Blatt 13. Abgabe bis 26.01.24

Additional problems are marked by *

Everywhere Ω is a domain in $\mathbb{R}^n$.

66. Prove the following version of the Poincaré inequality: for any ball B_R in $\mathbb{R}^n$ and for any $f \in W^{1,2}(B_R)$,
$$\int_{B_R} \left(f - \overline{f}\right)^2 dx \le CR^2 \int_{B_R} |\nabla f|^2 dx, \tag{61}$$
where $C = C(n)$ and
$$\overline{f} := \fint_{B_R} f(x)\, dx.$$
Hint: Use the Poincaré inequality of Theorem 3.10.

67. Let u be a function from $W_0^{1,2}(\Omega)$.

    (a) Let $\psi(t)$ be an odd C^1 function on $\mathbb{R}$ such that $\sup|\psi'| < \infty$. Prove that $\psi(u) \in W_0^{1,2}(\Omega)$ and
    $$\nabla\psi(u) = \psi'(u)\nabla u.$$
    Hint. Use the chain rule of Lemma 1.7 from lectures and approximation of ψ by C^∞ functions by means of mollifiers.

    (b) Prove that if in addition $u \in L^\infty(\Omega)$ and $\alpha \ge 1$ then the function $v = |u|^\alpha \operatorname{sgn} u$ belongs to $W_0^{1,2}(\Omega)$, and
    $$\nabla v = \alpha |u|^{\alpha-1}\nabla u.$$
    Hint. Apply (a) with an appropriate function ψ.

68. Let L be a uniformly elliptic operator
$$Lu = \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right)$$
with measurable coefficients in a bounded domain $\Omega \subset \mathbb{R}^n$, where $n > 2$. Consider the weak Dirichlet problem
$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{62}$$
Assume that $f \in L^\infty(\Omega)$ so that the solution u also belongs to $L^\infty(\Omega)$. For any q such that
$$\frac{2n}{n+2} \le q < n/2, \tag{63}$$
prove that
$$\|u\|_{L^r} \le C\|f\|_{L^q}, \tag{64}$$
where $C = C(\lambda, n, q)$ and
$$r = \frac{qn}{n-2q}.$$

Hint. By Exercise 67, the function $\varphi = |u|^{p-1} \operatorname{sgn} u$ (where $p \geq 2$ to be chosen later) belongs to $W_0^{1,2}(\Omega)$. Using φ as a test function (62) deduce that

$$\int_\Omega |u|^{p-2} |\nabla u|^2 \, dx \leq C' \int_\Omega |f| \, |u|^{p-1} \, dx.$$

Again by Exercise 67, the function $v = |u|^{p/2} \operatorname{sgn} u$ also belongs to $W_0^{1,2}(\Omega)$ and

$$|\nabla v|^2 = (p/2)^2 \, |u|^{p-2} \, |\nabla u|^2 \, ,$$

which implies

$$\int_\Omega |\nabla v|^2 \, dx \leq C'' \int_\Omega |f| \, |u|^{p-1} \, dx.$$

Next, use the Sobolev inequality in the left hand side and an appropriate Hölder inequality in the right hand side.

69. Prove the claim of Exercise 68 assuming that $f \in L^2(\Omega)$ instead of $f \in L^\infty(\Omega)$.

Hint. Approximate f by functions from $L^\infty(\Omega)$ and use Exercises 68, 22.

70. * Let Ω be a bounded domain in $\mathbb{R}^n$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Borel function such that

$$|f(x, v)| \leq C \, (1 + |v|) \tag{65}$$

for all $x \in \Omega$ and $v \in \mathbb{R}$, where $C > 0$. Assume that u is a solution of a semi-linear Dirichlet problem

$$\begin{cases} Lu = f(x, u) \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where L is a divergence form uniformly elliptic operator in Ω with measurable coefficients. Prove that $u \in C^\alpha(\Omega)$ with $\alpha = \alpha\,(n, \lambda) > 0$.

Hint. Assuming that the function $F_u := f(x, u(x))$ belongs to $L^q(\Omega)$ with some q as in (63), obtain by Exercise 69 that $u \in L^r(\Omega)$, whence also $F_u \in L^r(\Omega)$ with $r > q$. Therefore, the value of q in the relation $F_u \in L^q(\Omega)$ allows *self-improvement*. This procedure can be reiterated until you obtain $F_u \in L^q(\Omega)$ for a large enough q. The latter will imply that u is Hölder continuous by Theorem 3.12 from lectures.

Remark. Exercise 70 improves Theorem 3.16(c) from lectures. In particular, it implies that all the eigenfunctions of L in Ω are Hölder continuous.