Betti numbers over $\mathbb{F}_2$. They were able to compute Betti number β_5 and to show that $\beta_5 > 0$.

Problem 1.8 *Create computational tools capable of computing low dimensional Betti numbers for path homologies of digraphs of similar size.*

At present our program can compute β_1 on a digraph with $|V| \sim 7000$ and $|E| \sim 100,000$, and β_2 on a digraph with $|V| \sim 4000$ and $|E| \sim 25000$.

2 Combinatorial curvature of digraphs

2.1 Motivation

Let Γ be a finite planar graph. There is the following old notion of *combinatorial curvature* K_x at any vertex x of Γ:

$$K_x = 1 - \frac{\deg(x)}{2} + \sum_{f \ni x} \frac{1}{\deg(f)}, \tag{2.1}$$

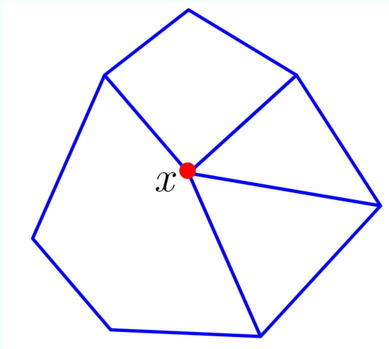
where the sum is taken over all faces f containing x and $\deg(f)$ denotes the number of vertices of f.

For example, for this graph we have

$$\deg(x) = 4$$

and

$K_x = 1 - \frac{4}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{7}{60}.$



A planar graph Γ

In particular, if all faces of Γ are triangles then we obtain from (2.1)

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3}, \tag{2.2}$$

where $\deg_\Delta(x)$ is the number of triangular faces having x as a vertex.

In general, denote by V, E and F the number of vertices, edges and faces of Γ, respectively, and observe that

$$\sum_x \deg(x) = 2E \quad \text{and} \quad \sum_x \sum_{f \ni x} \frac{1}{\deg(f)} = \sum_f \sum_{x \in f} \frac{1}{\deg(f)} = F.$$

Hence, we obtain from (2.1)

$$K_{total} := \sum_x K_x = V - E + F = \chi,$$

where χ is the Euler characteristic of the graph.

We try to realize this idea in order to define the curvature on an arbitrary digraph: to "distribute" the Euler characteristic over all vertices and, hence, to obtain an analog of the Gauss curvature that satisfies the Gauss-Bonnet formula.

2.2 Curvature operator

Let $G = (V, E)$ be a finite digraph and $\mathbb{K} = \mathbb{R}$. We would like to generalize (2.1) to arbitrary digraphs, so that the faces in (2.1) should be replaced by the elements of a basis in Ω_p. However, the result should be independent of the choice of a basis.

Fix $p \geq 0$. Any function $f : V \to \mathbb{R}$ induces a linear operator on the space $\mathcal{R}_p$ of regular p-paths

$$T_f : \mathcal{R}_p \to \mathcal{R}_p$$

as follows

$$T_f e_{i_0 \ldots i_p} = \left(f\left(i_0\right) + \ldots + f\left(i_p\right)\right) e_{i_0 \ldots i_p}.$$

For example, for a constant function $f = \mathbf{1}$ on V, we have $T_{\mathbf{1}} e_{i_0 \ldots i_p} = (p+1)\, e_{i_0 \ldots i_p}$ and, hence,

$$T_{\mathbf{1}} \omega = (p+1)\, \omega \quad \text{for any } \omega \in \mathcal{R}_p. \tag{2.3}$$

If $f = \mathbf{1}_x$ where $x \in V$, then

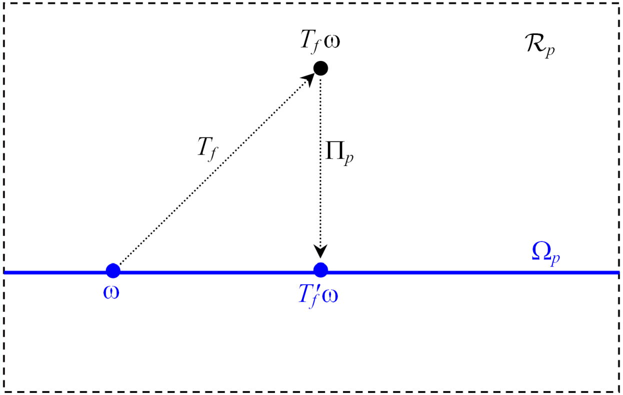
$$T_{\mathbf{1}_x} e_{i_0 \ldots i_p} = m e_{i_0 \ldots i_p}, \text{ where } m \text{ is the number of occurrences of } x \text{ in } i_0, \ldots, i_p. \tag{2.4}$$

Fix in $\mathcal{R}_p$ an inner product $(\cdot, \cdot)$. For example, this can be a *natural inner product* when all regular elementary paths $e_{i_0 \ldots i_p}$ form an orthonormal basis in $\mathcal{R}_p$.

Let $\Pi_p : \mathcal{R}_p \to \Omega_p$ be the orthogonal projection onto Ω_p in the space $\mathcal{R}_p$.

Considering T_f as an operator from Ω_p to $\mathcal{R}_p$, we obtain the following operator in Ω_p:

$$T_f' := \Pi_p \circ T_f : \Omega_p \to \Omega_p$$



Definition. Define the *incidence* of f and Ω_p by $\boxed{[f, \Omega_p] = \operatorname{trace} T_f'}$.

Example. Assume that $\Omega_2 = \langle e_{abc} - e_{ab'c} \rangle$ and let

$$f(a) = 2, f(b) = f(c) = 1, f(b') = 0.$$

Then for $\omega = e_{abc} - e_{ab'c}$ we have

$$T_f \omega = (f(a) + f(b) + f(c))\, e_{abc} - (f(a) + f(b') + f(c))\, e_{ab'c} = 4e_{abc} - 3e_{ab'c}.$$

Setting $\overline{\omega} = \frac{e_{abc} - e_{ab'c}}{\sqrt{2}}$ we obtain

$$T_f' \omega = (T_f \omega, \overline{\omega})\, \overline{\omega} = (4e_{abc} - 3e_{ab'c}, \tfrac{e_{abc} - e_{ab'c}}{\sqrt{2}})\overline{\omega} = \tfrac{7}{\sqrt{2}} \overline{\omega}.$$

It follows that $[f,\Omega_2] = \operatorname{trace} T'_f = \left(T'_f\overline{\omega},\overline{\omega}\right) = (\frac{7}{2}\overline{\omega},\overline{\omega}) = \frac{7}{2}$.

In order to compute $[f,\Omega_p]$ in general, we need the following notion.

Definition. For any $\omega \in \Omega_p$ define the *incidence* of f and ω by $\boxed{[f,\omega] = (T_f\omega,\omega)}$.

Lemma 2.1 *For any orthogonal basis* $\{\omega_k\}$ *in* Ω_p *we have*

$$[f,\Omega_p] = \sum_k \frac{[f,\omega_k]}{\|\omega_k\|^2}. \tag{2.5}$$

Proof. It suffices to prove (2.5) for orthonormal basis when $\|\omega_k\| = 1$ for all k. By the definition of the trace

$$\operatorname{trace} T'_f = \sum_k \left(T'_f\omega_k,\omega_k\right).$$

For any $\omega \in \Omega_p$ (in particular, for $\omega = \omega_p$) we have

$$\left(T'_f\omega,\omega\right) = (\Pi_p T_f\omega,\omega) = (T_f\omega,\Pi_p\omega) = (T_f\omega,\omega) = [f,\omega],$$

whence (2.5) follows. ■

Definition. For any $N \in \mathbb{N}$ define the *curvature operator* $K^{(N)} : \mathbb{R}^V \to \mathbb{R}$ of order N by

$$\boxed{K^{(N)} f = \sum_{p=0}^{N} \frac{(-1)^p}{p+1} [f, \Omega_p]}.$$

If $\Omega_p = \{0\}$ for all $p > N$, then write $K_f^{(N)} = K_f$.

For $f = \mathbf{1}_x$ where $x \in V$, we write

$$[x, \Omega_p] := [\mathbf{1}_x, \Omega_p] \quad \text{and} \quad [x, \omega] := [\mathbf{1}_x, \omega],$$

If $\{\omega_k\}$ is an orthogonal basis of Ω_p, then by (2.5)

$$\boxed{[x, \Omega_p] = \sum_k \frac{[x, \omega_k]}{\|\omega_k\|^2}}.$$

If the inner product is natural so that $\{e_{i_0 \ldots i_p}\}$ is orthonormal then by (2.4)

$$\boxed{\left[x, e_{i_0 \ldots i_p}\right] = m, \text{ where } m \text{ is the number of occurrences of } x \text{ in } i_0, \ldots, i_p}.$$

For example,

$$[a, e_{abca}] = 2, \qquad [b, e_{abca}] = 1, \quad [d, e_{abca}] = 0.$$

In this case, for $\omega = \sum \omega^{i_0 \dots i_p} e_{i_0 \dots i_p}$ we have

$$\boxed{[x, \omega] = \sum_{i_0 \dots i_p \in V} \left(\omega^{i_0 \dots i_p}\right)^2 \left[x, e_{i_0 \dots i_p}\right]}.$$

Definition. For any $N \in \mathbb{N}$ define the *curvature of order* N at a vertex x by

$$\boxed{K_x^{(N)} := K^{(N)} \mathbf{1}_x = \sum_{p=0}^{N} \frac{(-1)^p}{p+1} \left[x, \Omega_p\right]}.$$

Proposition 2.2 (Gauss-Bonnet) *For any choice of the inner product in* $\mathcal{R}_p$ *and for any* N *we have*

$$K_{total}^{(N)} := \sum_{x \in V} K_x^{(N)} = \sum_{p=0}^{N} (-1)^p \dim \Omega_p =: \chi^{(N)}.$$

Proof. Since $\sum_{x \in V} \mathbf{1}_x = \mathbf{1}$, we obtain that

$$K_{total}^{(N)} = \sum_{x \in V} K_x^{(N)} = \sum_{x \in V} K^{(N)} \mathbf{1}_x = K^{(N)} \mathbf{1} = \sum_{p=0}^{N} (-1)^p \frac{\left[\mathbf{1}, \Omega_p\right]}{p+1}.$$

On the other hand, by (2.3)

$$[\mathbf{1},\omega]=(T_{\mathbf{1}}\omega,\omega)=(p+1)\left\|\omega\right\|^{2}.$$

If $\{\omega_k\}$ is an orthogonal basis in Ω_p then by (2.5)

$$[\mathbf{1},\Omega_p]=\sum_k\frac{[\mathbf{1},\omega_k]}{\left\|\omega_k\right\|^2}=(p+1)\dim\Omega_p,$$

which implies

$$K_{total}^{(N)}=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\chi^{(N)}.$$

■

Remark. If $\Omega_p=\{0\}$ for all $p>N$ then

$$\chi:=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\sum_{p=0}^{N}(-1)^p\dim H_p.$$

Remark. It can happen that $\Omega_p\neq\{0\}$ for all p. For example, let $G=\{a\rightleftarrows b\}$. For this digraph we have

$$\Omega_0=\langle e_a,e_b\rangle,\quad \Omega_1=\langle e_{ab},e_{ba}\rangle,\quad \Omega_3=\langle e_{aba},e_{bab}\rangle,\quad \Omega_4=\{e_{abab},e_{baba}\},\quad \text{etc},$$

so that $|\Omega_p| = 2$ for all $p \geq 0$. Indeed, $e_{aba} \in \mathcal{A}_2$ and

$$\partial e_{aba} = e_{ba} - e_{aa} + e_{ab} = e_{ba} + e_{ab} \in \mathcal{A}_1$$

so that $e_{aba} \in \Omega_2$. Similarly, $e_{abab} \in \mathcal{A}_3$ and

$$\partial e_{abab} = e_{bab} - e_{aab} + e_{abb} - e_{aba} = e_{bab} - e_{aba} \in \mathcal{A}_2$$

so that $e_{abab} \in \Omega_3$, etc.

Problem 2.3 *How to decide whether the sequence $\{\Omega_p(G)\}$ vanishes for all large p?*

Alternatively, one can always truncate the chain complex to make it finite by setting by definition $\Omega_{N+1} = \{0\}$ for some N :

$$0 \leftarrow \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Omega_{N-1} \stackrel{\partial}{\leftarrow} \Omega_N \leftarrow 0$$

and work with homology groups of this complex. This corresponds to the following modification of the notion of allowed paths: all paths of length $> N$ are declared non-allowed.

2.3 Examples of computation of curvature

Let us fix in $\mathcal{R}_p$ the natural inner product. Using the orthonormal basis $\{e_i\}$ in Ω_0 we obtain

$$[x, \Omega_0] = \sum_i [x, e_i] = 1$$

and, using the orthonormal basis $\{e_{ij}\}$ with $i \to j$ in Ω_1, we obtain

$$[x, \Omega_1] = \sum_{i \to j} [x, e_{ij}] = \deg(x) .$$

Therefore,

$$K_x^{(1)} = 1 - \frac{\deg(x)}{2}$$

and, for any $N \geq 1$,

$$K_x^{(N)} = 1 - \frac{\deg(x)}{2} + \sum_{p=2}^{N} \frac{(-1)^p}{p+1} [x, \Omega_p] . \tag{2.6}$$

By Proposition 1.2, in the absence of double arrows the space Ω_2 has always a basis of triangles and squares (but this basis is not necessarily orthogonal).

For a triangle $e_{abc} \in \Omega_2$ we have

$$[x, e_{abc}] = \begin{cases} 1, & x \in \{a, b, c\} \\ 0, & \text{otherwise} \end{cases}$$

and for a square $e_{abc} - e_{ab'c} \in \Omega_2$

$$[x, e_{abc} - e_{ab'c}] = \begin{cases} 2, & x \in \{a, c\} \\ 1, & x \in \{b, b'\} \\ 0, & \text{otherwise} \end{cases}$$

In particular, if G has no square then Ω_2 has a basis $\{\omega_k\}$ that consists of all triangles in G. This basis is orthonormal and

$$[x, \Omega_2] = \sum_k [x, \omega_k] = \deg_\Delta(x) := \#\text{triangles containing } x.$$

It follows that

$$K_x^{(2)} = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3},$$

which matches (2.2).

Example. Let G be a line digraph, for example, $\cdots \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \ldots$. Then by (2.6) $K_x = \frac{1}{2}$ for the endpoints, and $K_x = 0$ for the interior points.

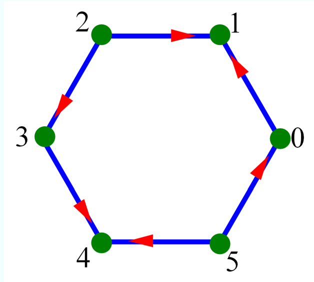
Example. Let G be a cyclic digraph (polygon) different from triangle or square:

Then we have $\Omega_p = \{0\}$ for $p > 1$.
Hence by (2.6), for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = 0.$$

and $K_{total} = 0$.
For comparison,

$$\chi = |\Omega_0| - |\Omega_1| = 6 - 6 = 0.$$



.

Example. Consider a dodecahedron (with any orientation of edges):

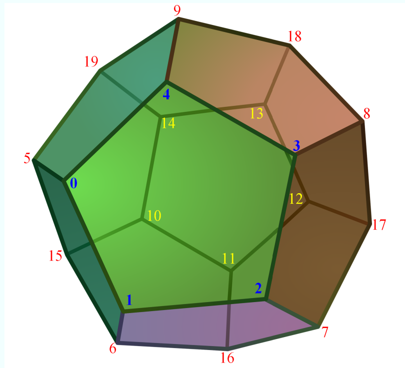
We have $|\Omega_0| = 20$, $|\Omega_1| = 30$, $|\Omega_2| = 0$,
and $|H_1| = 11$, $|H_p| = 0$ for $p > 1$.

