Eigenvalues of the Hodge Laplacian on digraphs

Alexander Grigor'yan Yong Lin S.-T. Yau Haohang Zhang

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Abstract

We prove explicit formulas for the spectra of the Hodge Laplacians of all dimensions on various classes of digraphs, including discrete n-cubes, n-tori, n-spheres.

Contents

1 Introduction

The purpose of this paper is to compute and estimate the eigenvalues of the Hodge Laplacians on certain classes of finite digraphs (=directed graphs). For any chain complex

$$\dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \Omega_{p+1} \overset{\partial}{\leftarrow} \dots \tag{1.1}$$

with the ground ring $\mathbb{R}$ and with a boundary operator ∂, the Hodge Laplacian Δ_p is defined as an operator in Ω_p by

$$\Delta_p = \partial\partial^* + \partial^*\partial,$$

where the adjoint operator ∂^* is defined with respect to chosen inner products in linear spaces Ω_p.

With any digraph G, one can associated a chain complex (1.1) whose elements are linear combinations of certain paths in G that go along arrows (see Section 2 for a detailed description). This complex that is referred to as a *path chain complex*, was introduced and investigated in a series of papers [1], [2], [4], [5], [6] etc. It reflects in a natural way the combinatorial structure of the underlying digraph and has appropriate functorial properties with respect to graph operations, such as Cartesian product, join, homotopy, etc. In particular, the path chain complex satisfies the Künneth formulas with respect to product and join (cf. (2.3) and (9.3)).

The main results of this paper include explicit computation of the spectra of Δ_p, for all relevant values of p, for some series of digraphs, in particular, for n-simplex, n-cube, n-torus and n-sphere for any dimension n.

The digraphs n-simplex and n-sphere are defined inductively in n by using the operation *join* of digraphs (see Section 9 for definition). As the operator Δ_p satisfies the product rule with respect to join (Proposition 9.1), one can use the method of separation of variables, which in conjunction with Künneth formula for join allows to compute inductively the spectra of Δ_p on n-simplex and n-sphere. These results are stated in Theorem 9.3.

The digraphs n-cube and n-torus are defined inductively by using Cartesian product of digraphs (see Section 2.4). However, the method of separation of variables *does not work* with the canonical inner product on spaces Ω_p where each path has by definition the norm 1, which makes the task of computing the Hodge spectra on such digraphs much more complicated.

We have devised a new method for computing Hodge spectra in this setting that has the following two ingredients.

(I) We have observed that the product rule does work for the so called *normalized* Hodge operator, denoted by $\Delta_p^{(a)}$, where a refers to the weight that is used to redefine the inner product in the spaces Ω_p. Namely, we use the weight where the norm of any path of length p is equal to

$$\frac{1}{\sqrt{p!}}.$$

This together with the Künneth formula for product allows us to compute inductively the spectra of all normalized Hodge operators $\Delta_p^{(a)}$ on Cartesian powers (Proposition 6.3, Theorem 7.2) including n-cubes and n-tori (Examples 7.3, 7.4).

(II) We relate in a certain way the spectra of Δ_p and $\Delta_p^{(a)}$ to those of operators $\mathcal{L}_p = \partial^* \partial$ also acting on Ω_p (Proposition 3.3, Lemma 3.5, Corollary 5.2). Knowing the spectra of $\Delta_p^{(a)}$ for all values of p, we compute the spectra of $\mathcal{L}_p$ and then the spectra of Δ_p. This program is fulfilled in Theorems 7.9, 8.1, 8.2, thus yielding the spectra of all operators Δ_p on all n-cubes and n-tori.

The paper is structured as follows. In Section 2 we provide an overview of the notions of path chain complex, path homology, products of paths and digraphs.

In Section 3 we establish the relations between the spectra of Δ_p and $\mathcal{L}_p$.

In Section 4 we prove an upper bound for the spectrum of Δ_1 in terms of combinatorial quantities (Theorem 4.1), using the results of Section 3.

In Section 5 we introduce the weighted Hodge Laplacian and obtain its general properties. In Section 6 we define the normalized Hodge Laplacian and apply the method of separation of variables in abstract form.

Sections 7 and 8 contain the central results of this paper. In Theorem 7.2 we obtain a rather general formula for the spectrum of $\Delta_p^{(a)}$ on Cartesian powers of digraphs, while in Theorems 8.1, 8.2 we combine this with the results of Section 3 in order to compute the spectra of Δ_p on n-cubes and n-tori.

Section 9 contains the aforementioned results about spectra on joins. In particular, Theorem 9.3 gives the spectra of Δ_p on n-simplices, n-spheres and many other digraphs.

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2 Path chain complex

In this section we revise the notion of a path chain complex on digraphs and some properties. The details can be found in [1], [2], [6], [4].

2.1 Paths and boundary operator

Let V be a finite set, whose elements are called vertices. For any $p \geq 0$, an *elementary p-path* is any sequence $\{i_k\}_{k=0}^p$ of $p+1$ vertices in V, which will be denoted by $e_{i_0 \ldots i_p} = i_0 \ldots i_p$. Define $\Lambda_p = \Lambda_p(V)$ as a linear space of all formal linear combinations of $e_{i_0 \ldots i_p}$ with coefficients from $\mathbb{R}$, that is,

$$\Lambda_p := \operatorname{span}_{\mathbb{R}}\{e_{i_0 \cdots i_p} \mid i_0, \ldots, i_p \in V\}.$$

The boundary operator $\partial_p : \Lambda_p \to \Lambda_{p-1}$ is defined as

$$\partial_p e_{i_0 \cdots i_p} = \sum_{k=0}^{p} (-1)^k e_{i_0 \cdots \widehat{i_k} \cdots i_p},$$

where the notation $\widehat{i_k}$ indicates the omission of the index i_k. Set also $\Lambda_{-1} = \{0\}$ and $\partial_0 e_i = 0$ for any 0-path.

Normally ∂_p will also be denoted by ∂ omitting the subscript p. It is a fundamental property that $\partial^2 = 0$; thus, the pair (Λ_*, ∂) constitutes a chain complex.

An elementary path $i_0 \dots i_p$ is called *regular* if $i_{k-1} \neq i_k$ for all $k = 1, \dots, p$, and as *non-regular* otherwise. The set of all regular elementary p-paths is denoted by $R_p = R_p(V)$, and the set of non-regular p-paths is defined as

$$\mathcal{N}_p = \mathcal{N}_p(V) := \mathrm{span}\{e_{i_0 \dots i_p} \mid i_0 \dots i_p \notin R_p\}.$$

Since $\partial \mathcal{N}_p \subset \mathcal{N}_{p-1}$, the boundary operator ∂ is well-defined on the quotient space $\mathcal{R}_p := \Lambda_p / \mathcal{N}_p$, thus giving rise to a chain complex $(\mathcal{R}_*, \partial)$. The elements of $\mathcal{R}_p$ are called *regularized* p-paths.

A p-path is called regular if it is a linear combination of elementary regular p-paths. It is easy to see that each regularized p-path as an equivalence class contains exactly one regular p-path; hence, for simplicity of notation we identify regularized p-paths with regular p-paths.

2.2 Path chain complex on digraph

Let $G = (V, E)$ be a digraph (=directed graph), where V is a finite set of vertices and E is the set of arrows, that is, ordered pairs (i, j) where i, j are distinct vertices. An arrow in E will be denoted by $i \to j$.

We say that an elementary p-path $i_0 \dots i_p$ is *allowed* if $i_{k-1} \to i_k$ for all $k = 1, \dots, p$. Let $A_p = A_p(G)$ be the set of all allowed elementary p-paths. A p-path is called allowed if it is a linear combination of allowed elementary p-paths. Denote by $\mathcal{A}_p = \mathcal{A}_p(G)$ the linear space of all allowed p-paths, that is,

$$\mathcal{A}_p = \mathcal{A}_p(G) := \mathrm{span}\{A_p\}.$$

It is clear that $\mathcal{A}_p \subset \mathcal{R}_p$ but the boundary operator ∂ does not have to take allowed paths to allowed.

Definition 2.1. A path $u \in \mathcal{R}_p$ is called ∂-invariant if $u \in \mathcal{A}_p$ and $\partial u \in \mathcal{A}_{p-1}$.

Denote by $\Omega_p = \Omega_p(G)$ the linear space of all ∂-invariant p-paths, that is,

$$\Omega_p = \Omega_p(G) := \{u \in \mathcal{A}_p \mid \partial u \in \mathcal{A}_{p-1}\}.$$

It is easy to see that $\partial \Omega_p \subset \Omega_{p-1}$ so that we obtain a chain complex $\{\Omega_*, \partial\}$.

Definition 2.2. The chain complex

$$0 \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots, \tag{2.1}$$

is called the path chain complex of the digraph G.

Observe that

$$\Omega_0 = \mathcal{A}_0 = \mathcal{R}_0 = \Lambda_0 = \operatorname{span}\{e_i : i \in V\}$$

and

$$\Omega_1 = \mathcal{A}_1 = \operatorname{span}\{e_{ij} : i \to j\}.$$

In general, Ω_p may be a proper subspace of $\mathcal{A}_p$ for $p \geq 2$.

We say that three distinct vertices a, b, c of a digraph form a *triangle* if $i \to j \to k$ and $i \to k$. Any triangle determines a ∂-invariant 2-path e_{ijk} as $e_{ijk} \in \mathcal{A}_2$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij} \in \mathcal{A}_1$. Such a path e_{ijk} is also referred to as a triangle.

We say that four distinct vertices i, j, j', k of a digraph form a *square* if $i \to j \to k$, $i \to j' \to k$ but $i \not\to k$. Any square determines a ∂-invariant 2-path

$$u = e_{ijk} - e_{ij'k}$$

as $u \in \mathcal{A}_2$ and $\partial u = e_{jk} + e_{ij} - e_{j'k} - e_{ij'} \in \mathcal{A}_1$. The path u is also called a square.

We say that two distinct vertices i and j form a *double arrow* if $i \to j \to i$. A double arrow determines ∂-invariant 2-paths e_{iji} and e_{jij}, that are also called double arrows.

These three configurations are shown on Fig. 1.

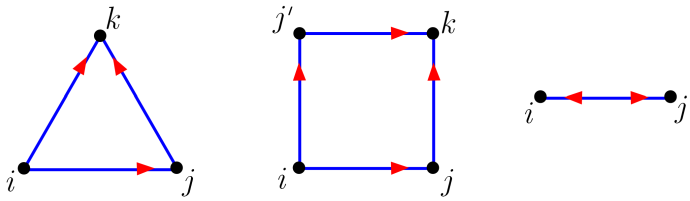

Figure 1: A triangle, a square and a double arrow

Proposition 2.3. ([3, Proposition 2.9], [1, Theorem 1.8]) *The space $\Omega_2(G)$ is spanned by all double arrows, triangles, and squares. In particular, if G contains neither double nor triangles nor squares then $\Omega_2(G) = \{0\}$.*

Note that all double arrows and triangles are linearly independent, while squares could be dependent. Indeed, consider a m-*square*, that is, a sequence of $m+3$ distinct vertices $a,\ b_0, b_1, ..., b_m,\ c$ such that $a \to b_k \to c$ for all $k = 0, \dots, m$, while $a \not\to c$ (see Fig. 2).

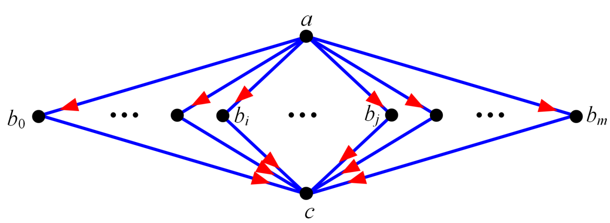

Figure 2: A multisquare

For example, 1-square is a square. If $m \geq 2$ then m-square is also called *multisquare*. An m-square determines the following squares

$$u_{ij} = e_{ab_ic} - e_{ab_jc} \in \Omega_2 \quad \text{for all } i, j = 0, ..., m,$$

and among them only m linearly independent (for example, $\{u_{0j}\}_{i=1}^{m}$). If G contains no multisquares then all double arrows, triangles and squares form a basis in Ω_2.

Proposition 2.4. [3, Proposition 3.23] *If $\Omega_q(G) = \{0\}$ for some q, then $\Omega_p(G) = \{0\}$ also for all $p > q$.*

2.3 Path homology

The chain complex (2.1) determines the homology groups

$$H_p = H_p(G) := \ker \partial_p / \operatorname{Im} \partial_{p+1}$$

that are called the *path homology groups* of G. The dimensions $\dim H_p$ are called the Betti numbers of G. It is possible to prove that $\dim H_0$ is equal to the number of (undirected) connected components of G ([1, Proposition 1.5]).

