Blatt 12. Abgabe bis 19.01.24

Additional problems are marked by *

Everywhere Ω is an open domain in $\mathbb{R}^n$.

60. (*Strong minimum principle for weak solutions*) Consider in Ω a uniformly elliptic operator
$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right)$$
with measurable coefficients. Let $u \in W^{1,2}(\Omega)$ satisfy $Lu = 0$ weakly in Ω. By De Giorgi's theorem, we can select a continuous version of u. Prove that if Ω is connected, $u \geq 0$ in Ω, and $u(x) = 0$ for some $x \in \Omega$ then $u \equiv 0$ in Ω.

    Hint: Use the weak Harnack inequality.

61. Consider in the unit ball B in $\mathbb{R}^n$ the following *semi-linear* Dirichlet problem:
$$\begin{cases} \Delta u = c\left(1 + |u|\right)^{\gamma}, \\ u \in W_0^{1,2}(B). \end{cases} \tag{54}$$
Prove that if $n > 4$ and $\gamma > \frac{n}{n-4}$ then, for some constant $c \in \mathbb{R}$, there exists a weak solution of (54) that has no continuous version.

    Hint: Look for the function u in the form $u(x) = |x|^s - 1$ with $s < 0$ and use Exercises 51(b) and 52(b).

    Remark: If $\gamma < 4/n$ then solutions of (54) are Hölder continuous by results of lectures.

62. Let K be a compact subset of $\mathbb{R}^n$. For any $p \in [1, \infty)$, define the *p-capacity* of K as follows:
$$\operatorname{cap}_p(K) := \inf_{\eta} \int_{\mathbb{R}^n} |\nabla \eta|^p \, dx, \tag{55}$$
where the infimum is taken over all functions $\eta \in Lip_c(\mathbb{R}^n)$ such that $0 \leq \eta \leq 1$ and $\eta|_K \equiv 1$ (any such function η is called a test function for $\operatorname{cap}_p(K)$).

    (a) *(Isocapacitory inequality)* Prove that if $n > p$ then
    $$\operatorname{cap}_p(K) \geq c\,|K|^{\frac{n-p}{n}}, \tag{56}$$
    where c is a positive constant depending only on n and p.

    Hint: Use the Sobolev inequality.

    (b) (*Capacity of balls*) Prove that if $n > p$ then, for any closed ball $\overline{B}_R$ in $\mathbb{R}^n$ of radius $R > 0$,
    $$\operatorname{cap}_p(\overline{B}_R) = c' R^{n-p}, \tag{57}$$
    where c' is a positive constant depending only on n and p

    Hint: In fact, $c' = \operatorname{cap}_p(\overline{B}_1)$ which is positive by (a).

Remark: Since $\left|\overline{B}_R\right| = \operatorname{const} R^n$, it follows from (57) that $\operatorname{cap}_p(\overline{B}_R) = c'' \left|\overline{B}_R\right|^{\frac{n-p}{p}}$, for some $c'' > 0$, which shows that the estimate (56) is sharp, up to the value of the constant c.

63. As in Exercise 62, let K be a compact subset of $\mathbb{R}^n$. Fix some $p \in (1, \infty)$.

    (a) Prove that if $n < p$ then $\operatorname{cap}_p(K) = 0$.

    (b) Prove that $\operatorname{cap}_p(K) = 0$ also in the case $n = p \geq 2$.

    Hint: For any $\varepsilon > 0$, find a test function η for $\operatorname{cap}_p(K)$ such that $\int_{\mathbb{R}^n} |\nabla \eta|^p \, dx < \varepsilon$. For that, assuming that $K \subset B_R$, choose η in the form $\eta(x) = f(|x|)$, where $f(r) = 1$ for $r \leq R$ and $f(r) = 0$ for $r \geq \rho$ where ρ is large enough. For $r \in [R, \rho]$ choose $f(r)$ to be linear in r in the case (a), and $f(r)$ to be linear in $\ln r$ in the case (b).

64. * Fix two functions $u, v \in W^{1,2}_{loc}(\Omega)$ and set $w := \max(u, v)$.

    (a) Prove that $w \in W^{1,2}_{loc}(\Omega)$.
    Hint: Use the identity $\max(u, v) = \frac{u+v+|u-v|}{2}$ and Exercise 15(a).

    (b) Prove that
    $$|\nabla w| \leq \max(|\nabla u|, |\nabla v|) \text{ a.e.}. \tag{58}$$
    Hint: Use Exercise 15(b) and the identities
    $$w = u \text{ on } \{u > v\} \quad \text{and} \quad w = v \text{ on } \{v \geq u\}.$$

    (c) (*Subadditivity of capacity*) Prove that if K_1 and K_2 are two compact subsets of $\mathbb{R}^n$ and $p \in [1, \infty)$ then
    $$\operatorname{cap}_p(K_1 \cup K_2) \leq \operatorname{cap}_p(K_1) + \operatorname{cap}_p(K_1). \tag{59}$$
    Hint: Use the definition of capacity (55) as well as (58).

65. * (*Liouville theorem for positive supersolutions in* $\mathbb{R}^2$) Consider in $\mathbb{R}^2$ a uniformly elliptic operator $Lu = \sum_{i,j=1}^{2} \partial_i (a_{ij} \partial_j u)$ with measurable coefficients. Let $u \in W^{1,2}_{loc}(\mathbb{R}^2)$ be a non-negative supersolution of L in $\mathbb{R}^2$. Prove that $u \equiv \operatorname{const}$.

    Hint: Assuming without loss of generality that $\operatorname{essinf}_{\mathbb{R}^2} u > 0$, set $v = \ln \frac{1}{u}$ and use the following inequality proved in lectures: for any $\eta \in Lip_c(\mathbb{R}^2)$
    $$\int_{\mathbb{R}^2} |\nabla v|^2 \eta^2 dx \leq C \int_{\mathbb{R}^2} |\nabla \eta|^2 dx, \tag{60}$$
    where $C = C(\lambda)$. Deduce from (60) that, for any ball B_R,
    $$\int_{B_R} |\nabla v|^2 dx \leq C \operatorname{cap}_2(B_R)$$
    and then use Exercise 63(b).

    Remark: By the classical Liouville theorem, any positive harmonic function in $\mathbb{R}^n$ is constant. The above statement in the case $L = \Delta$ means that in $\mathbb{R}^2$ any positive *superharmonic* function is constant. Note that the latter statement is not true in $\mathbb{R}^n$ with $n > 2$.