Then, for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = -\frac{1}{2}$$

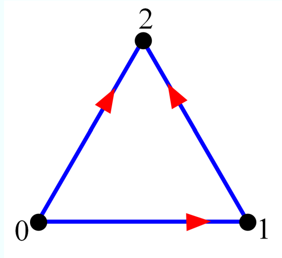
and $K_{total} = -10$.

For comparison,
$\chi = 1 - 11 = 20 - 30 = -10$.



Example. Let G be a triangle. We have $\Omega_2 = \langle e_{012} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.

Hence, for each vertex x,

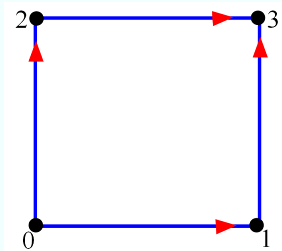


$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = \frac{1}{3}.$$

and $K_{total} = 1$.
For comparison, $\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = 3 - 3 + 1 = 1$.

Example. Let G be a square. Then $\Omega_2 = \langle e_{013} - e_{023} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.

Since $\|e_{013} - e_{023}\|^2 = 2$, we obtain



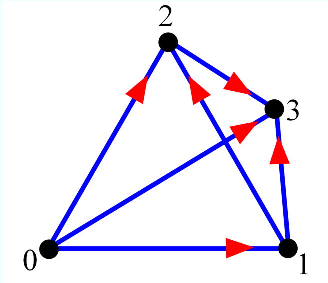
$$[0, \Omega_2] = \tfrac{1}{2}[0, e_{013} - e_{023}] = 1, \qquad [3, \Omega_2] = 1$$

$$[1, \Omega_2] = \tfrac{1}{2}[1, e_{013} - e_{023}] = \tfrac{1}{2}, \qquad [2, \Omega_2] = \tfrac{1}{2}$$

It follows that

$$K_3 = K_0 = 1 - \frac{\deg(0)}{2} + \frac{1}{3} = \frac{1}{3}, \qquad K_2 = K_1 = 1 - \frac{\deg(1)}{2} + \frac{1}{6} = \frac{1}{6}, \quad K_{total} = 1 = \chi.$$

Example. Let G be a 3-simplex



We have

$$\Omega_2 = \langle e_{012}, e_{013}, e_{023}, e_{123} \rangle$$

and

$$\Omega_3 = \langle e_{0123} \rangle,$$

while $\Omega_p = 0$ for $p > 3$. It follows that, for any vertex x,

$$[x, \Omega_2] = \deg_\Delta(x) = 3 \quad \text{and} \quad [x, \Omega_3] = 1$$

whence

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x, \Omega_2]}{3} - \frac{[x, \Omega_3]}{4} = \frac{1}{4}, \qquad K_{total} = 1 = \chi.$$

Example. Let G be an *n-simplex*, that is, a digraph with a set of vertices $\{0,1,...,n\}$ and edges $i \to j$ whenever $i<j$. Then, for any $p=0,1,...,n$

$$\Omega_p = \mathcal{A}_p = \langle e_{i_0...i_p} : i_0 < i_1 < ... < i_p \rangle$$

so that $\dim \Omega_p = \binom{n+1}{p+1}$. It follows that, for any vertex x,

$$[x, \Omega_p] = \# \left\{ e_{i_0...i_p} \text{ such that } x \in \{i_0, ..., i_p\} \right\} = \binom{n}{p},$$

and

$$K_x = \sum_{p=0}^{n} (-1)^p \frac{\binom{n}{p}}{p+1}.$$

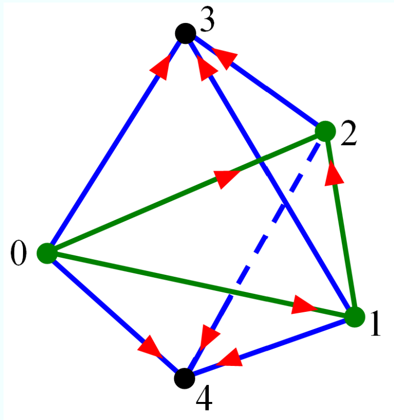
Change $j = p+1$ gives

$$(n+1)\, K_x = \sum_{j=1}^{n+1} (-1)^{j-1} \frac{(n+1)\binom{n}{j-1}}{j} = \sum_{j=1}^{n+1} (-1)^{j-1} \binom{n+1}{j} = 1,$$

whence

$$K_x = \frac{1}{n+1} \quad \text{and} \quad K_{total} = 1.$$

Example. Let G be a bipyramid:



We have $|\Omega_0| = 5, \quad |\Omega_1| = 9,$

$\Omega_2 = \langle e_{013}, e_{123}, e_{023}, e_{014}, e_{124}, e_{024}, e_{012} \rangle$

$\Omega_3 = \langle e_{0123}, e_{0124} \rangle$

and $|\Omega_p| = 0$ for $p \geq 4$.

Hence,
$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 5 - 9 + 7 - 2 = 1.$

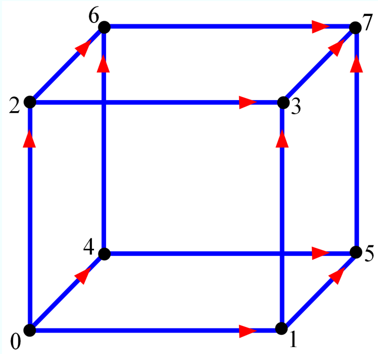
Let us compute the curvature:

x	$[x, \Omega_2]$	$[x, \Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$3, 4$	3	1	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$0, 1, 2$	5	2	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$

Hence, $K_{total} = \frac{2}{4} + \frac{3}{6} = 1.$

Example. Let G be a 3-cube. We have

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046}, \\ e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467}\rangle$$



(note that this above basis in Ω_2 is orthogonal)

$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}\rangle$$

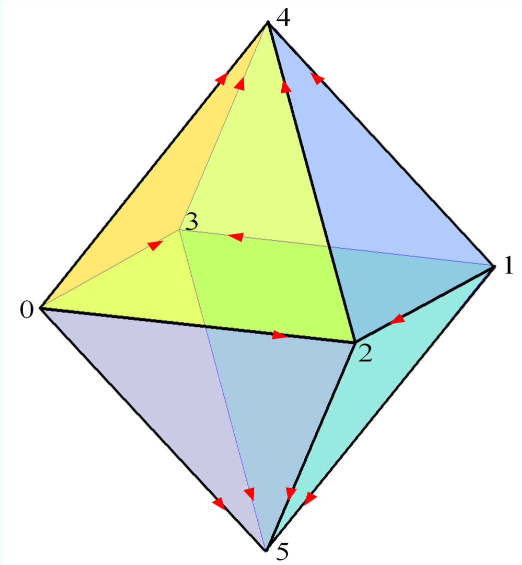
$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 8 - 12 + 6 - 1 = 1$$

Let us compute the curvature:

x	$\frac{[x,\Omega_2]}{}$	$\frac{[x,\Omega_3]}{}$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$0, 7$	$\frac{6}{2} = 3$	$\frac{6}{6} = 1$	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$1, 2, 3, 4, 5, 6$	$\frac{4}{2} = 2$	$\frac{2}{6} = \frac{1}{3}$	$1 - \frac{3}{2} + \frac{2}{3} - \frac{1}{12} = \frac{1}{12}$	$= \frac{1}{12}$

Consequently, $K_{total} = \frac{2}{4} + \frac{6}{12} = 1 = \chi$.

Example. Consider on octahedron:



We have

$$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$$

and $\Omega_p = \{0\}$ for all $p \geq 3$

For any vertex x we obtain

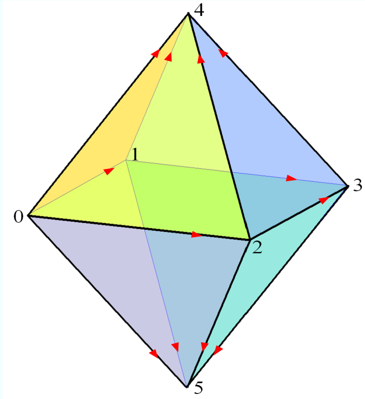
$$[x, \Omega_2] = \deg_\Delta(x) = 4$$

whence

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = 1 - \tfrac{4}{2} + \tfrac{4}{3} = \tfrac{1}{3}$$

In particular, $K_{total} = \frac{6}{3} = 2 = \chi$.

Example. Consider on octahedron with a different orientation:



We have the following orthogonal bases:

$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023} \rangle$

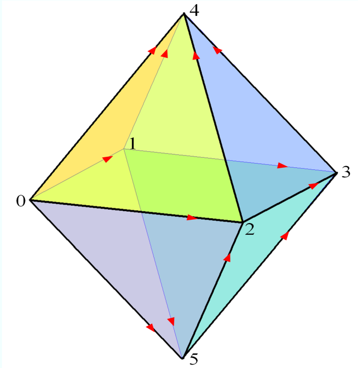
$\Omega_3 = \langle e_{0234} - e_{0134},\ e_{0235} - e_{0135} \rangle$

$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 6 - 12 + 9 - 2 = 1$

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
0	$4 + \frac{2}{2} = 1$	$\frac{4}{2} = 2$	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$
1	$4 + \frac{1}{2} = \frac{9}{2}$	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{9/2}{3} - \frac{1}{4}$	$= \frac{1}{4}$
2	$4 + \frac{1}{2} = \frac{9}{2}$	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{9/2}{3} - \frac{1}{4}$	$= \frac{1}{4}$
3	$4 + \frac{2}{2} = 5$	$\frac{4}{2} = 2$	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$
4	4	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{4}{3} - \frac{1}{4}$	$= \frac{1}{12}$
5	4	$\frac{2}{2} = 1$	$1 - \frac{4}{2} + \frac{4}{3} - \frac{1}{4}$	$= \frac{1}{12}$

$K_{total} = \frac{2}{6} + \frac{2}{4} + \frac{2}{12} = 1 = \chi.$

Example. Here is yet another octahedron. We have to orthogonalize the bases:



$$\begin{aligned}\Omega_2 &= \langle e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523},\\ &\qquad e_{013}-e_{023},\ e_{013}-e_{053},\ e_{524}-e_{534}\rangle \\ &= \langle e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523},\\ &\qquad e_{013}-e_{023},\ e_{013}+e_{023}-2e_{053},\ e_{524}-e_{534}\rangle\end{aligned}$$

$$\begin{aligned}\Omega_3 &= \langle e_{0153},\ e_{0523},\ e_{5234},\ e_{0134}-e_{0234},\ e_{0534}-e_{0134}-e_{0524}\rangle \\ &= \langle e_{0153},\ e_{0523},\ e_{5234},\ e_{0134}-e_{0234},\ e_{0134}+e_{0234}-2e_{0534}+2e_{0524}\rangle\end{aligned}$$

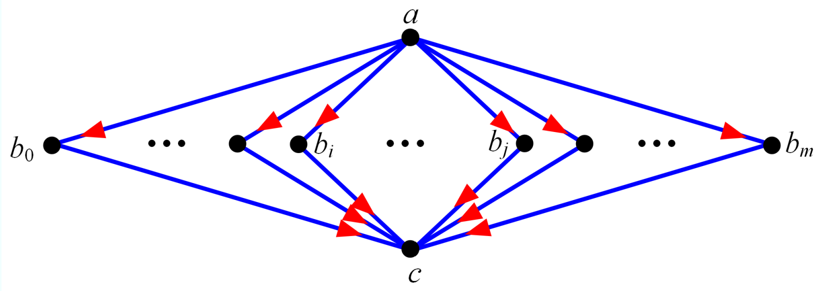
$\Omega_4 = \langle e_{05234}\rangle$, $\Omega_p = \{0\}$ for $p \geq 5$.

$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| + |\Omega_4| = 6 - 12 + 11 - 5 + 1 = 1.$$

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$[x,\Omega_4]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4} + \frac{[x,\Omega_4]}{5}$	$= K_x$
0	$4+\frac{2}{2}+\frac{6}{6}=6$	$2+\frac{2}{2}+\frac{10}{10}=4$	1	$1-\frac{4}{2}+\frac{6}{3}-\frac{4}{4}+\frac{1}{5}$	$=\frac{1}{5}$
1	$4+\frac{1}{2}+\frac{1}{6}=\frac{14}{3}$	$1+\frac{1}{2}+\frac{1}{10}=\frac{8}{5}$	0	$1-\frac{4}{2}+\frac{14/3}{3}-\frac{8/5}{4}$	$=\frac{7}{45}$
2	$4+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}=\frac{31}{6}$	$2+\frac{1}{2}+\frac{5}{10}=3$	1	$1-\frac{4}{2}+\frac{31/6}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{31}{180}$
3	$4+\frac{2}{2}+\frac{6}{6}+\frac{1}{2}=\frac{13}{2}$	$3+\frac{2}{2}+\frac{6}{10}=\frac{23}{5}$	1	$1-\frac{4}{2}+\frac{13/2}{3}-\frac{23/5}{4}+\frac{1}{5}=\frac{13}{60}$	$=\frac{13}{60}$
4	$4+\frac{2}{2}=5$	$1+\frac{2}{2}+\frac{10}{10}=3$	1	$1-\frac{4}{2}+\frac{5}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{7}{60}$
5	$4+\frac{4}{6}+\frac{2}{2}=\frac{17}{3}$	$3+\frac{8}{10}=\frac{19}{5}$	1	$1-\frac{4}{2}+\frac{17/3}{3}-\frac{19/5}{4}+\frac{1}{5}$	$=\frac{5}{36}$

$$K_{total} = \frac{1}{5}+\frac{7}{45}+\frac{31}{180}+\frac{13}{60}+\frac{7}{60}+\frac{5}{36} = 1 = \chi$$

Example. Consider the following spider-like digraph G:



The space Ω_2 consists of squares $e_{ab_ic} - e_{ab_jc}$ and their linear combinations, while $\Omega_p = \{0\}$ for all $p > 2$. It is easy to see that

$$\Omega_2 = \langle e_{ab_0c} - e_{ab_jc} \rangle_{j=1}^{m} \tag{2.7}$$

so that $|\Omega_2| = m$ and $K_{total} = \chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = (m+3) - 2\,(m+1) + m = 1.$

Orthogonalization of (2.7) gives the following orthogonal basis in Ω_2:

$$\begin{aligned}
\omega_1 &= e_{ab_0c} - e_{ab_1c} \\
\omega_2 &= e_{ab_0c} + e_{ab_1c} - 2e_{ab_2c} \\
&\dots
\end{aligned}$$

$$\omega_i = e_{ab_0c} + ... + e_{ab_{i-1}c} - ie_{ab_ic}$$
$$...$$
$$\omega_m = e_{ab_0c} + ... + e_{ab_{m-1}c} - me_{ab_mc}$$

We have $[a, \omega_i] = [c, \omega_i] = \|\omega_i\|^2 = i\,(i+1)$ while

$$[b_j, \omega_i] = \begin{cases} 0, & j > i \\ 1, & j < i \\ i^2, & j = i \end{cases}$$

which implies

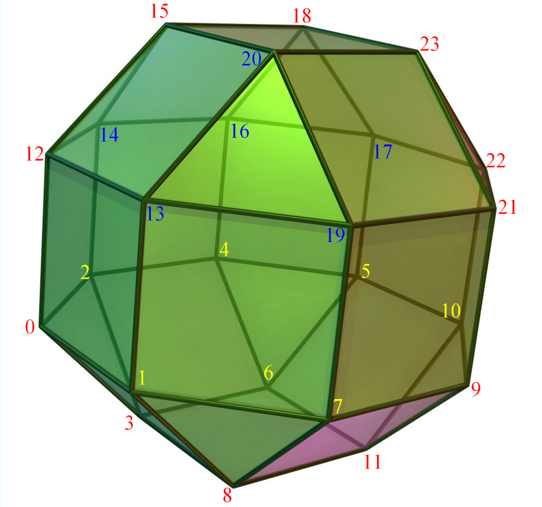
$$K_c = K_a = 1 - \frac{\deg(a)}{2} + \frac{1}{3}\sum_{i=1}^{m} \frac{[a, \omega_i]}{\|\omega_i\|^2} = 1 - \frac{m+1}{2} + \frac{m}{3} = \frac{5}{6} - \frac{m}{6}$$

and

$$K_{b_j} = 1 - \frac{\deg(b_j)}{2} + \frac{1}{3}\sum_{i=1}^{m} \frac{[b_j, \omega_i]}{i\,(i+1)} = \frac{1}{3}\frac{j^2}{j\,(j+1)} + \frac{1}{3}\sum_{i=j+1}^{m} \frac{1}{i\,(i+1)} = \frac{1}{3}\left(1 - \frac{1}{m+1}\right).$$

Example. Consider a rhombicuboctahedron:

It has 24 vertices, 48 edges and 26 faces, among them 8 triangular and 18 rectangular.