If the sequence $\{\Omega_p\}_{p \geq 0}$ is finite, that is, $\Omega_q = \{0\}$ for all large enough q, then we define the Euler characteristic of G by

$$\chi(G) = \sum_{k=0}^{\infty} (-1)^{p+1} \dim \Omega_p.$$

In this case, we have also

$$\chi(G) = \sum_{k=0}^{\infty} (-1)^{p+1} \dim H_p.$$

2.4 Product of paths and digraphs

Let X, Y be two finite sets. Consider their product

$$Z = X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Let $z = z_0 z_1 ... z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1, ..., r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1} z_k$ of consecutive vertices is either *vertical* (when $a_{k-1} = a_k$) or *horizontal* (when $b_{k-1} = b_k$).

For given elementary regular paths x on X and y on Y, denote by $\Sigma_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are x and y, respectively. For any path $z \in \Sigma_{x,y}$ define its *staircase* $S(z)$ as the image in $\mathbb{Z}_+^2$ as illustrated on Fig. 3, and the *elevation* $L(z)$ as the number of cells in $\mathbb{Z}_+^2$ below $S(z)$.

Definition 2.5. [1, Section 3.1] Define the *cross product* of elementary regular paths $e_x \in R_*(X)$ and $e_y \in R_*(Y)$ as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y := \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z,$$

and extend it by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

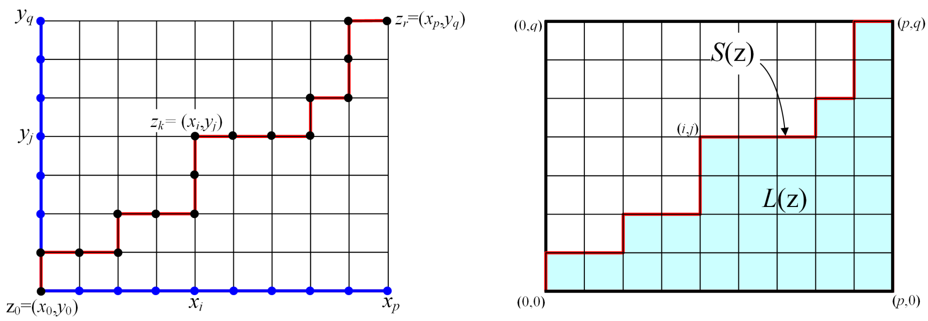

Figure 3: A stair-like path $z \in \Sigma_{x,y}$, its staircase $S(z)$ and its elevation $L(z)$.

Proposition 2.6. [6, Proposition 4.4] *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = \partial u \times v + (-1)^p u \times \partial v. \tag{2.2}$$

Let now X and Y be two digraphs (we denote their sets of vertices by the same letters X and Y, respectively). Define their Cartesian product $Z = X \Box Y$ as a digraph with the set of vertices $X \times Y$, and the arrows in Z are defined as follow:

$$(x_1, y_1) \to (x_2, y_2) \quad \Leftrightarrow \quad x_1 = x_2 \text{ and } y_1 \to y_2 \quad \text{or} \quad x_1 \to x_2 \text{ and } y_1 = y_2,$$

as is shown on the diagram:

$$\begin{array}{cccccc} y_2\bullet & \dots & \overset{(x_1,y_2)}{\bullet} & \longrightarrow & \overset{(x_2,y_2)}{\bullet} & \dots \\ \uparrow & & \uparrow & & \uparrow & \\ y_1\bullet & \dots & \overset{(x_1,y_1)}{\bullet} & \longrightarrow & \overset{(x_2,y_1)}{\bullet} & \dots \\ & & & & & \\ {}^{Y}\diagup{}_{X} & \dots & \underset{x_1}{\bullet} & \longrightarrow & \underset{x_2}{\bullet} & \dots \end{array}$$

Example 2.7. Consider an interval digraph $I = \{{}^0\bullet \to \bullet^1\}$ and define for any $n \geq 1$ the n-*cube* as the digraph

$$I^n = \underbrace{I \Box \ldots \Box I}_{n \text{ times}}.$$

The digraphs I^2 and I^3 are shown on Fig. 4.

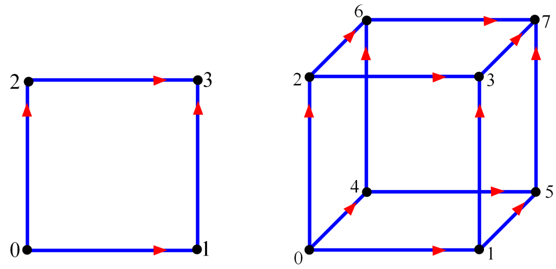

Figure 4: A square I^2 and a 3-cube I^3

Example 2.8. Consider a cyclic digraph $T = \{^0\bullet \to \bullet^1 \to \bullet^2 \to^0\}$ and define for any $n \geq 1$ the *n-torus* as the digraph

$$T^n = \underbrace{T\Box ... \Box T}_{n \text{ times}}.$$

The tori T and T^2 are shown on Fig. 5.

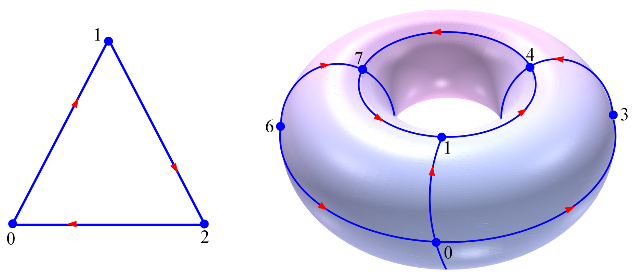

Figure 5: A 3-cycle T and a torus T^2 embedded on a topological torus

It is easy to see that if u and v are allowed paths on X and Y, respectively, then $u \times v$ is allowed on $Z = X\Box Y$. It follows from (2.2) that if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.

In fact, by the Künneth formula of [6, Theorem 4.7], we have, for any $r \geq 0$,

$$\Omega_r (X\Box Y) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (\Omega_p (X) \otimes \Omega_q (Y)), \tag{2.3}$$

where the isomorphism is given by $u \otimes v \mapsto u \times v$.

The identity (2.3) implies a similar isomorphism for homology groups:

$$H_r (X\Box Y) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} (H_p (X) \otimes H_q (Y)). \tag{2.4}$$

In particular, if X and Y are homologically trivial (that is, $H_p = \{0\}$ for all $p \geq 1$) then $X\Box Y$ is also homologically trivial.

3 Abstract Hodge Laplacian

In this section $\{\Omega_p\}_{p\geq 0}$ is any chain complex with a boundary operator $\partial : \Omega_p \to \Omega_{p-1}$, that is,

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots , \tag{3.1}$$

where each Ω_p is a finite-dimensional linear space over $\mathbb{R}$.

Let us fix an arbitrary inner product $\langle \cdot, \cdot \rangle$ on each Ω_p, which determines the adjoint boundary operators $\partial^* : \Omega_{p-1} \to \Omega_p$. Define the *Hodge Laplacian* on Ω_p by

$$\Delta_p = \partial\partial^* + \partial^*\partial.$$

It is a non-negative definite self-adjoint operator on Ω_p. It is known that 0 is the eigenvalue of Δ_p if and only if the homology group H_p of the chain complex (3.1) is non-trivial; moreover, the multiplicity of 0 (as the eigenvalue of Δ_p) is equal to dim H_p ([5, Corollary 3.4]).

Denote

$$\mathcal{K}_p = \partial\partial^*|_{\Omega_p} \quad \text{and} \quad \mathcal{L}_p = \partial^*\partial|_{\Omega_p}.$$

Clearly,

$$\Delta_p = \mathcal{K}_p + \mathcal{L}_p$$

and both $\mathcal{K}_p$ and $\mathcal{L}_p$ are non-negative definite self-adjoint operators in Ω_p (see Fig. 6).

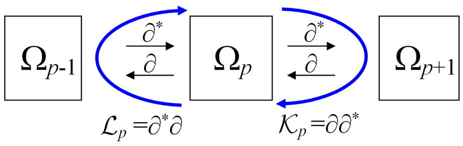

Figure 6: Operators $\mathcal{K}_p$ and $\mathcal{L}_p$

Note for future references that

$$\mathcal{L}_0 = 0 \quad \text{and} \quad \Delta_0 = \mathcal{K}_0.$$

Fix now a positive real λ and consider the subspaces of Ω_p:

$$E_p(\lambda) = \{\varphi \in \Omega_p : \Delta_p\varphi = \lambda\varphi\},$$

$$E'_p(\lambda) = \{\varphi \in \Omega_p : \mathcal{K}_p\varphi = \lambda\varphi\}, \tag{3.2}$$

$$E''_p(\lambda) = \{\varphi \in \Omega_p : \mathcal{L}_p\varphi = \lambda\varphi\}. \tag{3.3}$$

Lemma 3.1. [5, Lemma 4.9] *For any $\lambda > 0$ we have*

$$E_p(\lambda) = E'_p(\lambda) \oplus E''_p(\lambda). \tag{3.4}$$

Proof. Let us first verify that $E'_p(\lambda)$ and $E''_\lambda(\lambda)$ are subspaces of $E_p(\lambda)$. Indeed, if $\varphi \in E'_p(\lambda)$ then

$$\partial\partial^*\varphi = \lambda\varphi$$

whence

$$\partial\varphi = \frac{1}{\lambda}\partial\partial\partial^*\varphi = 0,$$

which implies that

$$\Delta_p\varphi = \partial\partial^*\varphi + \partial^*\partial\varphi = \partial\partial^*\varphi = \lambda\varphi$$

and, hence, $\varphi \in E_p(\lambda)$. Hence, $E'_p(\lambda) \subset E_p(\lambda)$ and in the same way $E''_p(\lambda) \subset E_p(\lambda)$.

Observe now that the subspace $E'_p(\lambda)$ and $E''_p(\lambda)$ of Ω_p are orthogonal because, for any $\varphi \in E'_p(\lambda)$ and $\psi \in E''_p(\lambda)$, we have

$$\langle\varphi,\psi\rangle = \frac{1}{\lambda^2}\langle\partial\partial^*\varphi, \partial^*\partial\psi\rangle = \frac{1}{\lambda^2}\langle\partial\partial\partial^*\varphi, \partial\psi\rangle = 0.$$

Finally, let us prove (3.4). For any $\varphi \in E_p(\lambda)$ we have

$$(\partial\partial^*)^2\varphi = \partial\partial^*(\partial\partial^*\varphi + \partial^*\partial\varphi) = \partial\partial^*\Delta\varphi = \lambda\partial\partial^*\varphi,$$

which implies that $\partial\partial^*\varphi \in E'_p$. Similarly, we have $\partial^*\partial\varphi \in E''_p$. Hence, for any $\varphi \in E_p(\lambda)$ we have

$$\varphi = \frac{1}{\lambda}\Delta\varphi = \frac{1}{\lambda}\partial\partial^*\varphi + \frac{1}{\lambda}\partial^*\partial\varphi,$$

whence (3.4) follows. ■

Lemma 3.2. *For any $\lambda > 0$, we have*

$$\dim E'_p(\lambda) = \dim E''_{p+1}(\lambda)$$

Proof. This follows from [5, Lemma 4.10] but we give here a short independent proof. If $\varphi \in E'_p(\lambda)$ then

$$\partial\partial^*\varphi = \mathcal{K}_p\varphi = \lambda\varphi$$

whence

$$\partial^*\partial(\partial^*\varphi) = \lambda\partial^*\varphi.$$

Since $\lambda \neq 0$, it follows that $\partial^*\varphi \neq 0$ and, hence, $\partial^*\varphi$ is an eigenvector of $\mathcal{L}_p$ with eigenvalue λ, that is, $\partial^*\varphi \in E''_{p+1}(\lambda)$. Hence, ∂^* is a monomorphism from $E'_p(\lambda)$ to $E''_{p+1}(\lambda)$. In the same way ∂ is a monomorphism from $E''_{p+1}(\lambda)$ to $E'_p(\lambda)$, which finishes the proof. ■

For any finite-dimensional self-adjoint operator A denote by $\operatorname{spec} A$ the spectrum of A, that is, an unordered sequence of eigenvalues with multiplicities. Observe that the problem of determination of the spectrum of A amounts to computation of dimensions of eigenspaces $\{Ax = \lambda x\}$ for all $\lambda \in \mathbb{R}$, that is, to multiplicities of all real λ.

