

Fundamental groupoids of digraphs and graphs

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December, 2016

Abstract

In this paper we introduce the notion of fundamental groupoid of a digraph and prove its basic properties. In particular, we obtain a product theorem and an analog of the Van Kampen theorem. Considering the category of (undirected) graphs as the full subcategory of digraphs we transfer the results to the category of graphs. As a corollary we obtain the corresponding results for the fundamental groups of digraphs and graphs. We give an application to graph coloring.

Keywords: digraph, fundamental group, fundamental groupoid, product of graphs.

Mathematics Subject Classification 2010: 05C25, 05C38, 05C76, 20L05, 57M15.

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1 Introduction

In this paper we develop further the homotopy theory for digraphs (=directed graphs) initiated in [9], [8], and [10]. In the category of digraphs, the homology and the homotopy theories were introduced in [8] in such a way that the homology groups are homotopy invariant and the first homology group of a connected digraph is isomorphic to the abelization of its fundamental group. By a natural way we can consider the category of nondirected graphs as a full subcategory of digraphs. Thus, the homology and homotopy theories of digraph can

be transferred to the category of nondirected graphs, thus leading to similar results for the latter category.

In the case of undirected graphs the fundamental group was first introduced in the papers [3] and [4], where they described the relation of the fundamental group of graph to the Atkin homotopy theory [1], [2]. Note that for undirected graphs the notions of fundamental groups of [8] and [3, 4] coincide.

In the present paper we introduce the notion of the fundamental groupoid of a digraph that is a natural generalization of the notion of fundamental group of digraph from [8]. Our definition of groupoid has essentially the origin in the discrete nature of graphs and is not related to the notion of fundamental groupoid of a graph as a topological space from [5] and [13].

We prove basic properties of the fundamental groupoid of digraphs, in particular, a product formula for the fundamental groupoids for various notions of product of digraphs as well as an analogous of the Van Kampen theorem for groupoids. Considering the category of nondirected graphs as a full subcategory of the category of digraphs we transfer these results to the category of nondirected graphs. Note, the Van Kampen theorem for the fundamental group of graphs was obtained also in [3] and [4].

The paper is organized as follows. In Section 2, we give a preliminary material, necessary definitions, and some useful constructions in the category of digraphs based on [8], [9], and [12].

In Section 3, we define the fundamental groupoid of a digraph and describe its basic properties. In fact, we define a functor from the category of digraphs to the category of groupoids. We prove the results concerning fundamental groupoids for various products of digraphs. We also give application to the first homology group of the products.

In Section 4, we construct a functor Δ (geometrical realization) from the category of digraphs to the category of 2-dimensional CW-complexes, that provides a natural equivalence of the corresponding fundamental groupoids on the vertices of digraphs. As a consequence of the geometric realization we obtain an analog of the Van Kampen theorem for groupoids of digraphs.

In Section 5, we transfer the aforementioned results to the category of nondirected graphs and compare our results with that of [3], [4].

In Section 6, we give an application to coloring of graphs.

Acknowledgements

The first and the third authors were partially supported by SFB 701 of German Research Council, the second and the third authors were partially supported by the CONACyT Grant 151338.

2 Category of digraphs and homotopy theory

In this Section we give necessary definitions and preliminary material (see [9] and [8]) which we need in the following sections. We prove also several technical results.

Definition 2.1 *A directed graph (digraph) G is a couple (V_G, E_G) consisting of a set V_G of vertices and of a subset $E_G \subset \{V_G \times V_G \setminus \text{diag}\}$ of ordered pairs. The elements of E_G are called *arrows*, which are denoted by $(v \rightarrow w)$, where the vertex $v = \text{orig}(v \rightarrow w)$ is the *origin of the arrow* and the vertex $w = \text{end}(v \rightarrow w)$ is the *end of the arrow*.*

A based digraph $G^ = (G, *)$ is a digraph G together with a *based vertex* $v = * \in V_G$.*

If there is an arrow from v to w , then we write $v \rightarrow w$. For two vertices $v, w \in V_G$, we write $v \cong w$ if either $v = w$ or $v \rightarrow w$.

Definition 2.2 A digraph H is called a subdigraph of G if $V_H \subset V_G$ and $E_H \subset E_G$.

Definition 2.3 A *digraph map* (or simply *map*) from a digraph G to a digraph H is a map $f: V_G \rightarrow V_H$ such that $v \rightarrow w$ on G implies $f(v) \cong f(w)$ on H .

A digraph map f is *non-degenerate* if $v \rightarrow w$ on G implies $f(v) \rightarrow f(w)$ on H .

A *digraph map of based digraphs* $f: (G, v) \rightarrow (H, w)$ has additional property that $f(v) = w$.

The set of all digraphs with digraph maps form a *category of digraphs* that will be denoted by \mathcal{D} . The set of all based digraphs with based digraph maps form a *category of based digraphs* that will be denoted by \mathcal{D}^* .

