Blatt 3. Abgabe bis 03.11.23

Additional problems are marked by *

In all Exercises, Ω is an open subset of $\mathbb{R}^n$.

14. Let $u \in W^{1,2}_{loc}(\Omega)$ and $\eta \in \mathcal{D}(\Omega)$. Prove that $\eta u \in W_0^{1,2}(\Omega)$ and, for any $i = 1, ..., n$,

$$\partial_i (\eta u) = (\partial_i \eta) u + \eta \partial_i u,$$

where the derivative ∂_i is understood in the weak sense.

Hint. Use Exercises 8 and 9.

15. Prove the following statements.

   (a) If $u \in W^{1,2}_{loc}(\Omega)$ then also $u_+ \in W^{1,2}_{loc}(\Omega)$ and

$$\nabla u_+ = \begin{cases} \nabla u & \text{a.e. on } \{u > 0\}, \\ 0 & \text{a.e. on } \{u \leq 0\}. \end{cases} \tag{15}$$

   Hint. The identity (15) was proved in lectures for $u \in W_0^{1,2}(\Omega)$. In order to extend it to $u \in W^{1,2}_{loc}(\Omega)$, use the function ηu from Exercise 14.

   (b) If $u \in W^{1,2}_{loc}(\Omega)$ then $\nabla u = 0$ a.e. on the set $\{u = \alpha\}$ for any $\alpha \in \mathbb{R}$.

16. Let $u \in W^{1,2}(\Omega)$.

   (a) Prove that $(u - \alpha)_+ \in W^{1,2}(\Omega)$ for any $\alpha \geq 0$.

   (b) Prove that

$$(u - \alpha)_+ \xrightarrow{W^{1,2}} u_+ \text{ as } \alpha \to 0+.$$

Hint. Use Exercise 15.

17. (*Chain rule in* $W^{1,2}_{loc}$ *and* $W^{1,2}$) Let J be a closed interval in $\mathbb{R}$ and $\psi : J \to \mathbb{R}$ be a C^∞-function such that

$$\sup_J |\psi'| < \infty.$$

Consider a function $u : \Omega \to J$ so that the composition $\psi(u) : \Omega \to \mathbb{R}$ is well-defined.

   (a) Prove that if $u \in W^{1,2}_{loc}(\Omega)$ then $\psi(u) \in W^{1,2}_{loc}(\Omega)$ and

$$\nabla \psi(u) = \psi'(u) \nabla u. \tag{16}$$

   Hint. A similar chain rule was proved in lectures assuming in addition that $u \in W_0^{1,2}(\Omega)$ and $\psi(0) = 0$. Reduce the claim of (a) to that case by adding to ψ a constant and by multiplying u by a test function η as in Exercise 14.

   (b) Prove that if Ω is bounded and $u \in W^{1,2}(\Omega)$ then also $\psi(u) \in W^{1,2}(\Omega)$.

18. * (*Nash inequality*) Prove that, for $n > 2$ and for any $u \in W_0^{1,2}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq c \left(\int_{\mathbb{R}^n} u^2 dx \right)^{\frac{n+2}{n}} \left(\int_{\mathbb{R}^n} |u| \, dx \right)^{-\frac{4}{n}}, \tag{17}$$

where $c = c(n) > 0$.

Hint. Use an appropriate Hölder inequality and the Sobolev inequality in the form

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}} \leq C \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \tag{18}$$

19. * Let

$$L = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x) \, \partial_j \right) + \sum_{i=1}^{n} b_i \partial_i u$$

be a uniformly elliptic operator in Ω with bounded measurable coefficients. We say a function $u \in W_{loc}^{1,2}(\Omega)$ satisfies $Lu \geq 0$ (resp. $Lu \leq 0$) weakly in Ω if

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx + \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, \varphi dx \geq 0 \quad (\text{resp.} \leq),$$

for any non-negative function $\varphi \in \mathcal{D}(\Omega)$. Prove that if

$$\operatorname*{essinf}_\Omega u > 0 \text{ and } Lu \leq 0 \text{ weakly in } \Omega$$

then

$$L \frac{1}{u} \geq 0 \text{ weakly in } \Omega.$$