We will use the following operations with unordered sequences:

- disjoint union: $\{\lambda_i\} \sqcup \{\mu_j\} = \{\lambda_i, \mu_j\}$ (if the same value λ occurs in the both sequences, then its multiplicities add up in the disjoint union);

- set subtraction: $\{\lambda_i\} \setminus \{\mu_j\}$ (the inverse operation to disjoint union);
- multiplication by constant: $c\{\lambda_i\} = \{c\lambda_i\}$;
- addition: $\{\lambda_i\} + \{\mu_j\} = \{\lambda_i + \mu_j\}$ (note that if one of the sequences is empty then the sum is also empty).

For any non-negative definite self-adjoint operator A, set

$$\operatorname{spec}_+ A = \operatorname{spec} A \setminus \{0\}.$$

The following statement provides a useful tool for computation of $\operatorname{spec} \Delta_p$.

Proposition 3.3. *For all $p \geq 0$, we have*

$$\operatorname{spec}_+ \Delta_p = \operatorname{spec}_+ \mathcal{K}_p \sqcup \operatorname{spec}_+ \mathcal{L}_p \tag{3.5}$$

and

$$\operatorname{spec}_+ \mathcal{K}_p = \operatorname{spec}_+ \mathcal{L}_{p+1}. \tag{3.6}$$

Consequently,

$$\operatorname{spec}_+ \Delta_p = \operatorname{spec}_+ \mathcal{L}_p \sqcup \operatorname{spec}_+ \mathcal{L}_{p+1}. \tag{3.7}$$

Proof. Let $m_p(\lambda)$ be for any $\lambda > 0$ the multiplicity of λ as an eigenvalue of Δ_p. In the same way, let $k_p(\lambda)$ be the multiplicity function of $\mathcal{K}_p$ and $\ell_p(\lambda)$ – that of $\mathcal{L}_p$. We have by Lemma 3.1 that, for any $\lambda > 0$,

$$\dim E_p(\lambda) = \dim E_p'(\lambda) + \dim E_p''(\lambda),$$

that is,

$$m_p(\lambda) = k_p(\lambda) + \ell_p(\lambda) \quad \text{for any } \lambda > 0.$$

This identity is equivalent to (3.5) because the disjoint union of positive spectra is exactly addition of the multiplicities of any $\lambda > 0$. Lemma 3.2 yields $k_p(\lambda) = \ell_{p+1}(\lambda)$, which is equivalent to (3.6).

Finally, (3.7) follows from (3.5) and (3.6). Equivalently, (3.7) can be stated as follows:

$$m_p(\lambda) = \ell_p(\lambda) + \ell_{p+1}(\lambda) \quad \text{for any } \lambda > 0. \tag{3.8}$$

■

Example 3.4. Assume that $\Omega_2 = \{0\}$. Since $\mathcal{L}_0 = 0$ and $\mathcal{L}_2 = 0$, we obtain from (3.7) with $p = 1$ and $p = 0$ that

$$\operatorname{spec}_+ \Delta_1 = \operatorname{spec}_+ \mathcal{L}_1 = \operatorname{spec}_+ \Delta_0.$$

Lemma 3.5. *For any finite digraph G and any $p \geq 1$, we have*

$$\operatorname{spec}_+ \mathcal{L}_p = \bigsqcup_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} \operatorname{spec}_+ \Delta_{p-(2k+1)} \setminus \bigsqcup_{k=1}^{\lfloor \frac{p}{2} \rfloor} \operatorname{spec}_+ \Delta_{p-2k} \tag{3.9}$$

Proof. Using the multiplicity functions from the proof of Proposition 3.3, let us rewrite (3.9) in an equivalent form:

$$\begin{aligned} \ell_p &= \sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} m_{p-(2k+1)} - \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} m_{p-2k} \\ &= \sum_{j=1}^{p} (-1)^{j+1} m_{p-j}. \end{aligned} \tag{3.10}$$

Hence, it suffices to prove (3.10). Using (3.8) with p replaced by $p-1$ and iterating, we obtain

$$\begin{aligned} l_p &= m_{p-1} - l_{p-1} \\ &= m_{p-1} - (m_{p-2} - l_{p-2}) \\ &= m_{p-1} - m_{p-2} + (m_{p-3} - l_{p-3}) \\ &\dots \\ &= m_{p-1} - m_{p-2} + m_{p-3} - \dots - (-1)^p (m_0 - l_0) \end{aligned}$$

whence (3.10) follows as $l_0 \equiv 0$. ■

4 Upper bound for $\lambda_{\max}(\Delta_1)$

In this section we use the techniques based on Proposition 3.3 in order to obtain a certain upper bound for spec Δ_1 on digraphs.

Here the chain complex $\{\Omega_p\}_{p\geq 0}$ is the path chain complex of a digraph $G = (V, E)$ as defined in Section 2.2. We endow each path space $\mathcal{R}_p$ with the *canonical* inner product $\langle \cdot, \cdot \rangle$ given by

$$\left\langle e_{i_0 \dots i_p}, e_{j_0 \dots j_p} \right\rangle = \delta_{i_0 \dots i_p}^{j_0 \dots j_p}. \tag{4.1}$$

In particular, each Ω_p is an inner product space. The corresponding Hodge Laplacians Δ_p on Ω_p are referred to as *canonical* Hodge Laplacians of the digraph G.

Given the inner product structure on $\mathcal{R}_p$, we introduce for each boundary operator $\partial : \mathcal{R}_p \to \mathcal{R}_{p-1}$ the dual operator $\partial^* : \mathcal{R}_{p-1} \to \mathcal{R}_p$, defined by

$$\langle \partial^* u, v \rangle = \langle u, \partial v \rangle, \quad \forall u \in \mathcal{R}_{p-1},\ v \in \mathcal{R}_p.$$

For any self-adjoint operator A in a finite-dimensional space, denote by $\lambda_{\max}(A)$ the maximal eigenvalue of A and by $\lambda_{\min}(A)$ – the minimal eigenvalue of A. In this section we prove an upper estimate of $\lambda_{\max}(\Delta_1)$.

Consider first Δ_0. For any vertex i of G, denote by $\deg(i)$ the (undirected) degree of i, that is, the total number of arrows having i at the one of the ends. It was proved in [1, Prop. 6.2] that

$$\lambda_{\max}(\Delta_0) \leq 2 \max_{i \in V} \deg i. \tag{4.2}$$

Let A denote the adjacency matrix of G, that is, $A = (a_{ij})_{i,j \in V}$ where

$$a_{ij} = \begin{cases} 1, & \text{if } i \to j \text{ or } j \to i \\ 0 & \text{otherwise.} \end{cases}$$

If G contains no double arrow then, in the basis $\{e_i\}$,

$$\text{matrix of } \Delta_0 = \operatorname{diag}(\deg(i)) - A \tag{4.3}$$

(see [1, Example 6.9]). In particular, Δ_0 does not depend on the choice of orientation of arrows.

For any arrow ξ in G, denote by $\deg_\Delta \xi$ the number of triangles containing the arrow ξ, and by $\deg_\square \xi$ – the number of squares containing ξ. Recall that the notion of multisquare was defined in Section 2.2 (see also Fig. 2).

Theorem 4.1. *Assume that digraph G contains neither multisquares nor double arrows. Then*

$$\lambda_{\max}(\Delta_1) \le \max\left(2\max_{i\in V}\deg i,\ \max_{\xi\in E}(3\deg_\Delta \xi + 2\deg_\square \xi)\right). \tag{4.4}$$

The estimate (4.4) improves significantly the estimate

$$\lambda_{\max}(\Delta_1) \le 2\max_{i\in V}\deg i + 3\max_{\xi\in E}\deg_\Delta \xi + 2\max_{\xi\in E}\deg_\square \xi \tag{4.5}$$

that was proved in [1, Theorem 6.20].

Proof. By (3.5) and (3.6) we have

$$\operatorname{spec}_+ \Delta_0 = \operatorname{spec}_+ \mathcal{K}_0$$

and

$$\operatorname{spec}_+ \Delta_1 = \operatorname{spec}_+ \mathcal{K}_0 \sqcup \operatorname{spec}_+ \mathcal{K}_1$$

whence

$$\operatorname{spec}_+ \Delta_1 = \operatorname{spec}_+ \Delta_0 \sqcup \operatorname{spec}_+ \mathcal{K}_1. \tag{4.6}$$

Hence, the estimate (4.4) will follow from (4.6), (4.2) and

$$\lambda_{\max}(\mathcal{K}_1) \le \max_{\xi\in E}(3\deg_\Delta \xi + 2\deg_\square \xi). \tag{4.7}$$

Let us now prove (4.7). Since the Rayleigh quotient of $\mathcal{K}_1$ is

$$\frac{\langle \mathcal{K}_1 u, u\rangle}{\|u\|^2} = \frac{\langle \partial\partial^* u, u\rangle}{\|u\|^2} = \frac{\|\partial^* u\|^2}{\|u\|^2},$$

it suffices to prove that, for any $u \in \Omega_1$,

$$\|\partial^* u\|^2 \le \max_{\xi\in E}(3\deg_\Delta \xi + 2\deg_\square \xi)\|u\|^2. \tag{4.8}$$

Let $u = \sum_{\xi\in E} u^\xi e_\xi$ and, hence,

$$\|u\|^2 = \sum_{\xi\in E}(u^\xi)^2.$$

By the hypothesis, all triangles and squares form an orthogonal basis in Ω_2, denote it by $\{\gamma_n\}$. Using this basis in Ω_2, we have

$$\|\partial^* u\|^2 = \sum_n \frac{\langle \partial^* u, \gamma_n \rangle^2}{\|\gamma_n\|^2} = \sum_n \frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2}.$$

If γ_n is a triangle e_{abc} then $\|\gamma_n\| = 1$ and

$$\langle u, \partial\gamma_n \rangle = \langle u, e_{bc} - e_{ac} + e_{ab} \rangle = u^{bc} - u^{ac} + u^{ab},$$

$$\frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2} \leq 3\left((u^{bc})^2 + (u^{ac})^2 + (u^{ab})^2\right). \tag{4.9}$$

If γ_n is a square $e_{abc} - e_{ab'c}$ then $\|\gamma_n\|^2 = 2$ and

$$\langle u, \partial\gamma_n \rangle = \langle u, e_{ab} + e_{bc} - e_{ab'} + e_{b'c} \rangle = u^{ab} + u^{bc} - u^{ab'} - u^{b'c},$$

$$\frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2} \leq 2\left((u^{ab})^2 + (u^{bc})^2 + (u^{ab'})^2 + (u^{b'c})^2\right). \tag{4.10}$$

Let us sum up (4.9) over all triangles γ_n and (4.10) over all squares γ_n and observe that, for each arrow $\xi \in E$, the component $\left(u^\xi\right)^2$ appears in the sum (4.9) for $\deg_\Delta \xi$ triangles γ_n and in the sum (4.10) for $\deg_\Box \xi$ squares γ_n. Hence, we obtain that

$$\begin{aligned}\sum_n \frac{\langle u, \partial\gamma_n \rangle^2}{\|\gamma_n\|^2} &\leq \sum_{\xi \in E} \left(3\deg_\Delta \xi + 2\deg_\Box \xi\right)\left(u^\xi\right)^2 \\ &\leq \max_{\xi \in E}\left(3\deg_\Delta \xi + 2\deg_\Box \xi\right) \sum_{\xi \in E}\left(u^\xi\right)^2,\end{aligned}$$

whence (4.8) follows. ■

Corollary 4.2. *Under the hypothesis of Theorem 4.1, if in addition* $\deg_\Delta \xi \leq 2$ *for all* $\xi \in E$ *then*

$$\lambda_{\max}(\Delta_1) \leq 2 \max_{i \in V} \deg i. \tag{4.11}$$

Proof. Let $\xi = ij \in E$ be an arrow where $\max_{\xi \in E}\left(3\deg_\Delta \xi + 2\deg_\Box \xi\right)$ is attained. The vertex i has adjacent arrow ξ; besides, each of $\deg_\Delta \xi$ triangles attached to ξ gives one more adjacent arrow to i, and so do all $\deg_\Box \xi$ squares attached to ξ. Hence, we obtain

$$\deg i \geq 1 + \deg_\Delta \xi + \deg_\Box \xi.$$

Using $\deg_\Delta \xi \leq 2$, we obtain

$$2\deg i \geq 2 + 2\deg_\Delta \xi + 2\deg \Box\xi \geq 3\deg_\Delta \xi + 2\deg \Box\xi.$$

Hence, (4.11) follows from (4.4). ■

Example 4.3. Let I be the interval digraph, that is, $I = \{{}^0\bullet \to \bullet^1\}$. As in Example 2.7, consider the n-cube

$$I^n = \underbrace{I\Box ...\Box I}_{n \text{ times}}.$$

It is easy to see that I^n contains no triangles. Since $\deg(i) = n$ for any vertex i, it follows from (4.11) that, for the digraph I^n,

$$\lambda_{\max}(\Delta_1) \le 2n.$$

As we will see in Section 8.3, for I^n we have, in fact, the equality $\lambda_{\max}(\Delta_1) = 2n$.

