Blatt 4. Abgabe bis 10.11.23

Additional problems are marked by *

In all Exercises Ω is an open subset of $\mathbb{R}^n$.

20. *(Integration by parts for weak derivatives)*

    (a) Prove the following identity for all $u \in W^{1,2}(\Omega)$, $v \in W_0^{1,2}(\Omega)$ and any $i = 1, ..., n$:

    $$\int_\Omega (\partial_i u) v \, dx = -\int_\Omega u \, \partial_i v \, dx. \tag{19}$$

    Hint. Consider first the case $v \in \mathcal{D}(\Omega)$.

    (b) Prove that (19) holds for all $u \in W_{loc}^{1,2}(\Omega)$ and $v \in W_c^{1,2}(\Omega)$.
    Hint: Use Exercise 8, where also the definition of $W_c^{1,2}(\Omega)$ was given.

21. *(Product rule in $W_{loc}^{1,2}$)* Prove that if $u, v \in W_{loc}^{1,2}(\Omega)$ then $uv \in W_{loc}^{1,1}(\Omega)$ and

    $$\partial_i(uv) = (\partial_i u)v + u\partial_i v. \tag{20}$$

    Hint: Use the definition of the distributional derivative ∂_i and Exercises 14, 20(b).

22. Consider in a bounded domain Ω a uniformly elliptic operator

    $$Lu = \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j u) \tag{21}$$

    with measurable coefficients $a_{ij}(x)$. Let u be a solution of the weak Dirichlet problem

    $$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

    Prove that

    $$\|u\|_{W^{1,2}} \leq \lambda \left(D + D^2\right) \|f\|_{L^2},$$

    where λ is the ellipticity constant of L and $D = \operatorname{diam}(\Omega)$.

    Hint: Use the Friedrichs inequality

    $$\int_\Omega u^2 dx \leq D^2 \int_\Omega |\nabla u|^2 \, dx$$

    for all $u \in W_0^{1,2}(\Omega)$.

23. Let Ω be bounded and L be a uniformly elliptic divergence form operator (21) with measurable coefficients $a_{ij}(x)$. Given $f \in L^2(\Omega)$ and some $k = 1, ..., n$, consider the following Dirichlet problem

    $$\begin{cases} Lu = \partial_k f \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{22}$$

    where both Lu and $\partial_k f$ are understood as distributions. Prove that the problem (22) has exactly one solution u.

    Hint: Rewrite the problem (22) in the form of an integral identity and use the same approach that was used in lectures for the equation $Lu = f$.

24. * Prove that the solution u of (22) satisfies the following estimate

$$\|u\|_{W^{1,2}} \leq C \|f\|_{L^2}, \tag{23}$$

where the constant C depends on $\operatorname{diam}(\Omega)$ and λ.

25. * Consider in a bounded domain Ω an operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^{n} b_i \partial_i u + cu,$$

where a_{ij}, b_i and c are measurable functions in Ω. Assume that L is uniformly elliptic with the ellipticity constant λ and that b_i, c are bounded, that is,

$$\sum_{i=1}^{n} |b_i| + |c| \leq b \text{ in } \Omega$$

for some constant b. Prove that if

$$|\Omega| < \delta, \tag{24}$$

for some $\delta = \delta\left(n, \lambda b\right) > 0$, then the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

has a unique solution for any $f \in L^2(\Omega)$.

Hint: Use (24) and the Faber-Krahn inequality to ensure that the bilinear form of L is coercive. Then apply the Lax-Milgram theorem.