Blatt 7. Abgabe bis 01.12.23

In all problems Ω is an open subset of $\mathbb{R}^n$.

35. Prove the following properties of Lipschitz functions.

    (a) If K be a compact subset of $\mathbb{R}^n$ and $f, g \in Lip(K)$, then $f+g$ and fg are also in $Lip(K)$.

    (b) If $f, g \in Lip_{loc}(\Omega)$ then $f+g$ and fg are also in $Lip_{loc}(\Omega)$.

    (c) Let Ω be convex. If $f \in C^1(\Omega)$ and $\sup_\Omega |\nabla f| \leq L$ for some constant L then f is a Lipschitz function in Ω with the Lipschitz constant L.

    (d) Any function from $C^1(\Omega)$ is locally Lipschitz in Ω, that is $C^1(\Omega) \subset Lip_{loc}(\Omega)$.

36. Prove the following properties of Lipschitz functions.

    (a) If $f \in Lip_{loc}(\Omega)$ then $f \in Lip(K)$ for any compact subset $K \subset \Omega$.

    (b) If $f \in Lip_{loc}(\Omega)$ and $\operatorname{supp} f$ is a compact subset of Ω then $f \in Lip(\Omega)$.

    (c) If $f \in Lip_c(\Omega)$ then, extending f by setting $f = 0$ in Ω^c, we obtain $f \in Lip_c(\mathbb{R}^n)$.

37. Prove that if $f \in W^{1,2}(\mathbb{R}^n)$ then, for any unit vector $e \in \mathbb{R}^n$,

    $$\partial_e^h f \xrightarrow{L^2} \partial_e f \quad \text{as } h \to 0,$$

    where the convergence is in the norm of $L^2(\mathbb{R}^n)$.

    Hint. Use Lemmas 2.3 and 2.5 from lectures and the following fact: if a sequence $\{u_k\}$ of elements of a Hilbert space H converges weakly to $u \in H$ and $\|u_k\| \to \|u\|$ as $k \to \infty$ then u_k converges to u strongly (that is, with respect to the norm).

38. Let f be a Lipschitz function in Ω with the Lipschitz constant L. Prove that, for any unit vector e, the distributional derivative $\partial_e f$ belongs to $L^\infty(\Omega)$ and $\|\partial_e f\|_{L^\infty} \leq L$.

    Remark: The case $\Omega = \mathbb{R}^n$ was considered in lectures.

    Hint: Use the following fact from lectures (Corollary 2.4): if $g \in Lip_c(\mathbb{R}^n)$ then $g \in W^{1,2}(\mathbb{R}^n)$. Then apply Exercise 37: $\partial_e^h g \xrightarrow{L^2} \partial_e g$ as $h \to 0$, which implies that $\partial_e^{h_k} g \xrightarrow{\text{a.e.}} \partial_e g$ for a subsequence $\{h_k\} \to 0$. Apply this to $g = f\psi$ where ψ is a cutoff function in Ω.