In fact, In all known examples (4.11) is satisfied even if $\deg_\Delta \xi > 2$ for some ξ.

5 Weighted Hodge Laplacian

As in Section 4, we denote by Δ_p the canonical Hodge Laplacian on a digraph G associated with the canonical inner product (4.1). As in section 3, we have

$$\Delta_p = \mathcal{K}_p + \mathcal{L}_p$$

where

$$\mathcal{K}_p = \partial\partial^*|_{\Omega_p} \quad \text{and} \quad \mathcal{L}_p = \partial^*\partial|_{\Omega_p}$$

and ∂^* is the adjoint to ∂ with respect to the canonical inner product.

Fix a sequence of positive numbers $a = \{a_p\}_{p=0}^{\infty}$ and define another another inner product in $\mathcal{R}_p(G)$ that will be referred to as *weighted inner product*:

$$\left\langle e_{i_0...i_p}, e_{j_0...j_p}\right\rangle_a := \frac{1}{a_p}\left\langle e_{i_0...i_p}, e_{j_0...j_p}\right\rangle = \frac{1}{a_p}\delta^{j_0...j_p}_{i_0...i_p}.$$

In particular, we have $\left\|e_{i_0...i_p}\right\|_a^2 = \frac{1}{a_p}$. Denote by ∂_a^* the adjoint operator of ∂ with respect to the weighted inner product, and by $\Delta_p^{(a)}$ the corresponding Hodge Laplacian of $\{\Omega_p\}_{p\ge 0}$:

$$\Delta_p^{(a)} = \partial\partial_a^* + \partial_a^*\partial = \mathcal{K}_p^{(a)} + \mathcal{L}_p^{(a)}$$

where

$$\mathcal{K}_p^{(a)} = \partial\partial_a^*|_{\Omega_p} \quad \text{and} \quad \mathcal{L}_p^{(a)} = \partial_a^*\partial|_{\Omega_p}$$

For convenience of notation, let us set $a_{-1} = 1$.

Lemma 5.1. *We have for $p \ge 0$*

$$\mathcal{K}_p^{(a)} = \frac{a_{p+1}}{a_p}\mathcal{K}_p \tag{5.1}$$

and

$$\mathcal{L}_p^{(a)} = \frac{a_p}{a_{p-1}}\mathcal{L}_p. \tag{5.2}$$

Consequently,

$$\Delta_p^{(a)} = \frac{a_{p+1}}{a_p}\mathcal{K}_p + \frac{a_p}{a_{p-1}}\mathcal{L}_p. \tag{5.3}$$

In particular, for $p=0$ we have $\mathcal{L}_0=0$ and, hence,

$$\Delta_0^{(a)}=\frac{a_1}{a_0}\mathcal{K}_0=\frac{a_1}{a_0}\Delta_0 \tag{5.4}$$

Proof. We have, for $u\in\Omega_p$ and $\omega\in\Omega_{p+1}$,

$$\langle\partial_a^* u,\omega\rangle_a=\langle u,\partial\omega\rangle_a=\frac{1}{a_p}\langle u,\partial\omega\rangle=\frac{1}{a_p}\langle\partial^* u,\omega\rangle=\frac{a_{p+1}}{a_p}\langle\partial^* u,\omega\rangle_a$$

whence

$$\partial_a^* u=\frac{a_{p+1}}{a_p}\partial^* u \quad \text{for all } u\in\Omega_p.$$

It follows that, for $u\in\Omega_p$,

$$\mathcal{K}_p^{(a)}u=\partial\partial_a^* u=\frac{a_{p+1}}{a_p}\partial\partial^* u=\frac{a_{p+1}}{a_p}\mathcal{K}_p u,$$

which proves (5.1).

If $p=0$ then (5.2) is trivial as $\mathcal{L}_p=0$. If $p\geq 1$ then we have

$$\mathcal{L}_p^{(a)}u=\partial_a^*\partial u=\frac{a_p}{a_{p-1}}\partial^*\partial u=\frac{a_p}{a_{p-1}}\partial^*\partial u=\frac{a_p}{a_{p-1}}\mathcal{L}_p u,$$

which proves (5.2) and, hence, (5.3). ∎

Corollary 5.2. *We have for all $p\geq 0$*

$$\operatorname{spec}_+\Delta_p^{(a)}=\frac{a_{p+1}}{a_p}\operatorname{spec}_+\mathcal{K}_p\sqcup\frac{a_p}{a_{p-1}}\operatorname{spec}_+\mathcal{L}_p \tag{5.5}$$

and

$$\operatorname{spec}_+\Delta_p^{(a)}=\frac{a_p}{a_{p-1}}\operatorname{spec}_+\mathcal{L}_p\sqcup\frac{a_{p+1}}{a_p}\operatorname{spec}_+\mathcal{L}_{p+1}. \tag{5.6}$$

Proof. Indeed, applying (3.5) and (3.7) to operator $\Delta_p^{(a)}$, we obtain

$$\operatorname{spec}_+\Delta_p^{(a)}=\operatorname{spec}_+\mathcal{K}_p^{(a)}\sqcup\operatorname{spec}_+\mathcal{L}_p^{(a)}$$

and

$$\operatorname{spec}_+\Delta_p^{(a)}=\operatorname{spec}_+\mathcal{L}_p^{(a)}\sqcup\operatorname{spec}_+\mathcal{L}_{p+1}^{(a)}.$$

which together with Lemma 5.1 yields (5.5) and (5.6). ∎

Proposition 5.3. *For any $p\geq 0$, $\operatorname{spec}\Delta_p^{(a)}$ is determined by the sequence $\{\operatorname{spec}\Delta_q\}_{q=0}^p$ and the weight a. Conversely, $\operatorname{spec}\Delta_p$ is determined by the sequence $\left\{\operatorname{spec}\Delta_q^{(a)}\right\}_{q=0}^p$ and a.*

Proof. By (5.6), $\operatorname{spec}_+\Delta_p^{(a)}$ is determined by $\operatorname{spec}_+\mathcal{L}_p$ and $\operatorname{spec}_+\mathcal{L}_{p+1}$ (together with a). By Lemma 3.5, $\operatorname{spec}_+\mathcal{L}_p$ is determined by $\left\{\operatorname{spec}_+\Delta_q\right\}_{q=0}^{p-1}$, and similarly $\operatorname{spec}_+\mathcal{L}_{p+1}$ is determined by $\left\{\operatorname{spec}_+\Delta_q\right\}_{q=0}^{p}$. Hence, $\operatorname{spec}_+\Delta_p^{(a)}$ is determined by $\left\{\operatorname{spec}_+\Delta_q\right\}_{q=0}^{p}$.

It remains to handle 0 as an eigenvalue of $\Delta_p^{(a)}$. The multiplicity of 0 is equal to $\dim H_p$ and is independent of the weight a. Hence, the full spectrum $\operatorname{spec}\Delta_p^{(a)}$ is determined by the sequence $\{\operatorname{spec}\Delta_q\}_{q=0}^p$.

The second claim is proved in the same way. ∎

Corollary 5.4. *Assume that $\Omega_r = \{0\}$ for all $r > p$. Then* $\operatorname{spec}\Delta_p^{(a)}$ *is determined by the sequence* $\{\operatorname{spec}\Delta_q\}_{q=0}^{p-1}$, *the Euler characteristic $\chi(G)$ and the weight a.*

Proof. Since $\Omega_{p+1} = \{0\}$, by the argument from the previous proof $\operatorname{spec}_+\Delta_p^{(a)}$ is determined by $\operatorname{spec}_+\mathcal{L}_p$ and, hence, by $\left\{\operatorname{spec}_+\Delta_q\right\}_{q=0}^{p-1}$. It remains to observe that the multiplicity of 0 as an eigenvalue of $\Delta_p^{(a)}$, that is, $\dim H_p$, is determined by $\{\operatorname{spec}\Delta_q\}_{q=0}^{p-1}$ and χ, which follows from

$$\chi = \dim H_0 - \dim H_1 + \ldots + (-1)^p \dim H_{p-1} + (-1)^{p+1} \dim H_p.$$

■

6 Normalized Hodge Laplacian

Consider now the following specific weight

$$a_p = p!. \tag{6.1}$$

Definition 6.1. The inner product $\langle\cdot,\cdot\rangle_a$ with the weight (6.1) will be referred to as the *normalized* inner product, and the corresponding weighted Hodge Laplacian will be called the *normalized* Hodge Laplacian.

For example, for the normalized Hodge Laplacian from (5.5) and (5.6) that

$$\operatorname{spec}_+\Delta_p^{(a)} = (p+1)\operatorname{spec}_+\mathcal{K}_p \sqcup p\operatorname{spec}_+\mathcal{L}_p \tag{6.2}$$

and

$$\operatorname{spec}_+\Delta_p^{(a)} = p\operatorname{spec}_+\mathcal{L}_p \sqcup (p+1)\operatorname{spec}_+\mathcal{L}_{p+1}. \tag{6.3}$$

Besides, it follows from (5.4) that

$$\Delta_0^{(a)} = \Delta_0 \tag{6.4}$$

The significance of the weight (6.1) is determined by the following statement, where we use the Cartesian product $X\Box Y$ of two digraphs.

Proposition 6.2. [5, Lemma 4.7] *Let a be the weight (6.1). Let X and Y be two digraphs. Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ (where $p, q \geq 0$) and $r = p + q$,*

$$\Delta_r^{(a)}(u \times v) = \Delta_p^{(a)}u \times v + u \times \Delta_q^{(a)}v. \tag{6.5}$$

Note that the proof of (6.5) in [5] is based on the following ingredients:

1. the product rule (2.2) for ∂;
2. the Künneth formula (2.3);

3. if $u\in\mathcal{A}_p(X)$, $v\in\mathcal{A}_q(Y)$, $\varphi\in\mathcal{A}_{p'}(X)$ and $\psi\in\mathcal{A}_{q'}(Y)$ then
$$\langle u\times v,\varphi\times\psi\rangle=\binom{p+q}{p}\langle u,\varphi\rangle\langle v,\psi\rangle,$$
which implies for the weight (6.1) that
$$\langle u\times v,\varphi\times\psi\rangle_a=\langle u,\varphi\rangle_a\langle v,\psi\rangle_a.$$

The following statement is a combination of the argument of separation of variables, based on (6.5), and the Künneth formula (2.3).

Proposition 6.3. *For the weight (6.1) we have, for any $r\geq 0$,*
$$\operatorname{spec}\Delta_r^{(a)}(X\Box Y)=\bigsqcup_{\{p,q\geq 0:\, p+q=r\}}\left(\operatorname{spec}\Delta_p^{(a)}(X)+\operatorname{spec}\Delta_q^{(a)}(Y)\right). \tag{6.6}$$

Proof. Observe that if $u\in\Omega_p(X)$ and $v\in\Omega_q(Y)$ are eigenvectors such that
$$\Delta_p^{(a)}u=\lambda u\ \text{ and }\ \Delta_q^{(a)}v=\mu v,$$
then we have by (6.5) for $r=p+q$:
$$\Delta_r^{(a)}(u\times v)=(\Delta_p^{(a)}u)\times v+u\times\Delta_q^{(a)}v=(\lambda+\mu)(u\times v),$$
that is, $u\times v$ is an eigenvector of $\Delta_r^{(a)}$ on $X\Box Y$ with the eigenvalue $\lambda+\mu$.