Let us make it into a digraph G by choosing direction $i \to j$ on an edge (i, j) if $i < j$. Then each face becomes a triangle or square.

For this digraph $|H_2| = 1$ and $H_p = \{0\}$ for $p = 1$ and $p > 2$.

Spaces Ω_p with $p \geq 3$ are trivial, while $|\Omega_2| = 26$. Space Ω_2 is generated by 8 triangles and 18 squares:

$$\begin{aligned}\Omega_2 \;\; = \;\; & \langle e_{023},\ e_{178},\ e_{456},\ e_{9\,10\,11},\ e_{12\,14\,15},\ e_{13\,19\,20},\ e_{16\,17\,18},\ e_{21\,22\,23}, \\ & e_{018} - e_{038},\ e_{0\,1\,13} - e_{0\,12\,13},\ e_{0\,2\,14} - e_{0\,12\,14},\ e_{1\,7\,19} - e_{1\,13\,19},\ e_{236} - e_{246}, \\ & e_{2\,4\,16} - e_{2\,14\,16},\ e_{3\,6\,11} - e_{3\,8\,11},\ e_{4\,5\,17} - e_{4\,16\,17},\ e_{5\,10\,11} - e_{5\,6\,11},\ e_{5\,10\,22} - e_{5\,17\,22}, \\ & e_{7\,8\,11} - e_{7\,9\,11},\ e_{7\,9\,21} - e_{7\,19\,21},\ e_{9\,10\,22} - e_{9\,21\,22},\ e_{12\,13\,20} - e_{12\,15\,20}, \\ & e_{14\,15\,18} - e_{14\,16\,18},\ e_{15\,18\,23} - e_{15\,20\,23},\ e_{17\,22\,23} - e_{17\,18\,23},\ e_{19\,20\,23} - e_{19\,21\,23}\rangle,\end{aligned}$$

while the generator of H_2 is a signed sum of all these 2-paths.

This basis in Ω_2 is orthogonal. Hence, we compute the curvature:

$x=$	0,11,23	1,3,4,6,8,9,12,13,15,16,18,20,21	2,5,7,14,17,19,22	10
$[x,\Omega_2]=$	$1+\frac{6}{2}=4$	$1+\frac{4}{2}=3$	$1+\frac{5}{2}=\frac{7}{2}$	$1+\frac{3}{2}=\frac{5}{2}$
$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}=$	$1-\frac{4}{2}+\frac{4}{3}$	$1-\frac{4}{2}+\frac{3}{3}$	$1-\frac{4}{2}+\frac{7/2}{3}$	$1-\frac{4}{2}+\frac{5/2}{3}$
K_x	$=\frac{1}{3}$	$=0$	$=\frac{1}{6}$	$=-\frac{1}{6}$

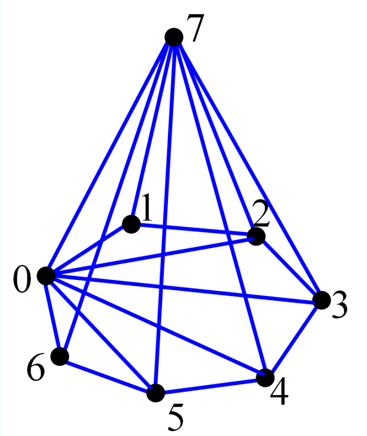
It follows that

$$K_{total}=\frac{3}{3}+\frac{7}{6}-\frac{1}{6}=2.$$

For comparison

$$\begin{aligned}\chi &= |\Omega_0|-|\Omega_1|+|\Omega_2| = 24-48+26=2\\ &= |H_0|-|H_1|+|H_2|\,.\end{aligned}$$

Example. Consider the following pyramid:



Let us make it into a digraph G by choosing direction $i \to j$ on an edge (i,j) if $i<j$.
We have $|\Omega_0| = 8, \quad |\Omega_1| = 18,$

$$\Omega_2 = \langle e_{017}, e_{027}, e_{037}, e_{047}, e_{057}, e_{067} \\ e_{012}, e_{023}, e_{034}, e_{045}, e_{056}, \; e_{127}, e_{237}, e_{347}, e_{457}, e_{567} \rangle$$

$$\Omega_3 = \langle e_{0127}, e_{0237}, e_{0347}, e_{0457}, e_{0567} \rangle$$

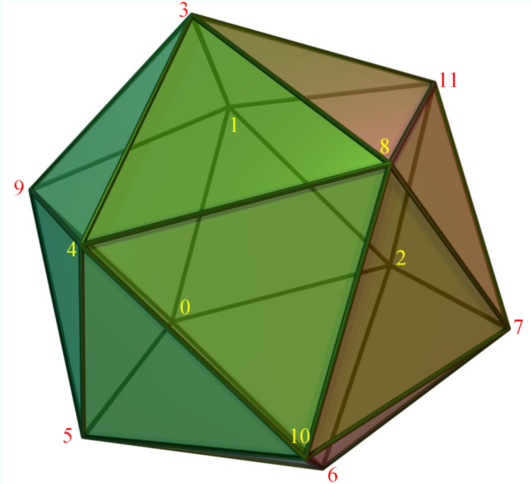
$$\Omega_p = \{0\} \text{ for } p \geq 4.$$

Let us compute the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}-\frac{[x,\Omega_3]}{4}$	$=K_x$
$0,7$	11	5	$1-\frac{7}{2}+\frac{11}{3}-\frac{5}{4}$	$=-\frac{1}{12}$
$1,6$	3	1	$1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}$	$=\frac{1}{4}$
$2,3,4,5$	5	2	$1-\frac{4}{2}+\frac{5}{3}-\frac{2}{4}$	$=\frac{1}{6}$

It follows that $K_{total} = -\frac{2}{12}+\frac{2}{4}+\frac{4}{6} = 1$. For comparison $\chi = 8-18+16-5 = 1$.

Example. Let us compute the curvature of icosahedron (cf. p. 28).



Here we choose direction $i \to j$ if $i < j$. We have

$|H_1| = 0,\ |H_2| = 1,\ |H_p| = 0$ for $p > 2$
$|\Omega_0| = 12,\quad |\Omega_1| = 30,\quad |\Omega_2| = 25,\quad |\Omega_3| = 6,$
$|\Omega_4| = 1$ and $\Omega_p = \{0\}$ for $p \geq 5$.

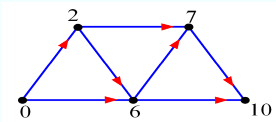
$$\begin{aligned} \text{Hence, } \chi &= |H_0| - |H_1| + |H_2| \\ &= |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| + |\Omega_4| = 2. \end{aligned}$$

We have

$$\begin{aligned} \Omega_2 = \langle & e_{0\,1\,9}, e_{0\,1\,2}, e_{1\,2\,11}, e_{0\,2\,6}, e_{0\,5\,9}, e_{0\,5\,6}, e_{5\,6\,10}, e_{1\,3\,9}, e_{1\,3\,11}, e_{2\,6\,7}, \\ & e_{6\,7\,10}, e_{2\,7\,11}, e_{3\,4\,9}, e_{3\,4\,8}, e_{4\,8\,10}, e_{3\,8\,11}, e_{4\,5\,9}, e_{4\,5\,10}, e_{7\,8\,10}, e_{7\,8\,11}, \\ & e_{0\,1\,11} - e_{0\,2\,11},\ e_{0\,5\,10} - e_{0\,6\,10},\ e_{2\,6\,10} - e_{2\,7\,10},\ e_{3\,4\,10} - e_{3\,8\,10},\ e_{0\,2\,7} - e_{0\,6\,7} \rangle \end{aligned}$$

$$\Omega_3 = \langle e_{0\,1\,2\,11},\ e_{0\,5\,6\,10},\ e_{3\,4\,8\,10},\ e_{0\,2\,6\,7},\ e_{2\,6\,7\,10},\ -e_{0\,6\,7\,10} + e_{0\,2\,7\,10} - e_{0\,2\,6\,10} \rangle$$

$$\Omega_4 = \langle e_{0\,2\,6\,7\,10} \rangle$$



a "snake like" path $e_{i_0 \ldots i_p}$
with $i_k \to i_{k+1}$ and $i_k \to i_{k+2}$
is ∂-invariant

Computation of the curvature:

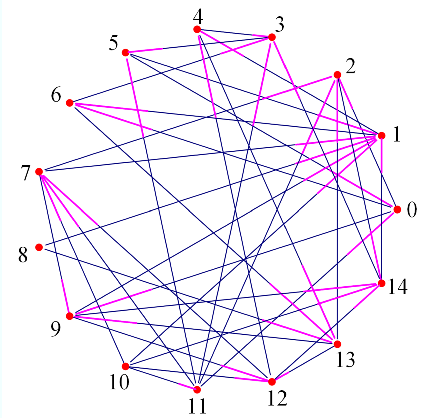
$x=$	0	1	2	3, 11
$[x,\Omega_2]=$	$6+\frac{4}{2}=8$	$5+\frac{1}{2}=\frac{11}{2}$	$5+\frac{4}{2}=7$	$5+\frac{2}{2}=6$
$[x,\Omega_3]=$	$3+\frac{3}{3}=4$	1	$3+\frac{2}{3}=\frac{11}{3}$	1
$[x,\Omega_4]=$	1	0	1	0
$\sum_{p=0}^{4}(-1)^p\frac{[x,\Omega_p]}{p+1}$	$1-\frac{5}{2}+\frac{8}{3}-\frac{4}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{11/2}{3}-\frac{1}{4}$	$1-\frac{5}{2}+\frac{7}{3}-\frac{11/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{6}{3}-\frac{1}{4}$
K_x	$=\frac{11}{30}$	$=\frac{1}{12}$	$=\frac{7}{60}$	$=\frac{1}{4}$

4, 5, 8	6	7	9	10
$5+\frac{1}{2}=\frac{11}{2}$	$5+\frac{3}{2}=\frac{13}{2}$	$5+\frac{3}{2}=\frac{13}{2}$	5	$5+\frac{6}{2}=8$
1	$3+\frac{2}{3}=\frac{11}{3}$	$2+\frac{2}{3}=\frac{8}{3}$	0	$3+\frac{3}{3}=4$
0	1	1	0	1
$1-\frac{5}{2}+\frac{11/2}{3}-\frac{1}{4}$	$1-\frac{5}{2}+\frac{13/2}{3}-\frac{11/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{13/2}{3}-\frac{8/3}{4}+\frac{1}{5}$	$1-\frac{5}{2}+\frac{5}{3}$	$1-\frac{5}{2}+\frac{8}{3}-\frac{4}{4}+\frac{1}{5}$
$=\frac{1}{12}$	$=-\frac{1}{20}$	$=\frac{1}{5}$	$=\frac{1}{6}$	$=\frac{11}{30}$

Note that $K_6=-\frac{1}{20}<0$.

The total curvature: $K_{total}=\frac{11}{30}\cdot 2+\frac{1}{12}\cdot 4+\frac{7}{60}+\frac{1}{4}\cdot 2-\frac{1}{20}+\frac{1}{5}+\frac{1}{6}=2.$

Example. Consider a randomly generated digraph:



We have $V = 15, \quad E = 39$
$|H_1| = 2, \quad |H_2| = 1, \quad H_p = \{0\}$ for $p \geq 3$
$|\Omega_2| = 28, \quad |\Omega_3| = 4, \quad \Omega_p = \{0\}$ for $p \geq 4$.

Hence, $\chi = |H_0| - |H_1| + |H_2|$
$= |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 0$

$$\begin{aligned}\Omega_2 \;=\; \langle\, & e_{13\,2\,14} - e_{13\,12\,14},\ e_{13\,2\,14} - e_{13\,9\,14},\ e_{0\,2\,14} - e_{0\,9\,14},\ e_{1\,4\,3} - e_{1\,6\,3}, \\ & e_{1\,4\,13} - e_{1\,6\,13},\ e_{5\,0\,6} - e_{5\,1\,6},\ e_{7\,2\,14} - e_{7\,9\,14},\ e_{9\,1\,4} - e_{9\,12\,4}, \\ & e_{10\,1\,4} - e_{10\,12\,4},\ e_{10\,7\,2} - e_{10\,11\,2},\ e_{10\,11\,3} - e_{10\,14\,3},\ e_{11\,0\,9} - e_{11\,7\,9}, \\ & e_{11\,5\,1} - e_{11\,7\,1},\ e_{12\,4\,3} - e_{12\,14\,3},\ e_{12\,7\,1} - e_{12\,14\,1},\ e_{7\,9\,1},\ e_{9\,12\,14},\ e_{9\,14\,1}, \\ & e_{10\,7\,1},\ e_{10\,11\,7},\ e_{10\,12\,7},\ e_{10\,12\,14},\ e_{10\,14\,1}, e_{11\,0\,2},\ e_{11\,3\,5},\ e_{11\,5\,0},\ e_{11\,7\,2},\ e_{13\,9\,12}\rangle\end{aligned}$$

$$\Omega_3 = \langle e_{10\,11\,7\,2}\,,\ e_{13\,9\,12\,14}\,,\ e_{10\,12\,7\,1} - e_{10\,12\,14\,1},\ e_{11\,0\,2\,14} - e_{11\,0\,9\,14} + e_{11\,7\,9\,14} - e_{11\,7\,2\,14}\,\rangle$$

$$\{K_x\}_{x=0}^{14} = \left\{-\tfrac{7}{24}, -\tfrac{1}{12}, -\tfrac{23}{72}, -\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, -\tfrac{1}{3}, \tfrac{1}{6}, 0, \tfrac{13}{72}, \tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{18}, -\tfrac{11}{12}, \tfrac{13}{24}\right\}.$$

2.4 Some problems

Problem 2.4 *Compare this notion of curvature with other definitions of curvature of graphs.*

Problem 2.5 *Is it true that for icosahedron (see p. 56)* $|\Omega_2| = 25$ *for any numbering of the vertices?*

Problem 2.6 *Devise an efficient algorithm/software for computation of the spaces* Ω_p *for arbitrary digraphs, possibly avoiding null-spaces of large matrices. Such algorithms exist for* Ω_2 *and* Ω_3*.*

Problem 2.7 *Let a digraph* G *be determined by a triangulation of* $\mathbb{S}^2$ *(see Section 1.7). Assume that* $\deg(x) \leq 4$ *for all* $x \in G$*. Is it true that* $K_x \geq 0$ *for all* $x \in G$*?*

For triangulations of $\mathbb{S}^1$ we have always $K_x \geq 0$: these are triangles and squares with $K_x > 0$ and other polygons with $K_x \equiv 0$.

For triangulations of $\mathbb{S}^2$ we have verified above that $K_x \geq 0$ for simplex, bipyramid, octahedron, but with specific orientations of edges (the question remains open when the numbering of vertices is arbitrary). All these digraphs have $\deg(x) \leq 4$. We have seen that $K_x < 0$ can occur for icosahedron with $\deg(x) = 5$ and for a pyramid with $\deg(x) = 7$.

Problem 2.8 *Denote $D = \max_{x \in G} \deg(x)$. Is it true that $|K_x| \leq C_D$ for some constant C_D depending only on D? The same question about $K_x^{(2)}$ and $K_x^{(3)}$.*

Note that K_x can be arbitrarily large, for example, for a strongly regular digraph satisfying ($B(k, m)$), we have

$$K_x = \frac{1 - (1 - m)^k}{km}$$

while $\deg(x) = (k - 1)m$.

