Blatt 1. Abgabe bis 20.10.23

5. Let L be a non-divergence form elliptic operator in $\mathbb{R}^n$, that is,

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{ij}u. \tag{7}$$

   (a) Prove that $L|x|^2 > 0$ in $\mathbb{R}^n$.

   (b) Prove that if L is uniformly elliptic then $L|x|^s > 0$ in $\mathbb{R}^n \setminus \{0\}$ provided $s < s_0$, where s_0 is a negative number depending on n and on the constant of ellipticity λ of L.

6. Let L be the operator (7) where $a_{ij}(x)$ are functions defined in an open subset D of $\mathbb{R}^n$. Assume that the operator L is elliptic at any point, that is, the matrix $(a_{ij}(x))_{i,j=1}^n$ is positive definite at any point $x \in D$. Prove that if u is a positive C^2 function in D such that $Lu \leq 0$ then $L\frac{1}{u} \geq 0$.

7. Let L be a divergence form uniformly elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^n$. Fix two functions $f \in L^2(\Omega)$, $g \in W^{1,2}(\Omega)$ and consider the following Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{8}$$

   that is understood in a weak sense as follows: the unknown function u is sought in the class $W^{1,2}(\Omega)$, the equation $Lu = f$ is satisfied weakly, and the boundary condition is understood as $u - g \in W_0^{1,2}(\Omega)$. Prove that the weak Dirichlet problem (8) has a unique solution.

   Hint. The case $g = 0$ was considered in lectures. In the general case, introduce a new unknown function $v = u - g \in W_0^{1,2}(\Omega)$ and apply the method from lectures.

8. Let Ω be an open subset of $\mathbb{R}^n$. Denote by by $W_0^{k,2}$ the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{k,2}(\Omega)$ and by $W_c^{k,2}(\Omega)$ – the set of functions from $W^{k,2}(\Omega)$ that have compact support in Ω. Prove that $W_c^{k,2}(\Omega) \subset W_0^{k,2}(\Omega)$.

   Hint. Extend a function $f \in W_c^{k,2}(\Omega)$ by setting $f = 0$ outside Ω and apply Exercise 4(c) to show that f can be approximated by functions from $\mathcal{D}(\Omega)$ in $W^{k,2}$-norm.