In each $\Omega_p(X)$ there is a basis that consists of eigenvectors of $\Delta_p^{(a)}$; denote by $\{u_k\}$ the union of all such bases of $\Omega_p(X)$ across all $p\geq 0$, with the corresponding eigenvalues $\{\lambda_k\}$. Let $\{v_l\}$ be a similar sequence on Y with the eigenvalues $\{\mu_l\}$. By the Künneth formula (2.3), we have, for any $r\geq 0$,
$$\Omega_r(X\Box Y)\cong\bigoplus_{\{p,q\geq 0: p+q=r\}}\left(\Omega_p(X)\otimes\Omega_q(Y)\right), \tag{6.7}$$
where the isomorphism is given by $u\otimes v\mapsto u\times v$. It follows that $\Omega_r(X\Box Y)$ has a basis
$$\{u_k\times v_l:|u_k|+|v_l|=r\},$$
where $|\cdot|$ denotes here the length of paths. The elements of this basis are the eigenvectors of $\Delta_r^{(a)}$ on $X\Box Y$ with eigenvalues $\lambda_k+\mu_l$, whence (9.5) follows. ∎

7 Hodge Laplacian on Cartesian products

7.1 Spectrum of the normalized Hodge Laplacian on G^n

Let $G=(V,E)$ be any finite connected digraph and a be the weight (6.1). Assuming that $\operatorname{spec}\Delta_p^{(a)}(G)$ is known for all $p\geq 0$, the spectrum $\operatorname{spec}\Delta_p^{(a)}(G^n)$ can be determined by induction using Proposition 6.3. We provide here an explicit expression for $\operatorname{spec}\Delta_p^{(a)}(G^n)$ in the case when the spaces $\Omega_p(G)$ are trivial for $p>1$. It follows from the Künneth formula 6.7 that $\Omega_r(G^n)=\{0\}$ for $r>n$.

In Theorem 7.2 below we obtain formulas for $\operatorname{spec}\Delta_r^{(a)}(G^n)$ for all $0\leq r\leq n$ using $\operatorname{spec}\Delta_0(G)$ only. We start with a lemma.

Lemma 7.1. *Assume that $\Omega_p(G) = \{0\}$ for all $p \geq 2$. Then, for all $0 \leq r \leq n$, we have*

$$\operatorname{spec} \Delta_r^{(a)}(G^n) = \Big\{ (\alpha_1 + \cdots + \alpha_{n-r} + \beta_1 + \cdots + \beta_r)_{\binom{n}{r}} \,\Big|\, \alpha_i \in \operatorname{spec} \Delta_0^{(a)}(G),\ \beta_j \in \operatorname{spec} \Delta_1^{(a)}(G) \Big\}. \tag{7.1}$$

Proof. This identity is trivial for $n = 1$. For the induction step from $n-1$ to n, we obtain, using (6.6) and the induction hypothesis, that

$$\begin{aligned}
\operatorname{spec} \Delta_r^{(a)}(G^n) &= \bigsqcup_{\{p,q \geq 0 :\, p+q=r\}} \left(\operatorname{spec} \Delta_p^{(a)}(G^{n-1}) + \operatorname{spec} \Delta_q^{(a)}(G) \right) \\
&= \left(\operatorname{spec} \Delta_r^{(a)}(G^{n-1}) + \operatorname{spec} \Delta_0^{(a)}(G) \right) \qquad (p = r, q = 0) \\
&\quad \sqcup \left(\operatorname{spec} \Delta_{r-1}^{(a)}(G^{n-1}) + \operatorname{spec} \Delta_1^{(a)}(G) \right) \qquad (p = r-1, q = 1) \\
&= \{ (\alpha_1 + \cdots + \alpha_{n-r-1} + \beta_1 + \cdots + \beta_r)_{\binom{n-1}{r}} + \alpha_{n-r} \} \\
&\quad \sqcup \{ (\alpha_1 + \cdots + \alpha_{(n-1)-(r-1)} + \beta_1 + \cdots + \beta_{r-1})_{\binom{n-1}{r-1}} + \beta_r \} \\
&= \{ (\alpha_1 + \cdots + \alpha_{n-r} + \beta_1 + \cdots + \beta_r)_{\binom{n}{r}} \},
\end{aligned}$$

where $\alpha_i \in \operatorname{spec} \Delta_0^{(a)}(G)$ and $\beta_j \in \operatorname{spec} \Delta_1^{(a)}(G)$. ■

Here is our main result about the spectrum of the normalized Hodge Laplacian.

Theorem 7.2. *Assume that G is a connected digraph containing neither double arrow, nor triangle nor square. Let $\lambda_1, \ldots, \lambda_s$ be all distinct positive eigenvalues of $\Delta_0(G)$, $0 < \lambda_1 < \ldots < \lambda_s$, and let $m_1, \ldots, m_s$ be their multiplicities.*

(a) Assume that G has v vertices and v arrows, where $v \geq 3$. Then, for all $0 \leq r \leq n$, we have

$$\operatorname{spec} \Delta_r^{(a)}(G^n) = \left\{ (k_1\lambda_1 + \ldots + k_s\lambda_s)_{m_1^{k_1} \ldots m_s^{k_s} \binom{n}{k_1 \ \ldots \ k_s} \binom{n}{r}} \,\middle|\, k_1, \ldots, k_s \geq 0,\ k_1 + \ldots + k_s \leq n \right\}, \tag{7.2}$$

where $\binom{n}{k_1 \ \ldots \ k_s}$ is a multinomial coefficient (see (7.8) below)*. Consequently,*

$$\lambda_{\max}(\Delta_r^{(a)}(G^n)) = (n\lambda_s)_{m_s^n \binom{n}{r}} \quad \text{and} \quad \lambda_{\min}(\Delta_r^{(a)}(G^n)) = 0_{\binom{n}{r}}.$$

(b) Assume that G has v vertices and $v-1$ arrows, where $v \geq 2$. Then, for all $0 \leq r \leq n$, we have

$$\operatorname{spec} \Delta_r^{(a)}(G^n) = \left\{ (k_1\lambda_1 + \ldots + k_s\lambda_s)_{m_1^{k_1} \ldots m_s^{k_s} \binom{n}{k_1 \ \ldots \ k_s} \binom{k_1 + \ldots + k_s}{r}} \,\middle|\, k_1, \ldots, k_s \geq 0,\ r \leq k_1 + \ldots + k_s \leq n \right\}. \tag{7.3}$$

Consequently,

$$\lambda_{\max}(\Delta_r^{(a)}(G^n)) = (n\lambda_s)_{m_s^n \binom{n}{r}} \quad \text{and} \quad \lambda_{\min}(\Delta_r^{(a)}(G^n)) = \{r\lambda_1\}_{m_1^r \binom{n}{r}}.$$

Proof. By (6.4) we have

$$\Delta_0^{(a)}(G) = \Delta_0(G)$$

so that we can replace in (7.1) $\operatorname{spec}\Delta_0^{(a)}(G)$ by $\operatorname{spec}\Delta_0(G)$.

Let us handle $\operatorname{spec}\Delta_1^{(a)}(G)$ in (7.1). By Proposition 2.3 we have $\Omega_p(G) = \{0\}$ for all $p \geq 2$. Since $\Omega_2(G) = \{0\}$ we have by Example 3.4 that

$$\operatorname{spec}_+ \Delta_1^{(a)}(G) = \operatorname{spec}_+ \Delta_0^{(a)}(G). \tag{7.4}$$

Since G is connected, we have $\dim H_0(G) = 1$, so that the eigenvalue 0 of $\Delta_0^{(a)}(G)$ has the multiplicity 1. The multiplicity of 0 as the eigenvalue of $\Delta_1^{(a)}(G)$ (and $\Delta_1(G)$) is equal to $\dim H_1(G)$. To determine it, we use the Euler characteristic χ of the chain complex Ω_* that in this case satisfies

$$\chi = \dim \Omega_0 - \dim \Omega_1 = \dim H_0 - \dim H_1.$$

It follows that

$$\dim H_1(G) = 1 - |V| + |E|. \tag{7.5}$$

(a) Since $|V| = |E| = v$, we obtain from (7.5) $\dim H_1 = 1$. Hence, 0 has the same multiplicity 1 in $\operatorname{spec}\Delta_1^{(a)}(G)$ and $\operatorname{spec}\Delta_0^{(a)}(G)$. Combining with (7.4) we obtain

$$\operatorname{spec}\Delta_1^{(a)}(G) = \operatorname{spec}\Delta_0^{(a)}(G) = \operatorname{spec}\Delta_0(G).$$

Therefore, we can replace in (7.1) the both spectra $\operatorname{spec}\Delta_0^{(a)}(G)$ and $\operatorname{spec}\Delta_1^{(a)}(G)$ by $\operatorname{spec}\Delta_0(G)$, so that

$$\operatorname{spec}\Delta_r^{(a)}(G^n) = \{(\alpha_1 + \cdots + \alpha_n)_{\binom{n}{r}} \mid \alpha_i \in \operatorname{spec}\Delta_0(G)\}. \tag{7.6}$$

Let k_l be the number of times the eigenvalue λ_l occurs in the sequence $\alpha_1, ..., \alpha_n$. Then

$$k_1 + ... + k_s \leq n$$

and

$$\alpha_1 + ... + \alpha_n = k_1\lambda_1 + ... + k_s\lambda_s. \tag{7.7}$$

The numbers of ways of inserting k_l values λ_l in the sequence $\alpha_1, ..., \alpha_n$ for all $l = 1, ..., n$ is equal to

$$\begin{pmatrix} & n & \\ k_1 & ... & k_s \end{pmatrix} = \frac{n!}{k_1!...k_s!(n - k_1 - ... - k_s)!} \tag{7.8}$$

(and the rest $n - k_1 - ... - k_s$ values are 0). Besides, within k_l already fixed positions of λ_l, there are $m_l^{k_l}$ ways of selecting this λ_l from m_l instances of λ_l in $\operatorname{spec}\Delta_0(G)$. Hence, the number of sequences $\alpha_1, ..., \alpha_n$ where λ_l occurs k_l times, is equal to

$$m_1^{k_1}...m_s^{k_s}\begin{pmatrix} & n & \\ k_1 & ... & k_s \end{pmatrix}.$$

It follows from (7.6) that $k_1\lambda_1 + ... + k_s\lambda_s$ has the multiplicity

$$m_1^{k_1}...m_s^{k_s}\begin{pmatrix} & n & \\ k_1 & ... & k_s \end{pmatrix}\binom{n}{r},$$

which finishes the proof of (7.2).

(b) Since $|V| = v$ and $|E| = v - 1$, we obtain from (7.5) $\dim H_1 = 0$. Hence, $0 \notin \Delta_1^{(a)}(G)$, and it follows from (7.4)

$$\operatorname{spec} \Delta_1^{(a)}(G) = \operatorname{spec}_+ \Delta_0^{(a)}(G) = \operatorname{spec} \Delta_0(G) \setminus \{0\} .$$

Substituting into (7.1) we obtain

$$\operatorname{spec} \Delta_r^{(a)}(G^n) = \Big\{ (\alpha_1 + \cdots + \alpha_{n-r} + \beta_1 + \cdots + \beta_r)_{\binom{n}{r}} \Big| \\ \alpha_i \in \operatorname{spec} \Delta_0(G),\ \beta_j \in \operatorname{spec} \Delta_0(G) \setminus \{0\} \Big\} . \tag{7.9}$$

Let λ_l occur k_l times in the sequence $\{\alpha_i, \beta_j\}$ so that

$$\alpha_1 + \cdots + \alpha_{n-r} + \beta_1 + \cdots + \beta_r = k_1\lambda_1 + ... + k_s\lambda_s.$$

Then there are $n - k_1 - ... - k_s$ values 0 in the sequence $\{\alpha_i, \beta_j\}$, and they may be chosen only within $\{\alpha_i\}$; hence, there are

$$\binom{n-r}{n-k_1-...-k_s} = \binom{n-r}{k_1+...+k_s-r}$$

ways of choosing positions of 0 in $\{\alpha_i, \beta_j\}$. In the remaining $k_1 + ... + k_s$ positions we can place k_l times λ_l in

$$\binom{k_1+...+k_s}{k_1 \ \ ... \ \ k_s}$$

ways. Taking into account the multiplicities m_l as in (a), we obtain that number of sequences $\alpha_1, ..., \alpha_n$ as above is equal to

$$m_1^{k_1}...m_s^{k_s}\binom{n-r}{k_1+...+k_s-r}\binom{k_1+...+k_s}{k_1 \ \ ... \ \ k_s},$$

which gives the multiplicity of $k_1\lambda_1 + ... + k_s\lambda_s$ as

$$m_1^{k_1}...m_s^{k_s}\binom{n-r}{k_1+...+k_s-r}\binom{k_1+...+k_s}{k_1 \ \ ... \ \ k_s}\binom{n}{r}.$$

Setting $k = k_1 + ... + k_s$, let us verify that

$$\binom{n-r}{k-r}\binom{k}{k_1 \ \ ... \ \ k_s}\binom{n}{r} = \binom{n}{k_1 \ ... \ k_s}\binom{k}{r},$$

which will conclude the proof of (7.3). Indeed, we have

$$\begin{aligned}\binom{n-r}{k-r}\binom{k}{k_1 \ \ ... \ \ k_s}\binom{n}{r} &= \frac{(n-r)!}{(n-k)!\,(k-r)!}\frac{k!}{k_1!...k_s!}\frac{n!}{(n-r)!r!} \\ &= \frac{n!}{(n-k)!k_1!...k_s!}\frac{k!}{(k-r)!r!} = \binom{n}{k_1 \ ... \ k_s}\binom{k}{r},\end{aligned}$$

which was claimed. ∎

In the next examples we compute the spectra $\operatorname{spec}\Delta_r^{(a)}(G^n)$ for six small digraphs G shown on Fig. 7, using Theorem 7.2 (a) and (b).