Problem 2.9 *What can be said about the curvature of random digraphs?*

3 Cartesian product of digraphs

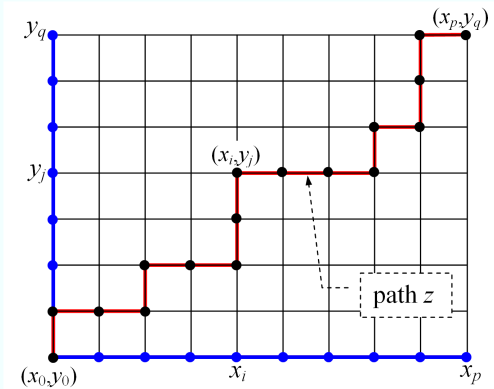
3.1 Cross product of paths

Given two finite sets X, Y, consider their product

$$Z = X \times Y = \{(a,b) : a \in X \text{ and } b \in Y\}.$$

Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1} z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

For any stair-like path z on Z, define its projection onto X as an elementary path x on X obtained from z by removing the Y-components in all the vertices of z and by collapsing in the resulting sequence of points of X consecutive repeated vertices to one vertex.

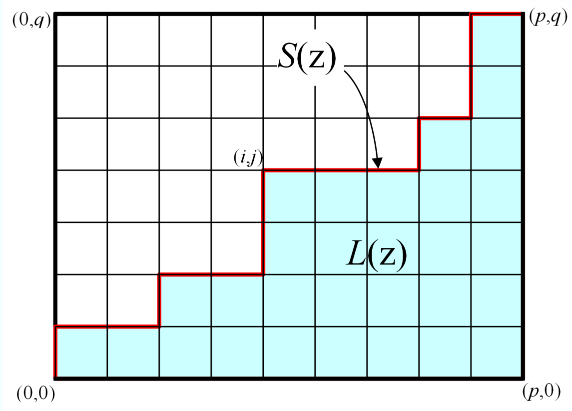


In the same way we define projection of z onto Y and denote it by y.

Projections $x = x_0 \ldots x_p$ and $y = y_0 \ldots y_q$ are regular elementary paths, and $p + q = r$.

Every vertex (x_i, y_j) of path z can be represented as a point (i, j) of $\mathbb{Z}^2$ so that path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting points $(0,0)$ and (p,q).

Define the *elevation* $L(z)$ of z as the number of cells in $\mathbb{Z}_+^2$ below the staircase $S(z)$.



For given elementary regular paths x on X and y on Y, denote by $\Pi_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y.

Definition. Define the *cross product* of the paths e_x and e_y as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z \in \Pi_{x,y}} (-1)^{L(z)} e_z \tag{3.1}$$

and it extend by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

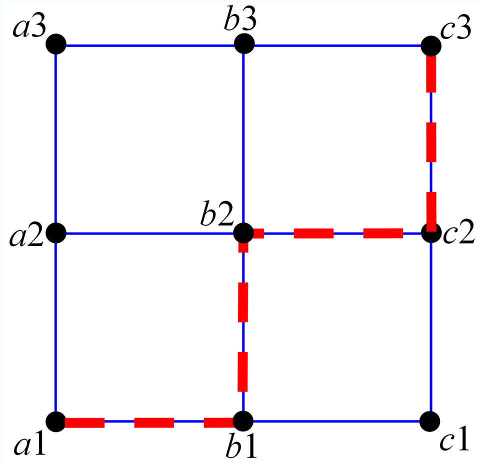
Example. Let us denote the vertices on X by letters a, b, c etc and the vertices on Y by integers $1, 2, 3$, etc so that the vertices on Z can be denoted as $a1, b2$ etc as the fields on the chessboard. Then we have

$$e_a \times e_{123} = e_{a1\,a2\,\,a3}, \quad e_{abc} \times e_1 = e_{a1\,b1\,\,c1}$$

$$e_{ab} \times e_{12} = e_{a1\,b1\,b2} - e_{a1\,a2\,b2}$$

$$e_{ab} \times e_{123} = e_{a1\,b1\,b2\,b3} - e_{a1\,a2\,b2\,b3} + e_{a1\,a2\,a3\,b3}$$

$$\begin{aligned} e_{abc} \times e_{123} = {} & e_{a1\,b1\,c1\,c2\,c3} - e_{a1\,b1\,b2\,c2\,c3} + e_{a1\,b1\,b2\,b3\,c3} \\ & + e_{a1\,a2\,b2\,c2\,c3} - e_{a1\,a2\,b2\,b3\,c3} + e_{a1\,a2\,a3\,b3\,c3} \end{aligned}$$



Lemma 3.1 *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v). \tag{3.2}$$

Example. For example, let $u = e_{ab}$ and $v = e_{123}$. We have

$$\begin{aligned}
\partial\left(u \times v\right) &= \partial\left(e_{a1\,b1\,b2\,b3} - e_{a1\,a2\,b2\,b3} + e_{a1\,a2\,a3\,b3}\right) \\
&= \overline{e_{b1\,b2\,b3}} - \underleftrightarrow{e_{a1\,b2\,b3}} + \underline{e_{a1\,b1\,b3}} - \underline{e_{a1\,b1\,b2}} \\
&\quad - \left(\underline{e_{a2\,b2\,b3}} - \underleftrightarrow{e_{a1\,b2\,b3}} + \underleftrightarrow{e_{a1\,a2\,b3}} - \underline{e_{a1\,a2\,b2}}\right) \\
&\quad + \underline{e_{a2\,a3\,b3}} - \underline{e_{a1\,a3\,b3}} + \underleftrightarrow{e_{a1\,a2\,b3}} - \overline{e_{a1\,a2\,a3}}
\end{aligned}$$

$$(\partial e_{ab}) \times e_{123} = (e_b - a_a) \times e_{123} = e_{b1\,b2\,b3} - e_{a1\,a2\,a3}$$

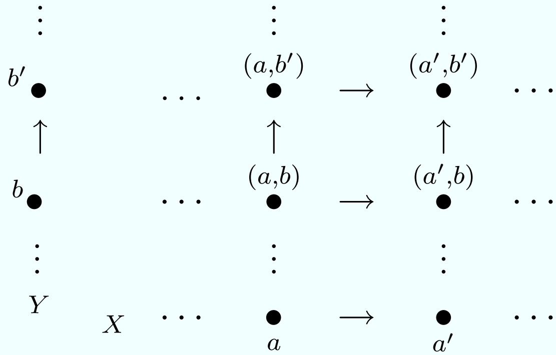
$$\begin{aligned}
(-1)^p\, e_{ab} \times \partial e_{123} &= -e_{ab} \times (e_{23} - e_{13} + e_{12}) \\
&= -\left(e_{a2\,b2\,b3} - e_{a2\,a3\,b3}\right) + \left(e_{a1\,b1\,b3} - e_{a1\,a3\,b3}\right) - \left(e_{a1\,b1\,b2} - e_{a1\,a2\,b2}\right).
\end{aligned}$$

The comparison of all terms shows that the identity (3.2) holds.

3.2 Cartesian product of digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define there Cartesian product as a digraph $Z = X \Box Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;
- the edges in Z are of two types: $(a, b) \to (a', b)$ where $a \to a'$ (a *horizontal* edge) and $(a, b) \to (a, b')$ where $b \to b'$ (a *vertical* edge):



It follows that any allowed elementary path in Z is stair-like.

Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed.

It follows from definition (3.1) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \quad \Rightarrow \quad u \times v \in \mathcal{A}_{p+q}(Z). \tag{3.3}$$

Furthermore, the following is true.

Lemma 3.2 *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.*

Proof. $u \times v$ is allowed by (3.3). Since ∂u and ∂v are allowed, by (3.3) also $\partial u \times v$ and $u \times \partial v$ are allowed. By (3.2), $\partial(u \times v)$ is also allowed. Hence, $u \times v \in \Omega_{p+q}(Z)$. ∎

Theorem 3.3 *Any ∂-invariant path w on $Z = X \Box Y$ admits a representation in the form*

$$w = \sum_{i=1}^{m} u_i \times v_i$$

for some finite m, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

3.3 Künneth formula

Here is the main result of this section.

Theorem 3.4 *Let X, Y be two finite digraphs. Then, for any $r \geq 0$,*

$$\Omega_r(X \Box Y) \cong \bigoplus_{\{p,q\geq 0:p+q=r\}} \Omega_p(X) \otimes \Omega_q(Y), \tag{3.4}$$

where the isomorphism is given by

$$u \otimes v \mapsto u \times v$$

for $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Consequently, we have

$$H_r(X \Box Y) \cong \bigoplus_{\{p,q\geq 0:p+q=r\}} H_p(X) \otimes H_q(Y) \tag{3.5}$$

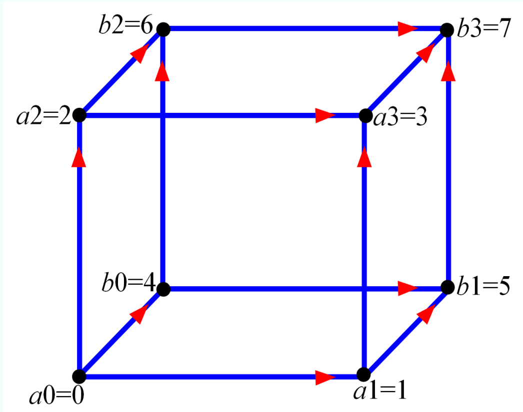
and

$$\beta_r(X \Box Y) = \sum_{\{p,q\geq 0:p+q=r\}} \beta_p(X)\, \beta_q(Y).$$

Example. Let X be an interval and Y be a square:

$$X = {}^{a}\bullet \rightarrow \bullet^{b} \quad \text{and} \quad Y = \begin{array}{ccc} {}_{2}\bullet & \longrightarrow & \bullet_{3} \\ \uparrow & & \uparrow \\ {}_{0}\bullet & \longrightarrow & \bullet_{1} \end{array}$$

Then $Z = X \Box Y$ is a cube:



$Z = X \Box Y$

We have:

$\Omega_1(X) = \langle e_{ab} \rangle$

$\Omega_p(X) = 0$ for $p \geq 2$

$\Omega_1(Y) = \langle e_{01}, e_{13}, e_{23}, e_{02} \rangle$

$\Omega_2(Y) = \langle e_{013} - e_{023} \rangle$

$\Omega_q(Y) = 0$ for $q \geq 3$.

By (3.4) we obtain

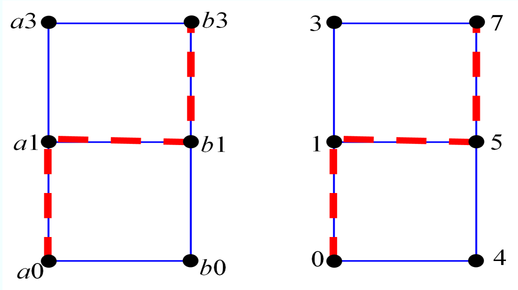
$$\Omega_3(Z) \cong \Omega_1(X) \otimes \Omega_2(Y) = \langle e_{ab} \times (e_{013} - e_{023}) \rangle.$$

Let us compute the cross-products:

$$
\begin{aligned}
e_{ab} \times e_{013} &= e_{a0\,b0\,b1\,b3} - e_{a0\,a1\,b1\,b3} + e_{a0\,a1\,a3\,b3} \\
&= e_{0457} - e_{0157} + e_{0137}
\end{aligned}
$$

and

$$
e_{ab} \times e_{023} = e_{0467} - e_{0267} + e_{0237}
$$



Hence, we obtain

$$
\Omega_3 (Z) = \langle e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237} \rangle
$$

that is the ∂-invariant 3-path associated with 3-cube.

Define n-cube as follows:

$$
n\text{- cube} = \underbrace{I \square I \square ... \square I}_{n} =: I^n,
$$

where $I = \{\bullet \to \bullet\}$. In particular, the square $= I^2$ and the above cube is I^3.

Similarly one shows that Ω_n (n- cube) is spanned by a single n-path that is an alternating sum of $n!$ elementary n-paths connecting the vertices 0 and $2^n - 1$. This corresponds to partitioning of a solid n-dim cube into $n!$ simplexes.

Proposition 3.5 *We have for any $p \geq 0$*

$$\dim \Omega_p(I^n) = 2^{n-p}\binom{n}{p}, \tag{3.6}$$

and

$$\beta_p\left(I^n\right) = \begin{cases} 1, & p = 0 \\ 0, & p > 0 \end{cases}. \tag{3.7}$$

Recall that $\beta_r = \dim H_p$.

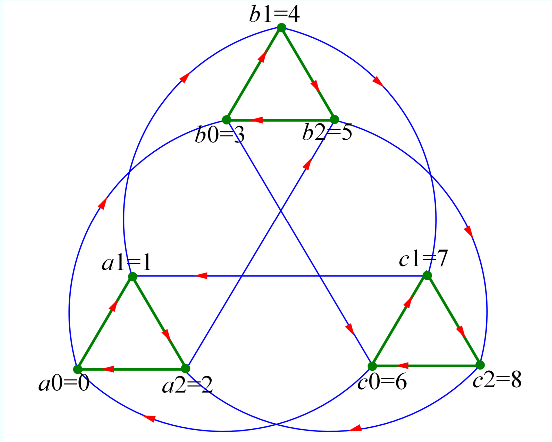
For example,

$$\dim \Omega_0\left(I^3\right) = 8, \quad \dim \Omega_1\left(I^3\right) = 12, \quad \dim \Omega_2\left(I^3\right) = 6, \quad \dim \Omega_3\left(I^3\right) = 1.$$

3.4 An example: 2-torus

Example. Denote by T the following 3-cycle (=1-torus):

$$T = \begin{tikzcd} & b \arrow[dr] & \\ a \arrow[ur] & & c \arrow[ll] \end{tikzcd} = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd}$$



Consider a 2-torus $T^2 = T \Box T$ shown here:

Let us compute $\Omega_r(G)$, $H_r(G)$, $K_x(G)$.

We know that

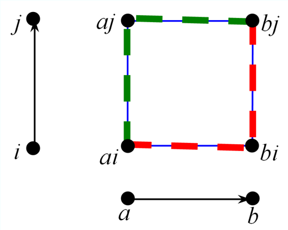
$$\Omega_0(T) = \langle e_0, e_1, e_2 \rangle, \quad \Omega_1(T) = \langle e_{01}, e_{12}, e_{20} \rangle, \quad \Omega_p(T) = \{0\} \text{ for } p \geq 2$$

By (3.4) we obtain $\Omega_r = \{0\}$ for $r \geq 3$ and

$$\Omega_2\left(T^2\right) = \Omega_1(T) \otimes \Omega_1(T)$$
$$= \langle e_{ab} \times e_{01}, e_{ab} \times e_{12}, e_{ab} \times e_{20}, e_{bc} \times e_{01}, e_{bc} \times e_{12}, e_{bc} \times e_{20}, e_{ca} \times e_{01}, e_{ca} \times e_{12}, e_{ca} \times e_{20} \rangle$$

Using

$$e_{ab} \times e_{ij} = e_{ai\,bi\,bj} - e_{ai\,aj\,bj}$$



we obtain that

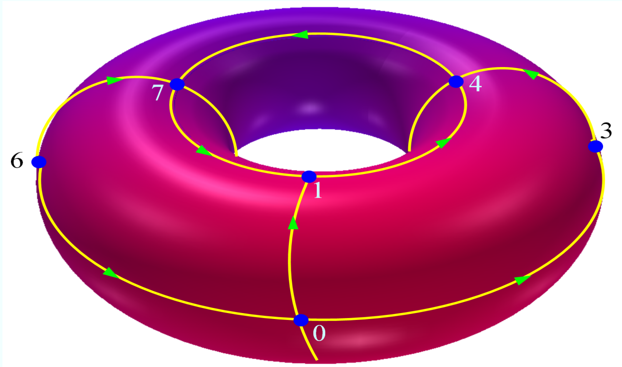
$$\begin{aligned}\Omega_2\left(T^2\right) = \langle & e_{a0\,b0\,b1} - e_{a0\,a1\,b1},\ e_{a1\,b1\,b2} - e_{a1\,a2\,b2},\ e_{a2\,b2\,b0} - e_{a2\,a0\,b0}, \\ & e_{b0\,c0\,c1} - e_{b0\,b1\,c1},\ e_{b1\,c1\,c2} - e_{b1\,b2\,c2},\ e_{b2\,c2\,c0} - e_{b2\,b0\,c0}, \\ & e_{c0\,a0\,a1} - e_{c0\,c1\,a1},\ e_{c1\,a1\,a2} - e_{c1\,c2\,a2},\ e_{c2\,a2\,a0} - e_{c2\,c0\,a0}\rangle\end{aligned}$$

that is,

$$\begin{aligned}\Omega_2\left(T^2\right) = \langle & e_{034} - e_{014},\ e_{145} - e_{125},\ e_{253} - e_{203}, \\ & e_{367} - e_{347},\ e_{478} - e_{458},\ e_{586} - e_{536} \\ & e_{601} - e_{671},\ e_{712} - e_{782},\ e_{820} - e_{860}\rangle.\end{aligned}$$

We see that $\Omega_2\left(T^2\right)$ is generated by 9 squares.

