Blatt 6. Abgabe bis 24.11.23

Additional problems are marked by *

In all Exercises Ω is an open subset of $\mathbb{R}^n$.

30. Let

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_i u\right)$$

be a uniformly elliptic operator in Ω with measurable coefficients. For any $\alpha > 0$ consider the following Dirichlet problem

$$\begin{cases} Lu - \alpha u = -f & \text{weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{28}$$

By Exercise 10 this problem has a unique solution u for any $f \in L^2(\Omega)$. Hence, define the *resolvent operator* $R_\alpha : L^2(\Omega) \to L^2(\Omega)$ as follows: if $f \in L^2(\Omega)$ then $R_\alpha f$ is equal the solution u of (28). Prove the following properties of R_α.

(a) R_α is a bounded linear operator in $L^2(\Omega)$ and $\|R_\alpha\| \le 1/\alpha$.

(b) R_α is a self-adjoint operator in $L^2(\Omega)$.

31. Under the hypotheses of Exercise 30 prove the following properties of the resolvent operator R_α.

(a) R_α is a positive definite operator in $L^2(\Omega)$.

(b) If Ω is bounded then R_α is a compact operator in $L^2(\Omega)$.
Hint: Use the compact embedding theorem.

32. Consider a non-divergence form operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}\left(x\right) \partial_{ij} u + \sum_{i=1}^{n} b_i \partial_i u$$

in a bounded domain Ω of $\mathbb{R}^n$. Assume that (a_{ij}) is uniformly elliptic with the ellipticity constant λ and that b_i are bounded: for some constant b

$$\sum_{i=1}^{n} |b_i| \le b \quad \text{pointwise in } \Omega.$$

Let $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be a classical solution of the Dirichlet problem

$$\begin{cases} Lu = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let D be the diameter of Ω. Prove that

$$\sup_{\Omega} u \leq C_1 \exp(C_2 D),$$

where C_1 and C_2 are constants depending on λ and b.

Hint: Use the same approach as in Exercise 2: compare u with a function

$$v(x) = -\alpha \exp(\gamma x_1) + \beta$$

with appropriate constants α, β, γ and apply the comparison principle.

33. Under the hypotheses of Exercise 32, assume that the diameter D of Ω is small enough, namely,

$$D < \frac{n}{2\lambda b}. \tag{29}$$

Prove that

$$\sup_{\Omega} u \leq \lambda D^2.$$

Hint: Assuming that the origin belongs to Ω, compare u with a function

$$v(x) = -\alpha |x|^2 + \beta$$

with appropriate constants α, β and apply the comparison principle of Exercise 2.

34. * Consider a ball B_R in $\mathbb{R}^n$, $n > 2$, and assume that a function $u \in C^2(B_R) \cap C(\overline{B}_R)$ solves the classical Dirichlet problem

$$\begin{cases} \Delta u = -f \text{ in } B_R, \\ u = 0 \text{ on } \partial B_R. \end{cases}$$

(a) Prove that, for any $q \in (n/2, \infty]$,

$$\|u\|_{L^\infty} \leq M \|f\|_{L^q}, \tag{30}$$

where $M = CR^{2-n/q}$ and C is a constant depending on n, q.

(b) Prove that the estimate (30) cannot hold for $q \leq n/2$ with any finite M.

Hint: Use the representation

$$u(x) = \int_{B_R} G(x,y) f(y) \, dy, \tag{31}$$

where $G(x,y)$ is the Green function of the ball B_R. Use also that

$$G(x,y) \leq \frac{1}{\omega_n (n-2) |x-y|^{n-2}} \tag{32}$$

and

$$G(0,y) = \frac{1}{\omega_n (n-2)} \left(\frac{1}{|y|^{n-2}} - \frac{1}{R^{n-2}} \right). \tag{33}$$