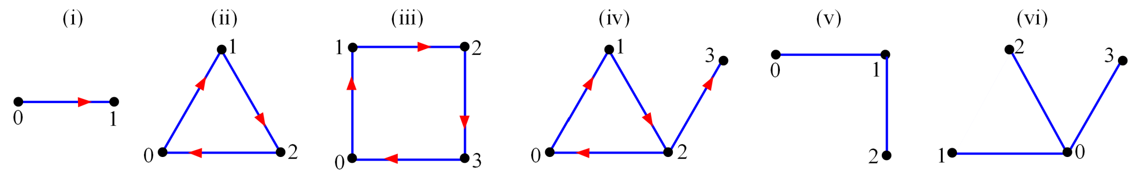

Figure 7: Some examples of digraphs

Example 7.3. Case (i). For the interval $G = I = \{0 \to 1\}$ the hypotheses of (b) are satisfied with $v = 2$. The matrix of $\Delta_0(I)$ is

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and $\operatorname{spec}\Delta_0(I) = \{0, 2\}$. Hence, we have only one positive eigenvalue $\lambda_1 = 2$ with multiplicity $m_1 = 1$. It follows from (7.3) that

$$\operatorname{spec}\Delta_r^{(a)}(I^n) = \left\{(2k)_{\binom{n}{k}\binom{k}{r}}\right\}_{k=r}^{n}. \tag{7.10}$$

In particular, have

$$\lambda_{\max}(\Delta_r^{(a)}(I^n)) = (2n)_{\binom{n}{r}} \quad \text{and} \quad \lambda_{\min}(\Delta_r^{(a)}(I^n)) = (2r)_{\binom{n}{r}}.$$

Example 7.4. Case (ii). For the torus $G = T = \{0 \to 1 \to 2 \to 0\}$ the hypotheses of (a) are satisfied with $v = 3$. The matrix of $\Delta_0(T)$ is

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

whence $\operatorname{spec}\Delta_0(T) = \{0, 3, 3\}$. Hence, we have only one positive eigenvalue $\lambda_1 = 3$ with multiplicity $m_1 = 2$. It follows from (7.2) that

$$\operatorname{spec}\Delta_r^{(a)}(T^n) = \left\{(3k)_{2^k\binom{n}{k}\binom{n}{r}}\right\}_{k=0}^{n}. \tag{7.11}$$

Clearly, we have

$$\lambda_{\max}(\Delta_r^{(a)}(T^n)) = (3n)_{2^n\binom{n}{r}} \quad \text{and} \quad \lambda_{\min}(\Delta_r^{(a)}(T^n)) = 0_{\binom{n}{r}}.$$

Example 7.5. Case (iii). Let $G = \{0 \to 1 \to 2 \to 3 \to 0\}$ so that the hypotheses of (a) are satisfied with $v = 4$. Then the matrix of $\Delta_0(G)$ is

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \tag{7.12}$$

whence $\operatorname{spec}\Delta_0(G)=\{0,2,2,4\}$ .Setting $\lambda_1=2,\ m_1=2$ and $\lambda_2=4,\ m_2=1$, we obtain by (7.2)

$$\operatorname{spec}\Delta_r^{(a)}(G^n)=\left\{(2k_1+4k_2)_{2^{k_1}\binom{n}{k_1\ k_2}\binom{n}{r}}\right\}_{0\le k_1+k_2\le n}.$$

It follows that

$$\lambda_{\max}(\Delta_r^{(a)}(G^n))=(4n)_{\binom{n}{r}}\quad\text{and}\quad\lambda_{\min}(\Delta_r^{(a)}(G^n))=0_{\binom{n}{r}}.$$

Example 7.6. Case (iv). Let $G=\{0\to1\to2\to0\to3\}$ so that G satisfies the hypothesis (a) with $v=4$. The matrix of $\Delta_0(G)$ is

$$\begin{pmatrix}3&-1&-1&-1\\-1&2&-1&0\\-1&-1&2&0\\-1&0&0&1\end{pmatrix},$$

the eigenvalues are $\{0,1,3,4\}$. We obtain by (7.2)

$$\operatorname{spec}\Delta_r^{(a)}(G^n)=\left\{(k_1+3k_2+4k_3)_{\binom{n}{k_1\,k_2\ k_3}\binom{n}{r}}\ \right\}_{0\le k_1+k_2+k_2\le n}.$$

Example 7.7. Case (v). Let $G=\{0\sim1\sim2\}$ where the orientation of edges is arbitrary. Then the hypotheses of (b) are satisfied with $v=3$. The matrix of $\Delta_0(G)$ is

$$\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix},$$

and the eigenvalues are $\{0,1,3\}$. Setting in (7.3) $\lambda_1=1,\ \lambda_2=3$ and $m_1=m_2=1$, we obtain

$$\operatorname{spec}\Delta_r^{(a)}(G^n)=\left\{(k_1+3k_2)_{\binom{n}{k_1\ k_2}\binom{k_1+k_2}{r}}\right\}_{r\le k_1+k_2\le n}.$$

In particular,

$$\lambda_{\max}(\Delta_r^{(a)}(G^n))=(3n)_{\binom{n}{r}}\quad\text{and}\quad\lambda_{\min}(\Delta_r^{(a)}(G^n))=r_{\binom{n}{r}}.$$

Example 7.8. Case (vi). Let $G=\{0\sim1,0\sim2,0\sim3\}$ where the orientation of edges is arbitrary. Then the hypotheses of (b) are satisfied with $v=4$. The matrix of $\Delta_0(G)$ is

$$\begin{pmatrix}3&-1&-1&-1\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{pmatrix},$$

and the eigenvalues of $\Delta_0(G)$ are $\{0,1,1,4\}$. Setting in (7.3) $\lambda_1=1,\ \lambda_2=4,\ m_1=2$ and $m_2=1$, we obtain

$$\operatorname{spec}\Delta_r^{(a)}(G^n)=\left\{(k_1+4k_2)_{2^{k_1}\binom{n}{k_1\ k_2}\binom{k_1+.k_2}{r}}\right\}_{r\le k_1+k_2\le n}.$$

In particular,

$$\lambda_{\max}(\Delta_r^{(a)}(G^n))=(4n)_{\binom{n}{r}}\quad\text{and}\quad\lambda_{\min}(\Delta_r^{(a)}(G^n))=\{r\}_{2^r\binom{n}{r}}.$$

7.2 Spectrum of the canonical Hodge Laplacian on G^n

Let us define the *Hodge spectrum* of a digraph G as a sequence

$$\operatorname{spec} G := \{\operatorname{spec} \Delta_p(G)\}_{p=0}^{\infty},$$

where Δ_p is the canonical Hodge Laplacian, as well as the *normalized Hodge spectrum* by

$$\operatorname{spec}^{(a)} G := \left\{\operatorname{spec} \Delta_p^{(a)}(G)\right\}_{p=0}^{\infty},$$

where a is the weight (6.1) and, hence, $\Delta_p^{(a)}$ is the normalized Hodge Laplacian.

Theorem 7.9. *For any finite digraph G and any $n \geq 1$, the following is true.*

(*a*) $\operatorname{spec} G^n$ *is determined by* $\operatorname{spec}^{(a)} G$.

(*b*) $\operatorname{spec} G^n$ *is determined by* $\operatorname{spec} G$.

(*c*) *If $\Omega_r(G) = \{0\}$ for some $r \geq 2$ then $\operatorname{spec} G^n$ is determined by the Euler characteristic $\chi(G)$ and the sequence*

$$\{\operatorname{spec} \Delta_q(G)\}_{q=0}^{r-2}. \tag{7.13}$$

Proof. (*a*) Applying Lemma 3.5 to operators $\Delta_p^{(a)}$ on G^n, we obtain that $\operatorname{spec}_+ \mathcal{L}_p^{(a)}(G^n)$ is determined by the sequence

$$\left\{\operatorname{spec}_+ \Delta_q^{(a)}(G^n)\right\}_{q=0}^{p-1}$$

and, hence, by $\operatorname{spec}^{(a)} G^n$. Since by (5.2)

$$\mathcal{L}_p^{(a)} = p\mathcal{L}_p,$$

we see that $\operatorname{spec}_+ \mathcal{L}_p(G^n)$ is also determined by $\operatorname{spec}^{(a)} G^n$.

On the other hand, as it follows from Proposition 6.3, $\operatorname{spec}^{(a)} G^n$ is determined by $\operatorname{spec}^{(a)} G$. Hence, $\operatorname{spec}_+ \mathcal{L}_p(G^n)$ is determined by $\operatorname{spec}^{(a)} G$.

By (3.7) we have

$$\operatorname{spec}_+ \Delta_p(G^n) = \operatorname{spec}_+ \mathcal{L}_p(G^n) \sqcup \operatorname{spec}_+ \mathcal{L}_{p+1}(G^n),$$

which implies that $\operatorname{spec}_+ \Delta_p(G^n)$ is also determined by $\operatorname{spec}^{(a)} G$.

Finally, the multiplicity of 0 as an eigenvalue of $\Delta_p(G^n)$ is equal to $\dim H_p(G^n)$, which by the Künneth formula (2.4) is determined by $\{\dim H_p(G)\}_{p\geq 0}$ and, hence, by $\operatorname{spec}^{(a)} G$. Hence, the spectrum $\operatorname{spec} \Delta_p(G^n)$ is determined by $\operatorname{spec}^{(a)} G$, which finishes the proof.

(*b*) This follows from (*a*) and Proposition 5.3.

(*c*) By Corollary 5.4, for any $p < r$, $\operatorname{spec} \Delta_p^{(a)}(G)$ is determined by $\chi(G)$ and the sequence $\{\operatorname{spec} \Delta_q(G)\}_{q=0}^{p-1}$ and, hence, by $\{\operatorname{spec} \Delta_q(G)\}_{q=0}^{r-2}$. Hence, the rest follows from (*a*). ∎

The argument in the proof of Theorem 7.9 theoretically allows to compute $\operatorname{spec} G^n$ knowing $\operatorname{spec}^{(a)} G$. However, practically it is more convenient to compute first $\operatorname{spec} \Delta_p^{(a)}(G^n)$ by means of Proposition 6.3 or Theorem 7.2, and then compute $\operatorname{spec} \mathcal{L}_p(G^n)$ and $\operatorname{spec} \Delta_p(G^n)$ using the above argument. In the next section we apply this approach in order to compute the Hodge spectrum for n-cube and n-torus.

7.3 Isospectral digraphs

We say that two digraphs G and $G'$are *Hodge isospectral* if $\operatorname{spec} G = \operatorname{spec} G'$. A natural question in the spirit of inverse spectral problems is whether Hodge isospectral digraphs are isomorphic. We show here that in general the answer is "no".

Let $\overline{G} = (V, E)$ be a connected undirected graph such that $\overline{G}$ contains neither 3-cycles nor 4-cycles. For example, $\overline{G}$ can be a polygon with $n \geq 5$ sides.

Let G be a digraph that is obtained from $\overline{G}$ by assigning arbitrarily orientation on each edge thus turning it into an arrow. Then $\Omega_2(G) = \{0\}$, and, by Theorem 7.9(c) with $r = 2$, we obtain that $\operatorname{spec} G^n$ is determined by $\operatorname{spec} \Delta_0$ and the Euler characteristic $\chi = |V| - |E|$. Since both Δ_0 and χ are independent of orientation of the edges (cf. (4.3)), we conclude that also $\operatorname{spec} G^n$ is independent of orientation of the edges.

Hence, for any two digraphs G_1 and G_2 that are obtained by assigning orientation of edges in $\overline{G}$, the digraphs G_1^n and G_2^n are Hodge isospectral for any $n \geq 1$, but obviously they do not have to be isomorphic.

However, G_1^n and G_2^n are still isomorphic as undirected graphs.

Problem. Is it true that Hodge isospectral digraphs are isomorphic as undirected graphs?