This can be visualized using the following embedding of $G = T^2$ on a topological torus:



Using $\Omega_2(G)$, let us compute the curvature K_x on G.
The above basis in $\Omega_2(G)$ is orthogonal and $\|\omega\|^2 = 2$ for any element ω of the basis.

Besides, for any vertex x, we have $[x, \omega] = 2$ for two of ω, $[x, \omega] = 1$ for two of ω, and $[x, \omega] = 0$ for the rest of ω. Hence,

$$[x, \Omega_2] = \sum_{\omega} \frac{[x, \omega]}{\|\omega\|^2} = \frac{2 \cdot 2 + 2 \cdot 1}{2} = 3$$

and

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x, \Omega_2]}{3} = 1 - \frac{4}{2} + \frac{3}{3} = 0.$$

Let us compute the homology groups of G. We know that

$$H_0(T) = \langle e_0 \rangle, \quad H_1(T) = \langle e_{01} + e_{12} + e_{20} \rangle, \quad H_p(T) = \{0\} \text{ for } p \geq 2.$$

By (3.5) we have

$$H_1(G) = H_0(T) \otimes H_1(T) + H_1(T) \otimes H_0(T) = \langle v_1, v_2 \rangle$$

where

$$v_1 = e_a \times (e_{01} + e_{12} + e_{20}) = e_{a0\,a1} + e_{a1\,a2} + e_{a2\,a0} = e_{01} + e_{12} + e_{20}$$

$$v_2 = (e_{ab} + e_{bc} + e_{ca}) \times e_0 = e_{a0\,b0} + e_{b0\,c0} + e_{c0\,a0} = e_{03} + e_{36} + e_{60}.$$

Again by (3.5)

$$H_2(G) = H_1(T) \otimes H_1(T) = \langle u \rangle,$$

where

$$u = (e_{ab} + e_{bc} + e_{ca}) \times (e_{01} + e_{12} + e_{20}),$$

and $H_r(Z) = 0$ for all $r \geq 2$.

Hence, we obtain

$$\begin{aligned} u = {} & e_{a0\,b0\,b1} - e_{a0\,a1\,b1} + e_{a1\,b1\,b2} - e_{a1\,a2\,b2} + e_{a2\,b2\,b0} - e_{a2\,a0\,b0} \\ & + e_{b0\,c0\,c1} - e_{b0\,b1\,c1} + e_{b1\,c1\,c2} - e_{b1\,b2\,c2} + e_{b2\,c2\,c0} - e_{b2\,b0\,c0} \\ & + e_{c0\,a0\,a1} - e_{c0\,c1\,a1} + e_{c1\,a1\,a2} - e_{c1\,c2\,a2} + e_{c2\,a2\,a0} - e_{c2\,c0\,a0} \end{aligned}$$

that is

$$\begin{aligned} u = {} & (e_{034} - e_{014}) + (e_{145} - e_{125}) + (e_{253} - e_{203}) + (e_{367} - e_{347}) + (e_{478} - e_{458}) \\ & + (e_{586} - e_{536}) + (e_{601} - e_{671}) + (e_{712} - e_{782}) + (e_{820} - e_{860}) \,. \end{aligned}$$

3.5 Cartesian product and curvature

Proposition 3.6 *Let X be any digraph with a finite chain sequence $\{\Omega_p\}$ and Y be a cyclic digraph*

$$Y = \{0 \to 1 \to 2 \to ... \to m \to 0\}$$

with $m \geq 2$. Then, with respect to the natural inner product,

$$K_z(X \Box Y) = 0$$

for any $z \in X \Box Y$. In particular, $K(T^n) = 0$ where T is an 1-torus.

Consider an n-cube$= I^n$ where $I = \{0 \to 1\}$. Then any vertex x of the n-cube is represented by a binary sequence $(x_1, ..., x_n)$. Set $|x| = x_1 + ... + x_n$.

Proposition 3.7 *For any vertex x of the n-cube we have*

$$K_x(n\text{-cube}) = \frac{1}{(n+1)\binom{n}{|x|}}.$$

Problem 3.8 *How to compute $K(X \Box Y)$ in general?*

3.6 Strong product

Define a *strong product* $X \boxslash Y$ of digraphs as follows: the set of vertices of $X \boxslash Y$ is $X \times Y$, while the arrows are defined as follows: $(a,b) \to (a',b)$ where $a \to a'$ (a horizontal edge), $(a,b) \to (a,b')$ where $b \to b'$ (a vertical edge), and $(a,b) \to (a',b')$ where $a \to a'$ and $b \to b'$ (a diagonal edge):

$$\begin{array}{ccc} \overset{(a,b')}{\bullet} & \longrightarrow & \overset{(a',b')}{\bullet} \\ \uparrow & \nearrow & \uparrow \\ \overset{(a,b)}{\bullet} & \longrightarrow & \overset{(a',b)}{\bullet} \end{array}$$

Conjecture 3.9 *The Künneth formula holds for the strong product:*

$$H_r\left(X \boxslash Y\right) \cong \bigoplus_{\{p,q \geq 0 : p+q=r\}} \left(H_p\left(X\right) \otimes H_q\left(Y\right)\right),$$

where the isomorphism is given by $u \otimes v \mapsto u \times v$.

It suffices to prove an analogue of the theorem of Eilenberg-Zilber: there are chain maps

$$F : \Omega_* (X \boxslash Y) \to \Omega_* (X) \otimes \Omega_* (Y)$$

and

$$G : \Omega_* (X) \otimes \Omega_* (Y) \to \Omega_* (X \boxslash Y)$$

such that $FG = \mathrm{id}$ and GF is *chain-homotopic* to id.

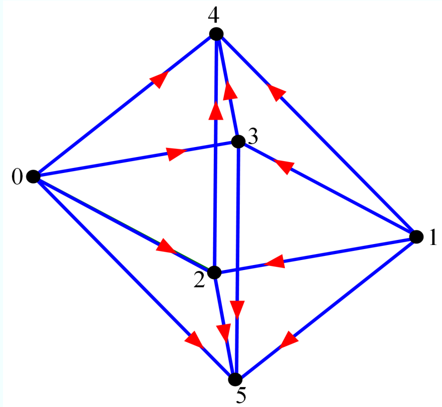
In fact, one can define G by $G (u \otimes v) = u \times v$, while the main difficulty is in construction of F. In the setting of Theorem 3.4, one uses Theorem 3.3 to show that G is bijective so that one can take $F = G^{-1}$.

4 Join of digraphs

Given two digraphs X, Y, define their *join* $X * Y$ as follows: take first a disjoint union $X \sqcup Y$ and add arrows from any vertex of X to any vertex of Y.

For example,

$$\{\bullet,\bullet\} * \{\bullet,\bullet\} = \begin{array}{ccc} \bullet & \leftarrow & \bullet \\ \uparrow & & \downarrow \\ \bullet & \rightarrow & \bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} \bullet & \leftarrow & \bullet \\ \uparrow & & \downarrow \\ \bullet & \rightarrow & \bullet \end{array} * \{\bullet,\bullet\} =$$



In order to compute homology of $X * Y$ we use the *augmented chain complex*

$$\mathbb{K} \stackrel{\partial}{\leftarrow} \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \ldots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \ldots \tag{4.1}$$

where $\partial e_i = e$ =the unity of $\mathbb{K}$.

The homology groups of (4.1) are called the reduced homology groups of G and are denoted by $\widetilde{H}_p(G)$. We have

$$\widetilde{H}_p(G) = H_p(G) \text{ for } p \geq 1 \text{ and } \widetilde{H}_0(G) = H_0(G)/\mathbb{K}.$$

Define the reduced Betti numbers: $\widetilde{\beta}_p(G) = \dim \widetilde{H}_p(G)$.

Define the join of elementary p-paths $u = e_{i_0 \ldots i_p}$ on X and $v = e_{j_0 \ldots j_q}$ on Y by

$$u * v = e_{i_0 \ldots i_p j_0 \ldots j_q}$$

so that $u * v$ is a $(p+q+1)$-path on $X * Y$. Then extend this definition by linearity to all paths u on X and v on Y.

If u and v are allowed then $u * v$ is also allowed. The join of paths satisfies the product rule

$$\partial (u * v) = (\partial u) * v + (-1)^{p+1} u * \partial v.$$

It follows that the join of ∂-invariant paths is ∂-invariant: if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u * v \in \Omega_r(X * Y)$ where $r = p + q + 1$.

Theorem 4.1 *We have the following isomorphism for any $r \geq -1$:*

$$\Omega_r(Z) \cong \bigoplus_{\{p,q \geq -1 : p+q=r-1\}} \Omega_p(X) \otimes \Omega_q(Y) \tag{4.2}$$

that is given by the map $u \otimes v \mapsto u * v$ with $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$.

Consequently, for any $r \geq 0$,

$$\widetilde{H}_r(X * Y) \cong \bigoplus_{\{p,q \geq 0 : p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y) \tag{4.3}$$

and

$$\widetilde{\beta}_r(X * Y) \cong \sum_{\{p,q \geq 0 : p+q=r-1\}} \widetilde{\beta}_p(X) \, \widetilde{\beta}_q(Y).$$

5 Digraphs of constant curvature

Fix a finite digraph $G = (V, E)$. The space $\mathcal{A}_p$ of allowed p-paths on G consists of all formal linear combinations of *elementary* allowed p-paths $e_{i_0 \dots i_p}$ (where $i_0 \to i_1 \to \dots \to i_p$ on G). Consider its subspace $\Omega_p = \{\omega \in \mathcal{A}_p : \partial\omega \in \mathcal{A}_{p-1}\}$. Then we have a path chain complex of digraph G:

$$0 \leftarrow \Omega_0 \stackrel{\partial}{\leftarrow} \Omega_1 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \Omega_{p+1} \stackrel{\partial}{\leftarrow} \dots$$

Fix in each $\mathcal{A}_p$ the *natural* inner product such that all allowed elementary p-paths $e_{i_0 \dots i_p}$ form an orthonormal basis in $\mathcal{A}_p$. This induces an inner product in all chain spaces Ω_p.

For any vertex x of G and any p-path $\omega = \sum \omega^{i_0 \dots i_p} e_{i_0 \dots i_p}$ we have

$$[x, \omega] = \sum_{i_0 \dots i_p \in V} \left(\omega^{i_0 \dots i_p}\right)^2 \left[x, e_{i_0 \dots i_p}\right],$$

where $\left[x, e_{i_0 \dots i_p}\right]$ =the number of occurrences of x in $i_0, \dots, i_p$.

If $\{\omega_k\}$ is an orthogonal basis of Ω_p, then by (2.5)

$$\boxed{[x, \Omega_p] = \sum_k \frac{[x, \omega_k]}{\|\omega_k\|^2}}.$$

If the sequence $\{\Omega_p\}$ is finite, that is, $\Omega_p = \{0\}$ for large enough p, then we define the combinatorial curvature of G at a vertex x by

$$K_x = \sum_{p=0}^{\infty} (-)^p \frac{[x, \Omega_p]}{p+1}.$$

Recall the Gauss-Bonnet formula:

$$K_{total} := \sum_{x \in V} K_x = \sum_{p=0}^{\infty} (-)^p \dim \Omega_p =: \chi.$$

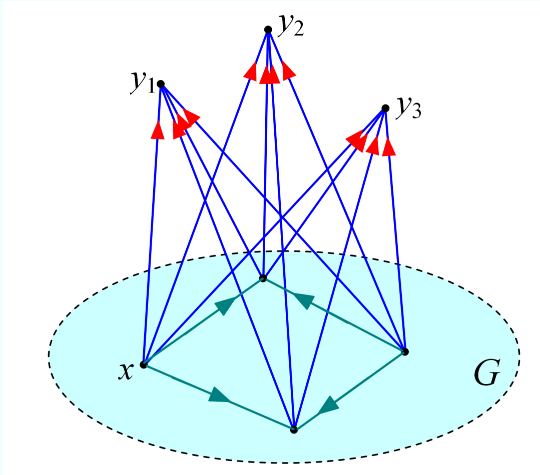
In this section we construct a two-parameter family of digraphs with $K_x = \mathrm{const}$.

Recall that a graph is called regular if $\deg(x)$ is constant. We say that a digraph G is *strongly regular* if the function $x \mapsto [x, \Omega_p]$ is constant for any p (and $\Omega_p = \{0\}$ for large enough p). In particular, a strongly regular digraph G is regular because $\deg(x) = [x, \Omega_1]$. In this case $K_x = \mathrm{const}$ and, hence,

$$K_x = \frac{K_{total}}{|V|} = \frac{\chi(G)}{|V|} =: K(G).$$

For any digraph G and any $m \in \mathbb{N}$ let us construct a new digraph by adding to G m new vertices $\{y_1, ..., y_m\}$ and all arrows $x \to y_i$ for all $x \in X$.
This digraph is called *m-suspension* of G and is denoted by $\operatorname{sus}_m G$.

In fact, $\operatorname{sus}_m G = G * \{y_1, ..., y_m\}$.



Theorem 5.1 *Let G be a strongly regular digraph, such that for some $k, m \in \mathbb{N}$ and, any $p \geq 0$,*

$$\dim \Omega_p(G) = \binom{k}{p+1} m^{p+1}. \tag{B(k,m)}$$

Then $\operatorname{sus}_m G$ is also strongly regular, and for all $p \geq 0$,

$$\dim \Omega_p(\operatorname{sus}_m G) = \binom{k+1}{p+1} m^{p+1}. \tag{B(k+1,m)}$$

For the digraph G as in Theorem 5.1 we have

$$\chi(G)=\sum_{p\geq 0}(-1)^p\dim\Omega_p=\sum_{p=0}^{k-1}(-1)^p\binom{k}{p+1}m^{p+1}=-\sum_{j=1}^{k}(-1)^j\binom{k}{j}m^j=1-(1-m)^k\,.$$

It follows that

$$K(G)=\frac{\chi(G)}{|V|}=\frac{\chi(G)}{\dim\Omega_0}=\frac{1-(1-m)^k}{km}.$$

Of course, the same formula is true for $K(\operatorname{sus}_m G)$ with k replaced by $k+1$:

$$K(\operatorname{sus}_m G)=\frac{1-(1-m)^{k+1}}{(k+1)\,m}$$

Let us now construct a family $\left\{D_m^k\right\}_{k,m\in\mathbb{N}}$ of digraphs, satisfying (B(k,m)). Denote by D_m the digraph that consists of m disjoint vertices and no arrows: $D_m=\{\underbrace{\bullet,...,\bullet}_{m\text{ vertices}}\}$. Then D_m is strongly regular and satisfies (B$\,(1,m)$) because

$$\dim\Omega_0\,(D_m)=m=\binom{1}{p+1}m^{p+1}\ \text{ for } p=0,\quad \dim\Omega_p\,(D_m)=0=\binom{1}{p+1}m^{p+1}\ \text{ for } p\geq 1$$

Define the digraph D_m^k by

$$D_m^k = \underbrace{D_m * ... * D_m}_{k}$$

that is, $D_m^{k+1} = \mathrm{sus}_m D_m^k$. From Theorem 5.1 we obtain by induction that D_m^k is strongly regular and satisfies (B(k,m)).

