Blatt 2. Abgabe bis 27.10.23

Additional problems are marked by *

9. Let Ω be an open set in $\mathbb{R}^n$

   (a) For any distribution $u \in \mathcal{D}'(\Omega)$ and for any smooth function $\psi \in C^\infty(\Omega)$ define the product ψu as a distribution in Ω by

   $$(\psi u, \varphi) = (u, \psi\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega)$$

   (observe that $\psi\varphi \in \mathcal{D}(\Omega)$). Prove the product rule:

   $$\partial_i(\psi u) = (\partial_i \psi) u + \psi \partial_i u. \tag{9}$$

   (b) Prove that if $u \in W^{1,p}_{loc}(\Omega)$ and $\psi \in C^\infty(\Omega)$ then $\psi u \in W^{1,p}_{loc}(\Omega)$.

   (c) Consider in Ω a divergence form operator

   $$Lu = \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j u)$$

   where the functions $a_{ij}(x)$ belong to $C^\infty(\Omega)$. By (a), the operator L is defined on all distributions $u \in \mathcal{D}'(\Omega)$. Prove that if $u \in W^{1,2}_{loc}(\Omega)$ and $f \in L^2_{loc}(\Omega)$ then the identity $Lu = f$ in the weak sense is equivalent to the same identity in the distributional sense.

10. Let Ω be an open subset in $\mathbb{R}^n$ and let

   $$L = \sum_{i,j=1}^n \partial_i(a_{ij}(x)\partial_j)$$

   be a divergence form uniformly elliptic operator in Ω with measurable coefficients $a_{ij}(x)$. Let $c(x)$ be a measurable function in Ω such that $c_1 \le c(x) \le c_2$ for almost all $x \in \Omega$ where c_1 and c_2 are two positive constants. Consider the following Dirichlet problem

   $$\begin{cases} Lu - cu = f & \text{in } \Omega \\ u \in W^{1,2}_0(\Omega) \end{cases} \tag{10}$$

   (a) Formulate the equation $Lu - cu = f$ in the weak sense.

   (b) Prove that the weak problem (10) has a unique solution $u \in W^1_0(\Omega)$ for any $f \in L^2(\Omega)$.

   Hint. Reduce the problem to the Riesz representation theorem.

11. (*Faber-Krahn inequality, case* $n > 2$) Let Ω be a bounded domain in $\mathbb{R}^n$ and u be a function from $W_0^{1,2}(\Omega)$. Consider the set

$$U = \{x \in \Omega : u(x) \neq 0\}.$$

Assuming that $n > 2$, prove that

$$\int_\Omega |\nabla u|^2 \, dx \geq c \, |U|^{-2/n} \int_\Omega u^2 dx, \tag{11}$$

where $c = c(n) > 0$ and $|U|$ is the Lebesgue measure of U.

Hint. Use the Hölder inequality and the Sobolev inequality

$$\left(\int_\Omega |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq C \int_\Omega |\nabla u|^2 \, dx. \tag{12}$$

where $C = C(n)$ and u is any function from $W_0^{1,2}(\mathbb{R}^n)$.

12. Let Ω be a bounded domain in $\mathbb{R}^n$. Prove that if $1 \leq p < q$ then

$$W^{1,q}(\Omega) \subset W^{1,p}(\Omega)$$

and

$$W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega). \tag{13}$$

Hint. Use the Hölder inequality.

13. * (*Faber-Krahn inequality, case* $n = 2$) Prove the inequality (11) of Exercise 11 in the case $n = 2$.
Hint. Use the Hölder inequality as well as the Sobolev inequality in the form

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}} \, dx \right)^{\frac{n-p}{n}} \leq C \int_{\mathbb{R}^n} |\nabla u|^p \, dx, \tag{14}$$

where $1 \leq p < n$, $C = C(n, p)$ and u is any function from $W_0^{1,p}(\mathbb{R}^n)$. In the case $n = 2$ use (14) with any $p \in (1, 2)$, for example, with $p = \frac{3}{2}$. Observe that $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$ by (13).