8 Spectrum of the Hodge Laplacian on cubes and tori

Recall that the n-cube I^n and n-torus T^n were defined in Examples 2.7 and 2.8, respectively. In this section we compute the Hodge spectra of I^n and T^n. Since for both digraphs $G = I$ and $G = T$ we have $\Omega_2(G) = \{0\}$, the space $\Omega_p(G^n)$ is trivial for $p > n$ and, hence, it suffices to compute $\operatorname{spec} \Delta_p(G^n)$ for $p \leq n$. For $p = 0$ we obtain by (7.10)

$$\operatorname{spec} \Delta_0(I^n) = \operatorname{spec} \Delta_0^{(a)}(I^n) = \left\{ (2k)_{\binom{n}{k}} \right\}_{k=0}^{n} \tag{8.1}$$

and by (7.11)

$$\operatorname{spec} \Delta_0(T^n) = \operatorname{spec} \Delta_0^{(a)}(T^n) = \left\{ (3k)_{2^k \binom{n}{k}} \right\}_{k=0}^{n}. \tag{8.2}$$

Hence, in what follows we restrict ourselves to $1 \leq p \leq n$.

Now we state and prove the main result about $\operatorname{spec} I^n$.

Theorem 8.1. *For all $1 \leq p \leq n$ we have*

$$\operatorname{spec} \Delta_p(I^n) = \left\{ \left(\frac{2k}{p} \right)_{\binom{n}{k}\binom{k-1}{p-1}} \right\}_{k=p}^{n} \sqcup \left\{ \left(\frac{2k}{p+1} \right)_{\binom{n}{k}\binom{k-1}{p}} \right\}_{k=p+1}^{n}. \tag{8.3}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(I^n)\right) = \left(\frac{2n}{p} \right)_{\binom{n-1}{p-1}} \quad \textit{and} \quad \lambda_{\min}\left(\Delta_p(I^n)\right) = 2_{\binom{n+1}{p+1}}$$

Proof. We start with computation of the spectrum of $\mathcal{L}_p(I^n)$. Namely, let us first prove that, for $1 \le p \le n$,

$$\operatorname{spec}_+ \mathcal{L}_p(I^n) = \left\{ \left(\frac{2k}{p}\right)_{\binom{n}{k}\binom{k-1}{p-1}} \right\}_{k=p}^{n}. \tag{8.4}$$

It follows from (8.1) that

$$\operatorname{spec}_+ \mathcal{L}_1(I^n) = \operatorname{spec}_+ \Delta_0(I^n) = \left\{ (2k)_{\binom{n}{k}} \right\}_{k=1}^{n},$$

which matches (8.4) for $p = 1$ and provides for the induction basis.

For the induction step from p to $p+1$, observe that, by (6.3),

$$\operatorname{spec}_+ \Delta_p^{(a)} = p \operatorname{spec}_+ \mathcal{L}_p \sqcup (p+1) \operatorname{spec}_+ \mathcal{L}_{p+1}, \tag{8.5}$$

whence

$$(p+1) \operatorname{spec}_+ \mathcal{L}_{p+1} = \operatorname{spec}_+ \Delta_p^{(a)} \setminus p \operatorname{spec}_+ \mathcal{L}_p. \tag{8.6}$$

By (7.10) we have

$$\operatorname{spec} \Delta_p^{(a)}(I^n) = \left\{ (2k)_{\binom{n}{k}\binom{k}{p}} \right\}_{k=p}^{n}. \tag{8.7}$$

Using also the induction hypothesis, we obtain

$$\begin{aligned} (p+1) \operatorname{spec}_+ \mathcal{L}_{p+1}(I^n) &= \left\{ (2k)_{\binom{n}{k}\binom{k}{p}} \right\}_{k=p}^{n} \setminus p \left\{ \left(\frac{2k}{p}\right)_{\binom{n}{k}\binom{k-1}{p-1}} \right\}_{k=p}^{n} \\ &= \left\{ (2k)_{\binom{n}{k}\binom{k-1}{p}} \right\}_{k=p}^{n}. \end{aligned}$$

It follows that

$$\operatorname{spec}_+ \mathcal{L}_{p+1} = \left\{ \left(\frac{2k}{p+1}\right)_{\binom{n}{k}\binom{k-1}{p}} \right\}_{k=p+1}^{n},$$

(where now k starts from $p+1$ because $\binom{k-1}{p} = 0$ for $k = p$), which concludes the induction step.

Now we can prove (8.3). Since I is homologically trivial, also all cubes $G = I^n$ are homologically trivial, which implies that, for all $1 \le p \le n$,

$$\operatorname{spec} \Delta_p(I^n) > 0.$$

Hence, by (3.7) we have

$$\operatorname{spec} \Delta_p(I^n) = \operatorname{spec}_+ \mathcal{L}_p(I^n) \sqcup \operatorname{spec}_+ \mathcal{L}_{p+1}(I^n).$$

Substituting $\operatorname{spec}_+ \mathcal{L}_p(I^n)$ and $\operatorname{spec}_+ \mathcal{L}_{p+1}(I^n)$ from (8.4), we obtain (8.3).

The maximal eigenvalue $\lambda_{\max} = \frac{2n}{p}$ comes from the first of two series in (8.3) for $k = n$, with the multiplicity $\binom{n}{n}\binom{n-1}{p-1} = \binom{n-1}{p-1}$. The minimal eigenvalue $\lambda_{\min} = 2$ comes from the both series, with $k = p$ and $k = p+1$, respectively, and its multiplicity is

$$\binom{n}{p}\binom{p-1}{p-1} + \binom{n}{p+1}\binom{p}{p} = \binom{n+1}{p+1}.$$

■

For example, in the case $p=1$ we obtain

$$\operatorname{spec}\Delta_1(I^n)=\left\{(2k)_{\binom{n}{k}}\right\}_{k=1}^{n}\sqcup\left\{k_{(k-1)\binom{n}{k}}\right\}_{k=2}^{n}, \tag{8.8}$$

$$\lambda_{\min}\left(\Delta_1(I^n)\right)=2_{\binom{n}{2}}\quad\text{and}\quad\lambda_{\max}\left(\Delta_1(I^n)\right)=(2n)_1\,,$$

which solves Problem 6.26 in [1].

Similarly, we can compute the Hodge spectrum of n-torus T^n.

Theorem 8.2. *For all $1\le p\le n$ we have*

$$\operatorname{spec}\Delta_p(T^n)=\left\{\left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}}\right\}_{k=0}^{n}\sqcup\left\{\left(\frac{3k}{p+1}\right)_{2^k\binom{n}{k}\binom{n-1}{p}}\right\}_{k=0}^{n}. \tag{8.9}$$

In particular,

$$\lambda_{\max}\left(\Delta_p(T^n)\right)=\left(\frac{3n}{p}\right)_{2^n\binom{n-1}{p-1}}\quad\textit{and}\quad\lambda_{\min}\left(\Delta_p(T^n)\right)=0_{\binom{n}{p}}.$$

Proof. Let us first prove that, for all $1\le p\le n$,

$$\operatorname{spec}_+\mathcal{L}_p(T^n)=\left\{\left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}}\right\}_{k=1}^{n}. \tag{8.10}$$

The induction basis for $p=1$ is given by (8.2):

$$\operatorname{spec}_+\mathcal{L}_1(T^n)=\operatorname{spec}_+\Delta_0(T^n)=\left\{(3k)_{2^k\binom{n}{k}}\right\}_{k=1}^{n}.$$

For the induction step from p to $p+1$, we use (8.6), the induction hypothesis, and (7.11) in the form

$$\operatorname{spec}_+\Delta_p^{(a)}(T^n)=\left\{(3k)_{2^k\binom{n}{k}\binom{n}{p}}\right\}_{k=1}^{n},$$

which yield

$$\begin{aligned}(p+1)\operatorname{spec}_+\mathcal{L}_{p+1}(T^n)&=\operatorname{spec}_+\Delta_p^{(a)}(T^n)\setminus p\operatorname{spec}_+\mathcal{L}_p(T^n)\\&=\left\{(3k)_{2^k\binom{n}{k}\binom{n}{p}}\right\}_{k=1}^{n}\setminus p\left\{\left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}}\right\}_{k=1}^{n}\\&=\left\{(3k)_{2^k\binom{n}{k}\binom{n-1}{p}}\right\}_{k=1}^{n},\end{aligned}$$

thus concluding the proof of (8.10).

Now we can prove (8.9). Indeed, by (3.7) and (8.10) we obtain

$$\operatorname{spec}_+\Delta_p(T^n)\quad=\quad\operatorname{spec}_+\mathcal{L}_p(T^n)\sqcup\operatorname{spec}_+\mathcal{L}_{p+1}(T^n)$$

$$= \left\{ \left(\frac{3k}{p}\right)_{2^k\binom{n}{k}\binom{n-1}{p-1}} \right\}_{k=1}^{n} \sqcup \left\{ \left(\frac{3k}{p+1}\right)_{2^k\binom{n}{k}\binom{n-1}{p}} \right\}_{k=1}^{n},$$

which matches the positive part of $\operatorname{spec}\Delta_p(T^n)$ in (8).

It remains to verify that 0 is the eigenvalue of $\Delta_p(T^n)$ with multiplicity

$$2^k\binom{n}{k}\binom{n-1}{p-1} + 2^k\binom{n}{k}\binom{n-1}{p}\bigg|_{k=0} = \binom{n}{p},$$

which amounts to

$$\dim H_p(T^n) = \binom{n}{p}. \tag{8.11}$$

Indeed, since

$$\dim H_0(T) = \dim H_1(T) = 1,$$

(8.11) follows from (2.4) by induction in n. ■

For example, for $p = 1$ we obtain

$$\operatorname{spec}\Delta_1(T^n) = \left\{ (3k)_{2^k\binom{n}{k}} \right\}_{k=0}^{n} \sqcup \left\{ \left(\frac{3k}{2}\right)_{2^k\binom{n}{k}(n-1)} \right\}_{k=0}^{n}.$$

9 Spectrum of the Hodge Laplacian on joins

In this section we use the augmented chain complex on a digraph G:

$$0 \leftarrow \mathbb{R} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{9.1}$$

where the boundary operator $\partial : \Omega_0 \to \Omega_{-1} := \mathbb{R}$ is now redefined[1] by $\partial e_i = e$ where e is the unity of $\mathbb{R}$.

In this section we use the canonical inner product $\langle \cdot, \cdot \rangle$ on each Ω_p as in (4.1). Denote by $\widetilde{\Delta}_p$ the Hodge Laplacian associated with the chain complex (9.1). Of course, $\widetilde{\Delta}_p$ coincides with Δ_p for $p \geq 1$ but is different for $p = -1$ and $p = 0$. The advantage of using the chain complex (9.1) is that the operator $\widetilde{\Delta}_p$ satisfies the product rule with respect to the operation *join* of paths.

Let us briefly recall this notion based on [1], [2], [5], [6].

For any two digraphs X and Y, define their join as the digraph $Z = X * Y$ whose set of vertices is a disjoint union of the set of vertices of X and Y, and the set of arrows consists of all arrows in X and Y as well as of all arrows $x \to y$ where $x \in X$ and $y \in Y$.

For any elementary paths $u = e_{i_0 \dots i_p}$ on X and $v = e_{j_0 \dots j_q}$ on Y, define their join $u * v$ as a path on Z by

$$u * v = e_{i_0 \dots i_p j_0 \dots j_q}.$$

[1] Recall that in Section 2 we defined $\partial e_i = 0$.

Observe that if u and v are allowed then $u * v$ is also allowed because of the presence of the arrow $i_p \to j_0$. Note also that the length of $u * v$ is $p + q + 1$. Using linearity, this definition of $u * v$ extends to all regular paths u on X and v on Y.

The operator ∂ of the augmented chain complex (9.1) satisfies the product rule: if $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with $p, q \geq -1$ then

$$\partial(u * v) = \partial u * v + (-1)^{p+1} u * \partial v. \tag{9.2}$$

It implies that if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u * v \in \Omega_{p+q+1}(Z)$.

The augmented chain complex (9.1) satisfies the Künneth formula with respect to join: for all $r \geq -1$

$$\Omega_r(X * Y) \cong \bigoplus_{\{p,q \geq -1:\, p+q+1=r\}} (\Omega_p(X) \otimes \Omega_q(Y)), \tag{9.3}$$

where the isomorphism is given by $u \otimes v \mapsto u * v$.

It is easy to see that, $u \in \mathcal{A}_p(X)$, $v \in \mathcal{A}_q(Y)$, $\varphi \in \mathcal{A}_{p'}(X)$ and $\psi \in \mathcal{A}_{q'}(Y)$ then

$$\langle u * v, \varphi * \psi \rangle = \langle u, \varphi \rangle \langle v, \psi \rangle.$$

This together with (9.2) and (9.3) allows to prove the product rule for $\widetilde{\Delta}_p$ with respect to join, as in the following statement.