Hence, D_m^k has a constant curvature

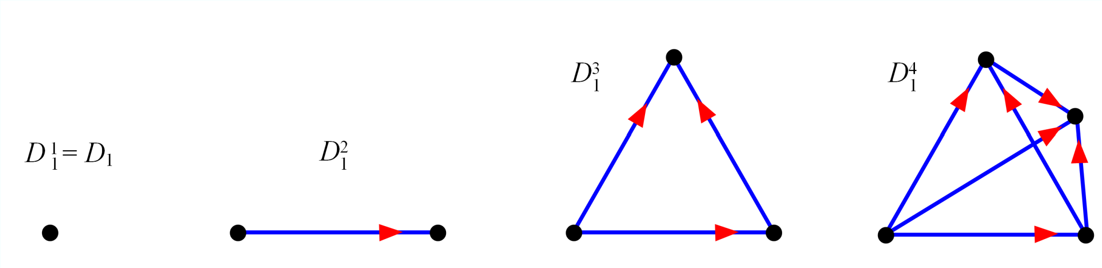
$$K(D_m^k) = \frac{1-(1-m)^k}{km}. \tag{5.1}$$

One can show that the only non-trivial Betti number of D_m^k with $k \geq 2$ is

$$\beta_{k-1} = (m-1)^k.$$

Example. For $m = 1$ we have by (5.1) $K(D_1^k) = \frac{1}{k}$.

Digraph D_1^k is a $(k-1)$-simplex:



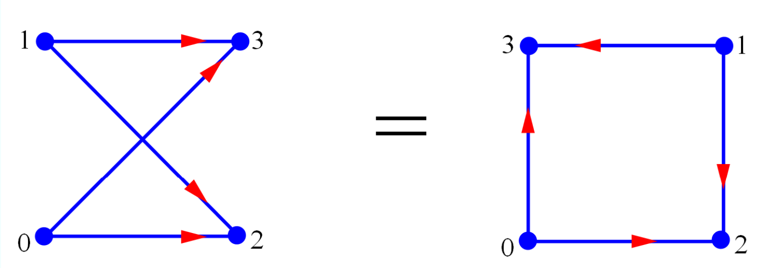
Example. For $m = 2$ we have by (5.1)

$$K(D_2^k) = \begin{cases} 0, & k \text{ even}, \\ \frac{1}{k}, & k \text{ odd}. \end{cases}$$

Digraph D_2^2 is a *diamond:*

It is an analogue of 1-sphere.

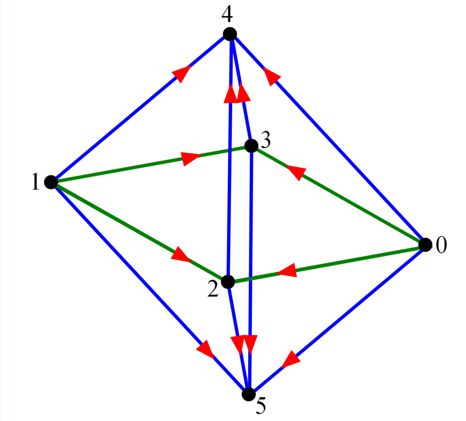
It has constant curvature 0.



D_2^3 is the octahedron:

It is an analogue of 2-sphere.

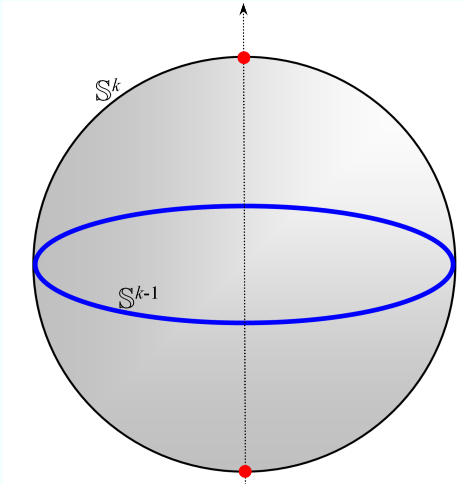
It has constant curvature $\frac{1}{3}$.



D_2^4 is an analogue of 3-sphere.

It has constant curvature 0.

D_2^{k+1} is a digraph analogue of a k-sphere $\mathbb{S}^k$ because D_2^{k+1} is obtained from D_2^k by 2-suspension.

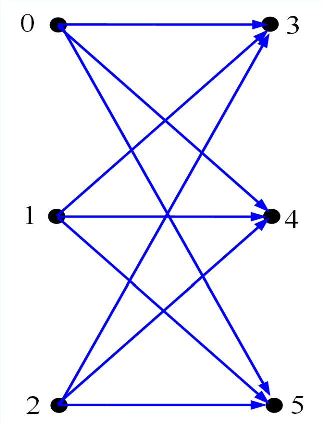


Besides, the only non-trivial Betti number of D_2^{k+1} is $\beta_k = 1$ like Betti numbers for $\mathbb{S}^k$.

Example. For $m = 3$ we have by (5.1)

$$K(D_3^k) = \frac{1 - (-2)^k}{3k} = \frac{1}{3k} \begin{cases} 1 - 2^k, & k \text{ even}, \\ 1 + 2^k, & k \text{ odd}. \end{cases}$$

For example, D_3^2 is a directed version of $K_{3,3}$:



D_3^2

We have

$$K(D_3^2) = -\frac{1}{2}$$

and

$$K(D_3^3) = 1.$$

6 Hodge Laplacian on digraphs

As above, we fix the natural inner product $\langle \cdot, \cdot \rangle$ in all spaces Ω_p.

6.1 Definition of Δ_p

For the operator $\partial : \Omega_p \to \Omega_{p-1}$ consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$ so that

$$\langle \partial u, v \rangle = \langle u, \partial^* v \rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition. Define the *Hodge-Laplace operator on paths* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^* \partial u + \partial \partial^* u. \tag{6.1}$$

Here we use the following operators ∂ and ∂^*: $\quad \Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p \quad$ and $\quad \Omega_p \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \Omega_{p+1}$.

Proposition 6.1 *The operator Δ_p is self-adjoint and non-negative definite.*

Denote by $\lambda_{\max}(\Delta_p)$ and $\lambda_{\min}(\Delta_p)$ the maximal and minimal eigenvalues of Δ_p. As we will see below, $\lambda_{\min}(\Delta_p) = 0$ if and only if $\beta_p > 0$; moreover, β_p is the multiplicity of the eigenvalue 0 of Δ_p.

Problem 6.2 *Find a reasonable upper bounds for* $\lambda_{\max}(\Delta_p)$. (Some upper bound for $\lambda_{\max}(\Delta_1)$ will be given below).

Problem 6.3 *Find lower bounds for* $\lambda_{\min}(\Delta_p)$ *when* $\beta_p = 0$.

Problem 6.4 *Devise a program for computing the spectrum of* Δ_p *for large digraphs.*

Problem 6.5 *For which classes of digraphs the spectrum of* Δ_p *can be computed exactly?* (Some partial answer will be given below).

We say that two digraphs G and $G'$are *Hodge isospectral* if $\operatorname{spec}\Delta_p(G) = \operatorname{spec}\Delta_p(G')$ for all $p \geq 0$. A natural question in the spirit of inverse spectral problems is *whether Hodge isospectral digraphs are isomorphic*. In general the answer is "no", but in the existing examples the digraph G' is obtained from G by changing orientation of some arrows.

Problem 6.6 *Is it true that Hodge isospectral digraphs are isomorphic as undirected graphs?*

6.2 Harmonic paths

A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$. One can easily verify that a path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.

Denote by $\mathcal{H}_p$ the set of all harmonic paths in Ω_p so that $\mathcal{H}_p$ is a subspace of Ω_p.

Theorem 6.7 (Hodge decomposition) *The space Ω_p is an orthogonal sum:*

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \partial^*\Omega_{p-1} \bigoplus \mathcal{H}_p \tag{6.2}$$

where ∂ and ∂^* are as follows:

$$\Omega_{p-1} \xrightarrow{\partial^*} \Omega_p \xleftarrow{\partial} \Omega_{p+1}.$$

Corollary 6.8 *There is a natural linear isomorphism*

$$H_p \cong \mathcal{H}_p. \tag{6.3}$$

In particular, $\dim \mathcal{H}_p = \beta_p$, that is, the multiplicity of 0 as an eigenvalue of Δ_p is equal to the Betti number β_p.

Proof. Consider the operators

$$\Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p.$$

It follows from (6.2) that

$$\ker \partial|_{\Omega_p} = (\partial^* \Omega_{p-1})^{\perp} = \partial \Omega_{p+1} \bigoplus \mathcal{H}_p \tag{6.4}$$

whence $H_p = \ker \partial|_{\Omega_p} / \partial \Omega_{p+1} \cong \mathcal{H}_p$. ∎

6.3 Matrix of Δ_p

Let $\{\alpha_i\}$ be an orthonormal basis in Ω_p, $\{\beta_m\}$ be an orthonormal basis in Ω_{p-1} and $\{\gamma_n\}$ be an orthonormal basis in Ω_{p+1} :

$$\begin{array}{ccccc} \Omega_{p-1} & \overset{\partial}{\underset{\partial^*}{\leftrightarrows}} & \Omega_p & \overset{\partial}{\underset{\partial^*}{\leftrightarrows}} & \Omega_{p+1} \\ \{\beta_m\} & & \{\alpha_i\} & & \{\gamma_n\} \end{array}$$

The operator $\partial : \Omega_p \to \Omega_{p-1}$ has in the bases $\{\alpha_i\}$ and $\{\beta_m\}$ the matrix

$$B = (\langle \beta_m, \partial \alpha_i \rangle)_{m,i} \tag{6.5}$$

where m is the row index and i is the column index.

Similarly, the operator $\partial^* : \Omega_p \to \Omega_{p+1}$ has the matrix

$$C = (\langle \gamma_n, \partial^* \alpha_i \rangle)_{n,i} = (\langle \partial \gamma_n, \alpha_i \rangle)_{n,i} \ . \tag{6.6}$$

Since $\Delta_p = \partial^* \partial + (\partial^*)^* \partial^*$, we obtain the matrix of Δ_p in the basis $\{\alpha_i\}$:

$$\boxed{\text{matrix of } \Delta_p = B^T B + C^T C}. \tag{6.7}$$

More explicitly, the (i,j)-entry of the matrix of Δ_p in the basis $\{\alpha_i\}$ is given by

$$\boxed{\langle \Delta_p \alpha_i, \alpha_j \rangle = \sum_m \langle \partial \alpha_i, \beta_m \rangle \langle \partial \alpha_j, \beta_m \rangle + \sum_n \langle \alpha_i, \partial \gamma_n \rangle \langle \alpha_j, \partial \gamma_n \rangle}. \tag{6.8}$$

Example. Recall that $\Omega_{-1} = \{0\}$, $\Omega_0 = \{e_i : i \in V\}$ and $\Omega_1 = \langle e_{kl} : k \to l \rangle$. Assuming that $\langle \cdot, \cdot \rangle$ is the natural inner product, we obtain by (6.8) that the matrix of Δ_0 is

$$\begin{aligned}
\langle \Delta_0 e_i, e_j \rangle &= \sum_{k \to l} \langle e_i, \partial e_{kl} \rangle \langle e_j, \partial e_{kl} \rangle \\
&= \sum_{k \to l} \langle e_i, e_l - e_k \rangle \langle e_j, e_l - e_k \rangle \\
&= \sum_{k \to l} (\delta_{il} - \delta_{ik}) (\delta_{jl} - \delta_{jk}) \\
&= \sum_{k \to i} \delta_{ij} + \sum_{i \to l} \delta_{ij} - \mathbf{1}_{\{i \to j\}} - \mathbf{1}_{\{j \to i\}} \\
&= \deg(i)\, \delta_{ij} - \mathbf{1}_{\{i \to j\}} - \mathbf{1}_{\{j \to i\}}.
\end{aligned}$$

If G has no double arrow then the matrix of $\Delta_0 = \operatorname{diag}(\deg(i)) - \mathbf{1}_{\{i \sim j\}}$ where $\mathbf{1}_{\{i \sim j\}}$ is the adjacency matrix of G. Hence, Δ_0 is the usual unnormalized Laplacian (=Kirchhoff operator) on functions on G.

Consequently, $\operatorname{trace} \Delta_0 = \sum_{i \in V} \deg(i) = 2E$.

6.4 Examples of computation of Δ_1

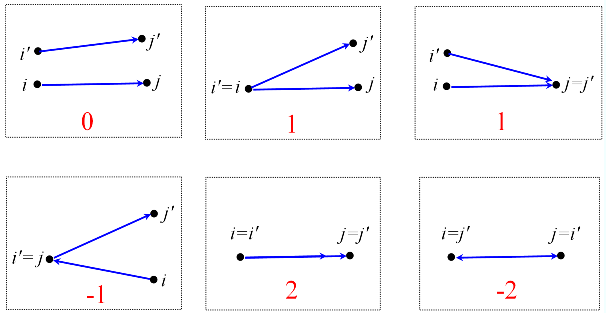
Let us compute Δ_1 for the natural inner product. We use the orthonormal bases $\{e_m\}$ in Ω_0 and $\{e_{ij} : i \to j\}$ in Ω_1. Let $\{\gamma_n\}$ be an orthonormal basis in Ω_2.
The matrix of Δ_1 has dimensions $E \times E$ and, by (6.8), its entries are

$$\langle \Delta_1 e_{ij}, e_{i'j'} \rangle = \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle + \sum_n \langle e_{ij}, \partial \gamma_n \rangle \langle e_{i'j'}, \partial \gamma_n \rangle \tag{6.9}$$

for all arrows $i \to j$ and $i' \to j'$. For the first sum in (6.9) we have

$$\begin{aligned} \sum_m \langle \partial e_{ij}, e_m \rangle \langle \partial e_{i'j'}, e_m \rangle &= \sum_m \langle e_j - e_i, e_m \rangle \langle e_{j'} - e_{i'}, e_m \rangle = \sum_m (\delta_{jm} - \delta_{im}) (\delta_{j'm} - \delta_{i'm}) \\ &= \delta_{jj'} - \delta_{ij'} - \delta_{ji'} + \delta_{ii'} =: [ij, i'j'] . \end{aligned}$$

The values of $[ij, i'j']$ are shown here:

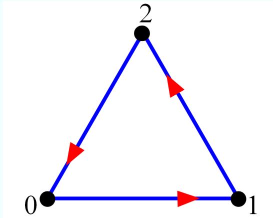


Hence, in the case $p = 1$, we have

$$B^T B = ([ij, i'j']) .$$

In particular, diagonal entries of $B^T B$ are 2.

Example. Consider an 1-torus

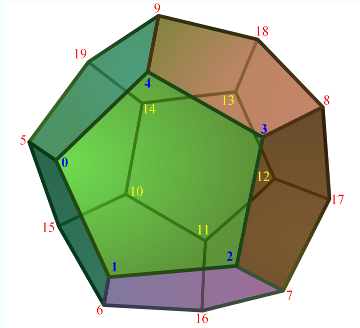
$$T =$$ 

In this case $\Omega_1 = \langle e_{01}, e_{12}, e_{20} \rangle$, $\Omega_2 = \{0\}$, $|H_1| = 1$. Hence, we obtain

$$
\begin{aligned}
\text{the matrix of } \Delta_1 = B^T B &= ([ij, i'j']) \\
&= \left(\begin{array}{cccc} & e_{01} & e_{12} & e_{20} \\ e_{01} & [01,01] & [01,12] & [01,20] \\ e_{12} & [12,01] & [12,12] & [12,20] \\ e_{20} & [20,01] & [20,12] & [20,20] \end{array}\right) \\
&= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}
\end{aligned}
$$

The eigenvalues of Δ_1 are $\{0, 3, 3\} = \{0, 3_2\}$.