Proposition 9.1. [5, Lemma 5.5] *Let X, Y be two digraphs. Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ with $p.q \geq -1$ we have*

$$\widetilde{\Delta}_r(u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v, \tag{9.4}$$

where $r = p + q + 1$.

Combining (9.4) with (9.3) as in the proof of Proposition 6.3, we obtain the following.

Proposition 9.2. [1, Theorem 6.36] *We have for any $r \geq -1$*

$$\operatorname{spec} \widetilde{\Delta}_r(X * Y) = \bigsqcup_{\{p,q \geq -1: p+q=r-1\}} \left(\operatorname{spec} \widetilde{\Delta}_p(X) + \operatorname{spec} \widetilde{\Delta}_q(Y) \right). \tag{9.5}$$

Let D_m denote the digraph that consists of $m \geq 1$ disjoint vertices and no arrows, that is,

$$D_m = \{\underbrace{\bullet, ..., \bullet}_{n \text{ vertices}}\}.$$

Consider for any $n \geq 1$ the digraph

$$D_m^n = \underbrace{D_m * ... * D_m}_{n \text{ times}}.$$

The next theorem is the main result of this section.

Theorem 9.3. *We have, for all $n, m \geq 1$ and $r \geq 0$,*

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^n) = \left\{ ((n-k)m)_{(m-1)^k \binom{r}{k} \binom{n}{r}} \right\}_{k=0}^{r}. \tag{9.6}$$

Consequently, for all $r \geq 2$,

$$\operatorname{spec} \Delta_{r-1}(D_m^n) = \left\{ ((n-k)m)_{(m-1)^k \binom{r}{k} \binom{n}{r}} \right\}_{k=0}^{r}. \tag{9.7}$$

More explicitly, (9.6) means the following: if $n < r$ then

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^n) = \emptyset,$$

while for $n \geq r$ the spectrum of $\widetilde{\Delta}_{r-1}(D_m^n)$ consists of the following $r+1$ eigenvalues

$$(n-r)m,\ (n-r+1)m,\ (n-r+2)m, ..., (n-1)m,\ nm, \tag{9.8}$$

with the multiplicities

$$(m-1)^r \binom{n}{r},\quad (m-1)^{r-1} r \binom{n}{r},\ (m-1)^{r-2} \binom{r}{2} \binom{n}{r}, ..., (m-1) r \binom{n}{r},\ \binom{n}{r}. \tag{9.9}$$

Example 9.4. Let $m = 1$, that is, $D_1 = \{\bullet\}$. Clearly, D_1^n coincides with a complete digraph K_n that consist of n vertices $\{0, ..., n-1\}$ and all arrows $i < j$. Clearly, K_n can be regarded an $(n-1)$-simplex digraph (see Fig. 8).

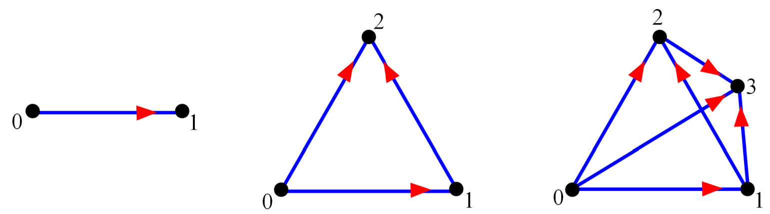

Figure 8: Simplices $K_2 = D_1^2$ (interval), $K_3 = D_1^3$ (triangle) and $K_4 = D_2^4$ (tetrahedron)

In this case all the multiplicities in (9.9) are 0 except for the last one $\binom{n}{r}$. Hence, $\operatorname{spec} \Delta_{r-1}(K_n)$ consists of a single eigenvalue n with the multiplicity $\binom{n}{r}$.

Example 9.5. Let $m = 2$. $D_2 = \{\bullet, \bullet\}$ and $D_2^n =: S^{n-1}$ can be regarded as a digraph sphere of dimension $n-1$. For example, S^1 is a *diamond* and S^2 is an *octahedron* as on Fig. 9.

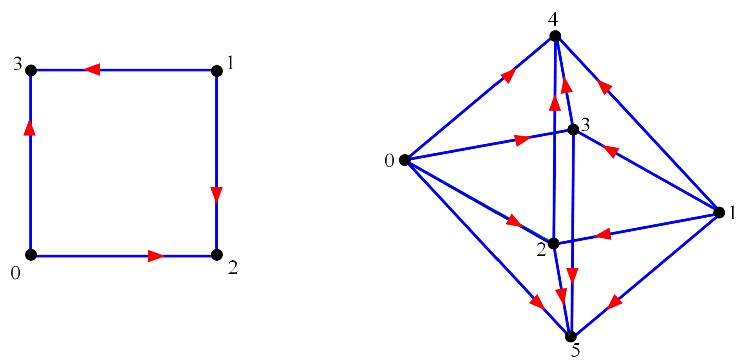

Figure 9: Digraph spheres: $S^1 = D_2^2$ is a diamond, and $S^2 = D_2^3$ is an octahedron.

In this case (9.7) becomes

$$\operatorname{spec}\Delta_{r-1}(S^{n-1}) = \left\{(2(n-k))_{\binom{r}{k}\binom{n}{r}}\right\}_{k=0}^{r}.$$

For example, for $r=2$ we have

$$\operatorname{spec}\Delta_1(S^{n-1}) = \left\{(2(n-2))_{\binom{n}{2}}, (2(n-1))_{2\binom{n}{2}}, (2n)_{\binom{n}{2}}\right\},$$

and for $r=3$

$$\operatorname{spec}\Delta_2(S^{n-1}) = \left\{(2(n-3))_{\binom{n}{3}}, (2(n-2))_{3\binom{n}{3}}, (2(n-1))_{3\binom{n}{3}}, (2n)_{\binom{n}{3}}\right\}.$$

Example 9.6. Let $m=3$ and $n=2$. Then we have $D_3^2 = K_{3,3}$ that is a complete bipartite digraph as on Fig. 10.

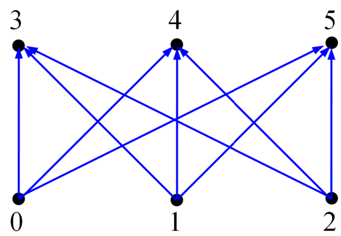

Figure 10: Digraph $K_{3,3}$

Then (9.7) yields for $r=2$ that

$$\operatorname{spec}\Delta_1(K_{3,3}) = \left\{(3(2-k))_{\binom{2}{k}\binom{2}{2}2^k}\right\}_{k=0}^{2} = \{0_4,\, 3_4,\, 6\}.$$

Proof of Theorem 9.3. If $r\geq 2$ then $r-1\geq 1$ and $\Delta_{r-1} = \widetilde{\Delta}_{r-1}$ so that (9.7) follows from (9.6).

Let us first prove (9.6) for $r=0$. We use the fact that the operator $\widetilde{\Delta}_{-1}$ on any digraph is one-dimensional, and it is multiplication by the number of vertices (see [1, Section 6.9]). Hence,

$$\operatorname{spec}\widetilde{\Delta}_{-1}(D_m^n) = \{|D_m^n|\} = \{nm\}, \tag{9.10}$$

which matches (9.6) for $r=0$.

Let us prove (9.6) for all $r\geq 1$ (and $m\geq 1$) by induction in n.

Induction basis for $n=1$. If $r=1$ then the right hand side of (9.6) is

$$\{0_{m-1}, m\}.$$

For any digraph G, the matrix of $\widetilde{\Delta}_0(G)$ in the basis $\{e_i\}$ is obtained from the matrix of $\Delta_0(G)$ (given by (4.3)) by adding 1 to all its entries (cf. [1, Section 6.9]). Since $\Delta_0(D_m)=0$ by (4.3), it follows that the matrix of $\widetilde{\Delta}_0(D_m)$ is an $m\times m$ matrix with all entries $=1$; hence

$$\operatorname{spec}\widetilde{\Delta}_0(D_m) = \{0_{m-1}, m\}, \tag{9.11}$$

which proves (9.6) in this case. If $r \geq 2$ then

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m) = \emptyset,$$

and the right hand side of (9.6) is also empty as all the multiplicities vanish. Hence, we have verified (9.6) for $n = 1$.

For the induction step from n to $n+1$, let us use that

$$D_m^{n+1} = D_m^n * D_m,$$

and

$$|D_m| = m, \quad |D_m^n| = nm.$$

Let us apply (9.5) and rewrite it in the form

$$\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^{n+1}) = \bigsqcup_{\{p,q \geq 0: p+q=r\}} \left(\operatorname{spec} \widetilde{\Delta}_{p-1}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)\right).$$

The spectrum $\operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)$ is empty if $q \geq 2$. Hence, the values of q here should be restricted to $q = 0$ and $q = 1$. Applying (9.10) and (9.11) to compute $\operatorname{spec} \widetilde{\Delta}_{q-1}\left(D_m\right)$ for $q = 0$ and $q = 1$ as well as the induction hypothesis (9.6) to compute $\operatorname{spec} \widetilde{\Delta}_{p-1}\left(D_m^n\right)$ for $p = r$ and $p = r-1$, we obtain

$$\begin{aligned}
\operatorname{spec} \widetilde{\Delta}_{r-1}(D_m^{(n+1)}) &= \left(\operatorname{spec} \widetilde{\Delta}_{r-1}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{-1}\left(D_m\right)\right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_{r-2}\left(D_m^n\right) + \operatorname{spec} \widetilde{\Delta}_{0}\left(D_m\right)\right) \\
&= \left\{ ((n-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k} + m \right\}_{k=0}^{r} \qquad (9.12) \\
&\quad \sqcup \left\{ ((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l} + \{0_{m-1}, m\} \right\}_{l=0}^{r-1}. \qquad (9.13)
\end{aligned}$$

The sequence in (9.12) is equal to

$$\left\{ ((n+1-k)m)_{\binom{r}{k}\binom{n}{r}(m-1)^k} \right\}_{k=0}^{r}, \tag{9.14}$$

and the sequence in (9.13) is equal to

$$\begin{aligned}
&\left\{ ((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^{l+1}}, ((n-l+1)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l} \right\}_{l=0}^{r-1} \\
&= \left\{ ((n-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^{l+1}} \right\}_{l=0}^{r-1} \\
&\qquad \sqcup \left\{ ((n+1-l)m)_{\binom{r-1}{l}\binom{n}{r-1}(m-1)^l} \right\}_{l=0}^{r-1} \\
&= \left\{ ((n+1-k)m)_{\binom{r-1}{k-1}\binom{n}{r-1}(m-1)^k} \right\}_{k=1}^{r} \\
&\qquad \sqcup \left\{ ((n+1-k)m)_{\binom{r-1}{k}\binom{n}{r-1}(m-1)^k} \right\}_{k=0}^{r-1} \\
&= \left\{ ((n+1-k)m)_{\binom{r-1}{k-1}\binom{n}{r-1}(m-1)^k} \right\}_{k=0}^{r}
\end{aligned}$$

$$\sqcup \left\{ ((n+1-k)m)_{\binom{r-1}{k}\binom{n}{r-1}(m-1)^k} \right\}_{k=0}^{r}. \tag{9.15}$$

Combining (9.14) and (9.15), we conclude that spec $\widetilde{\Delta}_{r-1}(D_m^{(n+1)})$ consists of the eigenvalues

$$\lambda_k = (n+1-k)m, \quad k = 0, ..., r,$$

where the multiplicity of λ_k is

$$\begin{aligned} &\left[\binom{r}{k}\binom{n}{r} + \binom{r-1}{k-1}\binom{n}{r-1} + \binom{r-1}{k}\binom{n}{r-1}\right](m-1)^k \\ &= \left[\binom{r}{k}\binom{n}{r} + \binom{r}{k}\binom{n}{r-1}\right](m-1)^k \\ &= \binom{r}{k}\binom{n+1}{r}(m-1)^k, \end{aligned}$$

which proves the induction step. ∎

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AG: Department of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany

grigor@math.uni-bielefeld.de

YL: Yau Mathematical Sciences Center and Department of Mathematics, Tsinghua University, Beijing, 100084, China

yonglin@tsinghua.edu.cn

STY: Yau Mathematical Sciences Center and Department of Mathematics, Tsinghua University, Beijing, 100084, China

styau@tsinghua.edu.cn

HZ: Yau Mathematical Sciences Center and Department of Mathematics, Tsinghua University, Beijing, 100084, China

zhanghh22@mails.tsinghua.edu.cn