Example. Consider a dodecahedron (like on p.2.3):



We have $V = 20,\ E = 30,$
$\Omega_2 = \{0\}$ and $|H_1| = 11.$
In particular, $C^T C = 0.$

The matrix of $\Delta_1 = B^T B$ is shown here:

2	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	0	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	2	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-1	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	-1	0	2	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-1	0	1	2	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	-1	0	2	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	2	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	-1	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
0	0	0	0	-1	0	0	0	0	0	1	0	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	-1	0	1	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	-1	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	1	0	0	0	0	1	0	-1
0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	1	2	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	2	1	-1	-1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	-1	0	0	1	2	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	2	1	-1	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	-1	0	2	1	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	1	2	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	-1	0	0	0	0	0	-1	0	2

The eigenvalues of Δ_1 are:

$$0_{11},\ 2_5,\ 3_4,\ 5_4,\ \left(3 \pm \sqrt{5}\right)_3,$$

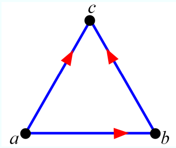
where the subscripts show multiplicity.

For a general digraph G with $\Omega_2 \neq \{0\}$, let us compute the entry $\langle e_{ij}, \partial\gamma_n\rangle$ of the matrix C assuming that $\gamma_n = \gamma$ is a triangle or square (note that although Ω_2 has always a basis of triangles and squares, the squares in this basis do not have to be orthogonal).
If $\gamma = e_{abc}$ is a triangle then we have

$$\langle e_{ij}, \partial\gamma\rangle = \langle e_{ij}, e_{ab} + e_{bc} - e_{ac}\rangle = [ij, \gamma]\,,$$

where

$$\boxed{[ij,\gamma] := \begin{cases} 1, & \text{if } ij \in \{ab, bc\} \\ -1 & \text{if } ij = ac \\ 0, & \text{otherwise.} \end{cases}}$$

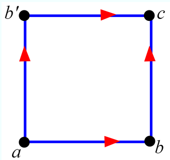


If $\gamma = \frac{e_{abc} - e_{ab'c}}{\sqrt{2}}$ is a (normalized) square then

$$\langle e_{ij}, \partial\gamma\rangle = \frac{1}{\sqrt{2}}\langle e_{ij},\, e_{ab} + e_{bc} - e_{ab'} - e_{b'c}\rangle = \frac{1}{\sqrt{2}}[ij, \gamma]\,,$$

where

$$\boxed{[ij,\gamma] = \begin{cases} 1, & \text{if } ij \in \{ab\,, bc\} \\ -1 & \text{if } ij \in \{ab', b'c\} \\ 0, & \text{otherwise.} \end{cases}}$$



Example. Let G be a triangle $\{0 \to 1 \to 2, 0 \to 2\}$. Then $\Omega_1 = \langle e_{01}, e_{12}, e_{02} \rangle$ and

$$B^T B = ([ij, i'j']) = \left(\begin{array}{cccc} & e_{01} & e_{12} & e_{02} \\ e_{01} & [01,01] & [01,12] & [01,20] \\ e_{12} & [12,01] & [12,12] & [12,20] \\ e_{02} & [02,01] & [02,12] & [02,02] \end{array}\right) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single triangle $\gamma = e_{012}$ so that

$$C = \left(\begin{array}{cccc} & e_{01} & e_{12} & e_{02} \\ e_{012} & [01,\gamma] & [12,\gamma] & [02,\gamma] \end{array}\right) = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

$$C^T C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example. Let G be a square $\{0 \to 1 \to 3, 0 \to 2 \to 3\}$. Then $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23} \rangle$ and

$$B^T B = ([ij, i'j']) = \begin{array}{cc} & \begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{13} \\ e_{23} \end{array} & \begin{pmatrix} [01,01] & [01,02] & [01,13] & [01,23] \\ [02,01] & [02,02] & [02,13] & [02,23] \\ [12,01] & [13,02] & [13,13] & [13,23] \\ [23,01] & [23,02] & [23,13] & [23,23] \end{pmatrix} \end{array} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

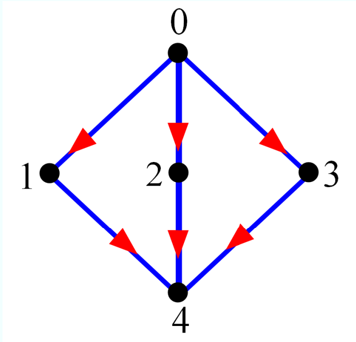
The basis $\{\gamma_n\}$ of Ω_2 consists of a single square $\gamma = \frac{1}{\sqrt{2}}(e_{013} - e_{023})$ so that

$$C = \frac{1}{\sqrt{2}} \begin{array}{cc} & \begin{array}{cccc} e_{01} & e_{02} & e_{13} & e_{23} \end{array} \\ \gamma & \begin{pmatrix} [01,\gamma] & [02,\gamma] & [13,\gamma] & [23,\gamma] \end{pmatrix} \end{array} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix}$$

$$C^T C = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix}, \quad \text{the eigenvalues are } \{2_3, 4\}.$$

Example. Consider a following digraph:



Here $|\Omega_1| = E = 6, \quad |\Omega_2| = 2$ and

$\Omega_2 = \langle e_{014} - e_{024}, e_{014} - e_{034} \rangle$

However, this basis is not orthogonal.

Orthogonalization gives an orthonormal basis in Ω_2:

$$
\begin{aligned}
\gamma_1 &= \frac{1}{\sqrt{2}} \left(e_{014} - e_{024} \right), \\
\gamma_2 &= \frac{1}{\sqrt{6}} \left(e_{014} + e_{024} - 2e_{034} \right).
\end{aligned}
$$

Since

$$
\begin{aligned}
\partial\gamma_1 &= \frac{1}{\sqrt{2}} \left(e_{01} + e_{14} - e_{02} - e_{24} \right), \\
\partial\gamma_2 &= \frac{1}{\sqrt{6}} \left(e_{01} + e_{04} + e_{02} + e_{24} - 2e_{03} - 2e_{34} \right),
\end{aligned}
$$

we compute the matrix C:

$$C = (\langle e_{ij}, \partial\gamma_n \rangle) = \begin{array}{c} \\ \partial\gamma_1 \\ \partial\gamma_2 \end{array} \begin{array}{cccccc} e_{01} & e_{14} & e_{02} & e_{24} & e_{03} & e_{34} \\ \end{array}$$

$$\begin{array}{c|cccccc} & e_{01} & e_{14} & e_{02} & e_{24} & e_{03} & e_{34} \\ \hline \partial\gamma_1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \partial\gamma_2 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{array}$$

and

$$C^T C = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

We compute also B:

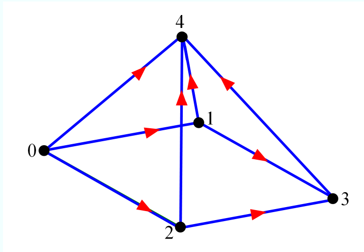
$$B^T B = ([e_{ij}, e_{i'j'}]) = \begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}$$

whence

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} \end{pmatrix}.$$

The spectrum of Δ_1 is $\{2_4, 3, 5\}$.

Example. Consider the following pyramid:



Here $|\Omega_0| = 5$, $|\Omega_1| = 8$, $|\Omega_2| = 5$,

and

$$\Omega_2 = \langle e_{014}, e_{024}, e_{134}, e_{234}, e_{013} - e_{023} \rangle .$$

We have

$$B^T B = ([ij, i'j']) = \begin{array}{cc} & \begin{array}{cccccccc} e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \end{array} \\ \begin{array}{c} e_{01} \\ e_{02} \\ e_{13} \\ e_{23} \\ e_{04} \\ e_{14} \\ e_{24} \\ e_{34} \end{array} & \begin{pmatrix} 2 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

$$C = \begin{array}{cc} & \begin{array}{cccccccc} e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \end{array} \\ \begin{array}{c} e_{014} \\ e_{024} \\ e_{134} \\ e_{234} \\ \frac{1}{\sqrt{2}}\left(e_{013} - e_{023}\right) \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

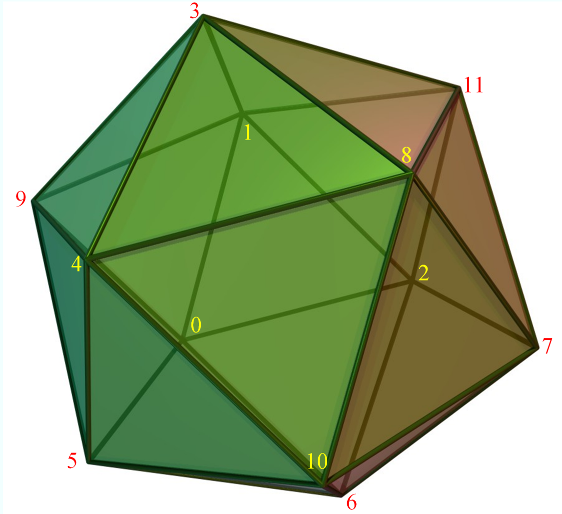
$$C^TC = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 & 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\text{matrix of } \Delta_1 = B^TB+C^TC = \begin{pmatrix} \frac{7}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 \end{pmatrix} \tag{6.10}$$

The eigenvalues of Δ_1 are $\{3_5,\ 5_3\}$.

Example. Consider the icosahedron:

Here $V = 12, \quad E = 30, \quad |\Omega_2| = 25$



Space Ω_2 is generated by 20 triangles and 5 squares (see p.56).

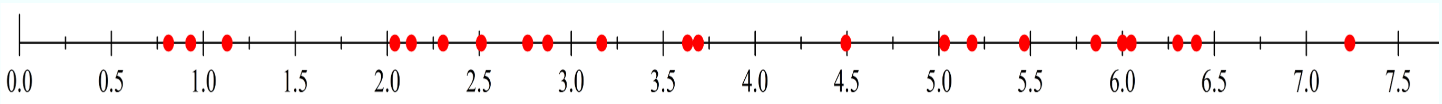
Computation shows that

$$\lambda_{\min} = 0.810... \text{ and } \lambda_{\max} = \left(5 + \sqrt{5}\right)_3$$

Other multiple eigenvalues are

$$6_5 \text{ and } \left(5 - \sqrt{5}\right)_3.$$

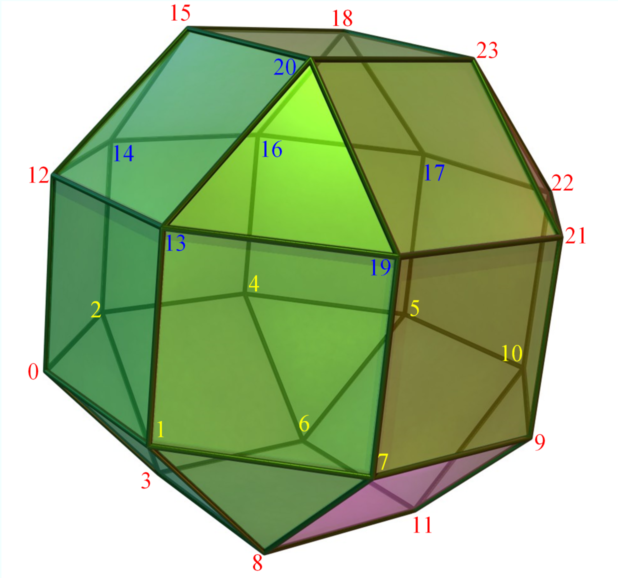
The full spectrum of Δ_1 is shown here:



For icosahedron
the matrix of $\Delta_1 =$

4.5	-0.5	1.0	0.0	1.0	0.0	-1.0	0.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	5.0	1.0	1.0	-0.5	0.0	0.0	0.0	-0.5	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0
1.0	1.0	4.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0
0.0	1.0	0.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	-0.5	-0.5	1.0	5.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	4.0	1.0	1.0	0.0	-1.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-1.0	0.0	0.0	0.0	0.0	1.0	4.0	0.0	0.0	0.0	0.0	0.0	-1.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	0.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	1.0	4.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0
0.0	-0.5	0.0	0.0	-0.5	-1.0	0.0	0.0	0.0	5.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	-1.0	-0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	-0.5	4.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0
-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	1.0	4.5	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	4.5	-0.5	0.0	1.0	-1.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	-0.5	4.5	1.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	-0.5	0.0
0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	1.0	4.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	4.0	1.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-1.0
0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	1.0	4.5	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	4.0	1.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.5	0.0	-0.5	0.0	1.0	0.0	1.0	0.0
0.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	4.5	0.0	-1.0	0.0	-1.0	0.0	0.0
0.0	0.0	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	-0.5	0.0	5.0	0.0	-0.5	0.0	1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	4.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	-0.5	0.0	4.5	1.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	1.0	4.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	1.0	0.0	1.0	0.0	0.0	0.0	4.5	1.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	4.0

Example. Consider a rhombicuboctahedron (see also p.53):



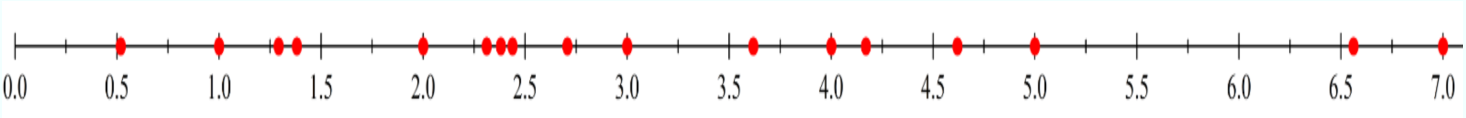
Here $V = 24$, $E = 48$, $|\Omega_2| = 26$.

Space Ω_2 is generated by 8 triangles and 18 squares.

We have $\lambda_{\max} = 7_2$ and $\lambda_{\min} = 0.518...$, and there are many multiple eigenvalues:

$$5_6, \;\; 4_4, 3_3, 2_3, 1_3 \text{ etc.}$$

The spectrum of Δ_1 is shown here:



For rhombicuboctahedron the matrix of $\Delta_1 =$

3.0	1.0	0.5	-1.0	-0.5	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	3.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.0	3.5	0.0	-0.5	0.0	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-1.0	0.0	0.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	0.0	-0.5	0.0	3.5	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	3.5	0.5	-0.5	-1.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	-1.0	0.0	0.0	0.0	0.5	3.0	-0.5	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	-1.0	0.0	0.0	-0.5	-0.5	3.0	0.5	0.0	0.5	0.0	1.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	0.0	-0.5	0.0	0.5	-1.0	0.0	0.5	3.0	0.0	0.0	0.0	0.0	-0.5	1.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	3.5	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.5	0.0	0.0	3.5	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	3.0	0.5	-0.5	0.0	0.0	0.0	1.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.5	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.5	3.5	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	-1.0	-0.5	-0.5	3.0	0.0	0.0	0.5	0.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	3.5	0.5	-0.5	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	3.0	-0.5	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	-1.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.5	-0.5	-0.5	3.0	0.0	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-1.0	0.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	-0.5	-0.5	0.5	0.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.5	0.0	0.0	1.0	0.0	0.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.5	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-0.5	0.0	0.0	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	3.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.5	0.0	0.0	-0.5	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	-0.5	0.0	0.0	0.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	3.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	3.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	-0.5	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	3.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	-0.5	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	3.0	-0.5	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.5	1.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	3.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	-0.5	-1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	1.0	0.0	0.0	0.0	0.0	-0.5	0.0	-0.5	3.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.5	0.0	-1.0
-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	3.0	1.0	0.5	-1.0	-0.5	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.5	0.0	0.0	0.0	0.0	0.0	1.0	3.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	0.0	3.5	0.0	-0.5	0.0	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.5	0.0	0.0	-1.0	0.0	0.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	3.5	0.0	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	3.5	0.5	-0.5	-1.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.5	3.0	-0.5	0.0	-1.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	-0.5	-0.5	3.0	0.5	0.0	0.5	0.0	1.0	-0.5	0.0	0.0	-0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	-0.5	0.0	0.5	-1.0	0.0	0.5	3.0	0.0	0.0	0.0	0.0	-0.5	1.0	0.0	-0.5	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	3.5	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.5	0.0	0.0	3.5	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	3.0	0.5	-0.5	0.0	0.0	0.0	1.0	0.0	-0.5
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.5	3.5	-0.5	0.0	0.0	0.0	0.0	0.0	-0.5
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.0	-1.0	-0.5	-0.5	3.0	0.0	0.0	0.5	0.0	1.0	0.5
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	3.5	0.5	-0.5	0.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.5	3.0	-0.5	-1.0	-0.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	-0.5	-0.5	0.0	0.0	0.0	0.0	0.5	-0.5	-0.5	3.0	0.0	0.5	1.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0	-1.0	0.0	3.5	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	-0.5	-0.5	0.5	0.0	3.5	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.5	-0.5	0.5	0.0	0.0	1.0	0.0	0.0	3.5

6.5 Trace of Δ_1

Recall that, for any digraph G,

$$\operatorname{trace}\Delta_0 = \sum_{i\in V} \deg(i) = 2E.$$

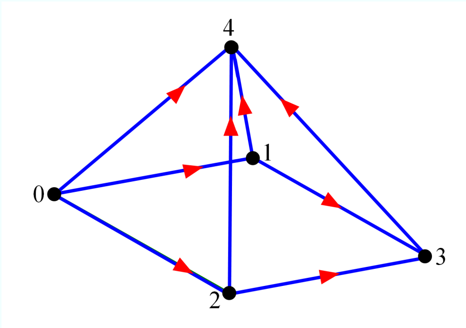
There is a similar result for the trace of Δ_1.

Theorem 6.9 *Let T be the number of triangles in Ω_2, S be the number of linearly independent squares in Ω_2, and D be the number of double arrows $a \rightleftarrows b$. Then*

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S + 4D. \tag{6.11}$$

By a square here we mean an allowed 2-path $e_{abc} - e_{ab'c}$ such that $a \neq c$ and $a \not\to c$.

For example, for this pyramid (as on p.104) we have $E = 8$, $T = 4$, $S = 1$ and $D = 0$, whence $\operatorname{trace}\Delta_1 = 2\cdot 8 + 3\cdot 4 + 2\cdot 1 = 30$, which matches the sum of the eigenvalues as well as the sum of the diagonal entries of the matrix of Δ_1 in (6.10).



6.6 An estimate of $\lambda_{\max}(\Delta_1)$

Denote by $\lambda_{\max}(A)$ the maximal eigenvalue of a symmetric operator A. It is easy to prove that

$$\lambda_{\max}(\Delta_0) \leq 2 \max_i \deg(i).$$

For any arrow ξ in G denote by $\deg_\Delta \xi$ the number of triangles containing ξ and by $\deg_\square \xi$ the number of squares containing ξ.

Theorem 6.10 *Assume that there is an orthogonal basis in Ω_2 that consists of triangles and squares. Then*

$$\lambda_{\max}(\Delta_1) \leq \max\left(2 \max_{i \in V} \deg i, \quad \max_{\xi \in E} \left(3 \deg_\Delta \xi + 2 \deg_\square \xi\right)\right). \tag{6.12}$$

Problem 6.11 *Extend* (6.12) *to the general case.*

Problem 6.12 *Is it true that in fact*

$$\lambda_{\max}(\Delta_1) \leq 2 \max_{i \in V} \deg i \ ?$$

This is the case in all known examples.

6.7 Exact computation of $\operatorname{spec} \Delta_p$ on some digraphs

6.7.1 Spectrum of Δ_p on digraphs spheres

Let $D_m = \{ \underbrace{\bullet, ..., \bullet}_{m \text{ vertices}} \}$ be the digraph that consists of $m \geq 1$ disjoint vertices and no arrows. Consider for any $n \geq 1$ the digraph $D_m^n = \underbrace{D_m * ... * D_m}_{n \text{ times}}$.

Theorem 6.13 *We have, for all $n, m \geq 1$ and $r \geq 2$,*

$$\operatorname{spec} \Delta_{r-1}(D_m^n) = \left\{ ((n-k)m)_{(m-1)^k \binom{r}{k} \binom{n}{r}} \right\}_{k=0}^{r}. \tag{6.13}$$

More explicitly, (6.13) means the following: if $r > n$ then $\operatorname{spec} \Delta_{r-1}(D_m^n) = \emptyset$, while for $r \leq n$ the spectrum of $\Delta_{r-1}(D_m^n)$ consists of the following $r+1$ eigenvalues

$$(n-r)m, \; (n-r+1)m, \; (n-r+2)m, ..., (n-1)m, \; nm, \tag{6.14}$$

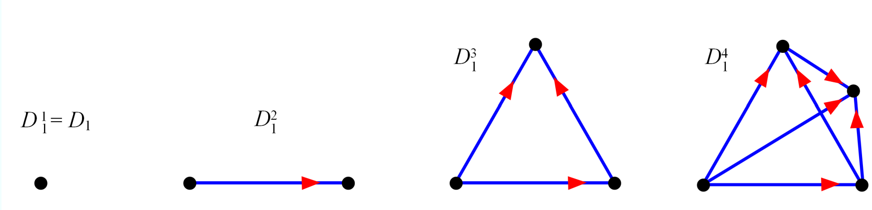
with the multiplicities

$$(m-1)^r \binom{n}{r}, \quad (m-1)^{r-1} r \binom{n}{r}, \; (m-1)^{r-2} \binom{r}{2} \binom{n}{r}, ..., (m-1) r \binom{n}{r}, \; \binom{n}{r}. \tag{6.15}$$

Example. Let $m = 1$, that is, $D_1 = \{\bullet\}$. Clearly, $D_1^n = K_n$ where K_n is a complete digraph that consist of n vertices

$$\{0, ..., n-1\}$$

and all arrows $i \to j$ for $i < j$, that is, a directed $(n-1)$-simplex.

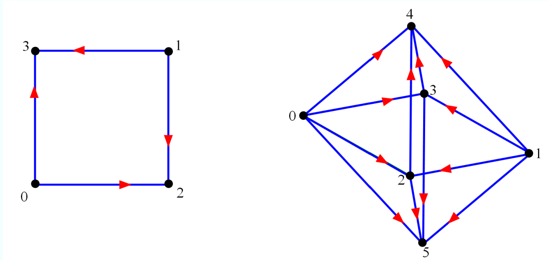


In this case all the multiplicities in (6.15) are 0 except for the last one $\binom{n}{r}$. Hence,

$$\operatorname{spec} \Delta_{r-1}(K_n) = \{n_{\binom{n}{r}}\}.$$

Example. Let $m = 2$, that is, $D_2 = \{\bullet, \bullet\}$. Then $D_2^n =: S^{n-1}$ can be regarded as a digraph sphere of dim $= n - 1$.

For example, S^1 is a *diamond* and S^2 is an *octahedron*.



In this case (6.13) becomes

$$\operatorname{spec}\Delta_{r-1}(S^{n-1}) = \left\{(2(n-k))_{\binom{r}{k}\binom{n}{r}}\right\}_{k=0}^{r}.$$

Consequently, if $2 \leq r \leq n$ then

$$\lambda_{\max}\left(\Delta_{r-1}(S^{n-1})\right) = 2n_{\binom{n}{r}} \quad \text{and} \quad \lambda_{\min}\left(\Delta_{r-1}(S^{n-1})\right) = (2\,(n-r))_{\binom{n}{r}}.$$

For example, for $r = 2$ we have

$$\operatorname{spec}\Delta_{1}(S^{n-1}) = \left\{(2(n-2))_{\binom{n}{2}}, (2(n-1))_{2\binom{n}{2}}, (2n)_{\binom{n}{2}}\right\},$$

and for $r = 3$

$$\operatorname{spec}\Delta_{2}(S^{n-1}) = \left\{(2(n-3))_{\binom{n}{3}}, (2(n-2))_{3\binom{n}{3}}, (2\,(n-1))_{3\binom{n}{3}}, (2n)_{\binom{n}{3}}\right\}.$$

For the octahedron S^2 (that is $n = 3$) we obtain

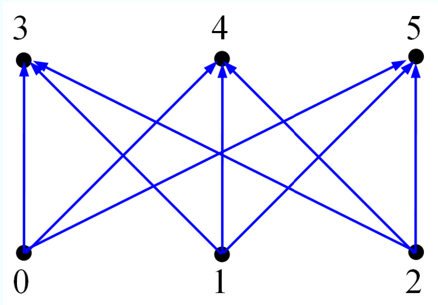
$$\operatorname{spec}\Delta_{1}(S^{2}) = \{2_3, 4_6, 6_3\}$$

and

$$\operatorname{spec}\Delta_{2}(S^{2}) = \{0, 2_3, 4_3, 6\}.$$

Example. Let $m = 3$ and $n = 2$.

Then D_3^2 coincides with the complete bipartite digraph $K_{3,3}$:



Then (6.13) yields for $r = 2$ that

$$\operatorname{spec} \Delta_1(K_{3,3}) = \left\{ (3(2-k))_{\binom{2}{k}\binom{2}{2}2^k} \right\}_{k=0}^{2} = \{0_4,\ 3_4,\ 6\}\,.$$

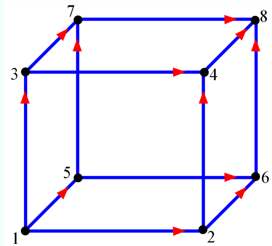
6.7.2 Spectrum of Δ_p on cubes

Recall that the n-cube I^n is defined by

$$I^n = \underbrace{I \Box I \Box \ldots \Box I}_{n}$$

where $I = \{\bullet \rightarrow \bullet\}$.

$$I^3 =$$



The operator $\Delta_p(I^n)$ is non-trivial if $0 \leq p \leq n$. It is possible to prove that

$$\operatorname{spec} \Delta_0(I^n) = \left\{ (2k)_{\binom{n}{k}} \right\}_{k=0}^{n}. \tag{6.16}$$

Theorem 6.14 *For all* $1 \leq p \leq n$ *we have*

$$\operatorname{spec} \Delta_p(I^n) = \left\{ \left(\frac{2k}{p} \right)_{\binom{n}{k}\binom{k-1}{p-1}} \right\}_{k=p}^{n} \bigsqcup \left\{ \left(\frac{2k}{p+1} \right)_{\binom{n}{k}\binom{k-1}{p}} \right\}_{k=p+1}^{n}. \tag{6.17}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(I^n)\right) = \left(\frac{2n}{p} \right)_{\binom{n-1}{p-1}} \quad \textit{and} \quad \lambda_{\min}\left(\Delta_p(I^n)\right) = 2_{\binom{n+1}{p+1}}$$

For example, for $p = 1$ we obtain

$$\operatorname{spec} \Delta_1(I^n) = \left\{ (2k)_{\binom{n}{k}} \right\}_{k=1}^{n} \bigsqcup \left\{ k_{(k-1)\binom{n}{k}} \right\}_{k=2}^{n}, \tag{6.18}$$

and

$$\lambda_{\min} \left(\Delta_1(I^n) \right) = 2_{\binom{n}{2}} \quad \text{and} \quad \lambda_{\max} \left(\Delta_1(I^n) \right) = (2n)_1 \, .$$

Example. For a 3-cube we obtain

$$\operatorname{spec} \Delta_1(I^3) = \left\{ (2k)_{\binom{3}{k}} \right\}_{k=1}^{3} \bigsqcup \left\{ k_{(k-1)\binom{3}{k}} \right\}_{k=2}^{3} = \{2_3, 4_3, 6\} \sqcup \{2_3, 3_2\} = \{2_6, 3_2, 4_3, 6\} \, ,$$

$$\operatorname{spec} \Delta_2 \left(I^3 \right) = \left\{ k_{\binom{3}{k}\binom{k-1}{1}} \right\}_{k=2}^{3} \bigsqcup \left\{ \left(\frac{2}{3} k \right)_{\binom{3}{k}\binom{k-1}{2}} \right\}_{k=3}^{3} = \{2_3, 3_2\} \sqcup \{2_1\} = \{2_4, 3_2\} \, ,$$

$$\operatorname{spec} \Delta_3 \left(I^3 \right) = \left\{ \left(\frac{2k}{3} \right)_{\binom{3}{k}\binom{k-1}{2}} \right\}_{k=3}^{3} = \{2_1\}$$

It follows from (6.18) that

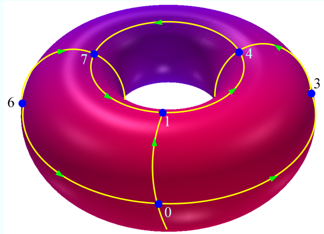
$$\operatorname{spec} \Delta_1(I^4) = \{2_{10}, \ 3_8, \ 4_9, \ 6_4, \ 8\} \, , \quad \operatorname{spec} \Delta_1 \left(I^5 \right) = \{2_{15}, \ 3_{20}, \ 4_{25}, \ 5_4, \ 6_{10}, \ 8_5, \ 10\} \, .$$

6.7.3 Spectrum of Δ_p on tori

Recall that the n-torus T^n is defined by

$$T^n = \underbrace{T\square T\square \ldots \square T}_{n}$$

$$T^2 =$$



where $T = \{0 \to 1 \to 2 \to 0\}$.

We have

$$\operatorname{spec}\Delta_0(T^n) = \left\{(3k)_{2^k\binom{n}{k}}\right\}_{k=0}^{n}. \tag{6.19}$$

Theorem 6.15 *For all $1 \le p \le n$ we have*

$$\operatorname{spec}\Delta_p(T^n) = \left\{\left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}}\right\}_{k=0}^{n} \bigsqcup \left\{\left(\frac{3k}{p+1}\right)_{2^k\binom{n}{k}\binom{n-1}{p}}\right\}_{k=0}^{n}. \tag{6.20}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(T^n)\right) = \left(\frac{3n}{p}\right)_{2^n\binom{n-1}{p-1}} \quad \text{and} \quad \lambda_{\min}\left(\Delta_p(T^n)\right) = 0_{\binom{n}{p}}.$$

For example, for $p = 1$ we obtain

$$\operatorname{spec}\Delta_1(T^n) = \left\{(3k)_{2^k\binom{n}{k}}\right\}_{k=0}^{n} \bigsqcup \left\{\left(\frac{3k}{2}\right)_{2^k\binom{n}{k}(n-1)}\right\}_{k=0}^{n}.$$

Example. For T^2 we obtain

$$\begin{aligned}\operatorname{spec}\Delta_1(T^2) &= \left\{(3k)_{2^k\binom{2}{k}}\right\}_{k=0}^{2} \bigsqcup \left\{\left(\frac{3k}{2}\right)_{2^k\binom{2}{k}}\right\}_{k=0}^{2} \\ &= \{0_1, 3_4, 6_4\} \bigsqcup \left\{0_1, \left(\frac{3}{2}\right)_4, 3_4\right\} = \left\{0_2,\ \left(\frac{3}{2}\right)_4,\ 3_8,\ 6_4\right\}\end{aligned}$$

$$\operatorname{spec}\Delta_2\left(T^2\right) = \left\{\left(\frac{3}{2}k\right)_{2^k\binom{2}{k}}\right\}_{k=0}^{2} = \left\{0, \left(\frac{3}{2}\right)_4, 3_4\right\},$$

and for T^3

$$\operatorname{spec}\Delta_1(T^3) = \left\{0_3,\ \left(\frac{3}{2}\right)_{12},\ 3_{30},\ \left(\frac{9}{2}\right)_{16},\ 6_{12},\ 9_8\right\}.$$