For two digraphs G and H we denote by $\text{Hom}(G, H)$ the set of all digraph maps from G to H . For two based digraphs G^* and H^* we denote by $\text{Hom}(G^*, H^*)$ the set of all based digraph maps from G^* to H^* .

Definition 2.4 For digraphs G, H define two notions of their product.

i) Define a \square -*product* $\Pi = G \square H$ as a digraph with a set of vertices $V_\Pi = V_G \times V_H$ and a set of arrows E_Π given by the rule

$$(x, y) \rightarrow (x', y') \text{ if } x = x' \text{ and } y \rightarrow y', \text{ or } x \rightarrow x' \text{ and } y = y',$$

where $x, x' \in V_G$ and $y, y' \in V_H$. The \square -product is also referred to as the Cartesian product.

ii) Define a \times -*product* $P = G \times H$ as a digraph with a set of vertices $V_P = V_G \times V_H$ and a set of arrows E_P given by the rule

$$(x, y) \rightarrow (x', y') \text{ if } x = x' \text{ and } y \rightarrow y', \text{ or } x \rightarrow x' \text{ and } y = y', \text{ or } x \rightarrow x' \text{ and } y \rightarrow y'.$$

Let G and H be digraphs. For any vertex $v \in V_H$ there are natural inclusions $i_v: G \rightarrow P$ and $j_v: G \rightarrow \Pi$ given on the set of vertices by the rules

$$i_v(x) = (x, v) \in V_P, \quad j_v(x) = (x, v) \in V_\Pi \text{ for } x \in V_G.$$

Similarly, there are natural inclusions $i_w: H \rightarrow P$ and $j_w: H \rightarrow \Pi$ for any $w \in V_G$.

Also we have natural projections $p: P \rightarrow G$, $q: P \rightarrow H$ given on the set of vertices by the rule

$$p(x, y) = x \in V_G, \quad q(x, y) = y \in V_H \text{ for } x \in V_G, y \in V_H.$$

Similarly, there are projections $\Pi \rightarrow G$ and $\Pi \rightarrow H$.

In what follows we use the sign $\dot{\cup}$ to denote a disjoint union.

Definition 2.5 i) Let $f: G \rightarrow H$ be a digraph map of digraphs G, H . Define a digraph $C_f = (V_C, E_C)$ as follows

$$V_C = V_G \dot{\cup} V_H, \quad E_C = E_G \dot{\cup} E_H \dot{\cup} E_I, \text{ where } E_I = \{(v \rightarrow f(v)) \mid v \in V_G\}.$$

The digraph C_f is called the *direct cylinder of the map f* . The *inverse cylinder* C_f^- of the map f has the same set of vertices V_C as C_f and the set of arrows

$$E_{C^-} = E_G \dot{\cup} E_H \dot{\cup} E_{I^-}, \text{ where } E_{I^-} = \{(f(v) \rightarrow v) \mid v \in V_G\}.$$

Let us recall now the basic notions of the homotopy theory of [8]. Let I_n , $n \geq 0$, denote a digraph with the set of vertices $V_n = \{0, 1, \dots, n\}$ and the set of arrows E_{I_n} that contains exactly one of the arrows $(i \rightarrow (i+1))$ or $((i+1) \rightarrow i)$ for any $i = 0, 1, \dots, n-1$, and no other arrow. The digraph I_n is called a *line digraph*. There are only two line digraphs with two vertices, which will be denoted by $I = (0 \rightarrow 1)$ and $I^- = (1 \rightarrow 0)$.

Denote by I_n^* the based digraph $(I_n, 0)$. Let \mathcal{I}_n (or \mathcal{I}_n^*) be the set of all line digraphs (or based line digraphs) with the vertex set V_n and set

$$\mathcal{I} = \bigcup_{n \geq 0} \mathcal{I}_n, \quad \mathcal{I}^* = \bigcup_{n \geq 0} \mathcal{I}_n^*.$$

Definition 2.6 Let G, H be digraphs.

i) Two digraph maps $f_i: G \rightarrow H$, $i = 0, 1$, are called *homotopic* if there exists a line digraph $I_n \in \mathcal{I}$ and a digraph map $F: G \square I_n \rightarrow H$ such that

$$F|_{G \square \{0\}} = f_0: G \square \{0\} \rightarrow H, \quad F|_{G \square \{n\}} = f_1: G \square \{n\} \rightarrow H.$$

In this case we shall write $f_0 \simeq f_1$. If $I_n = I_1$ then we shall refer to F as an *one-step homotopy* from f_0 to f_1 and to the maps f_i as *one-step homotopic*.

ii) Two digraphs G and H are *homotopy equivalent* if there exist digraph maps

$$f: G \rightarrow H, \quad g: H \rightarrow G$$

such that

$$f \circ g \simeq \text{Id}_H, \quad g \circ f \simeq \text{Id}_G.$$

In this case, we write $H \simeq G$, and the maps f and g are called *homotopy inverses* to each other.

iii) A digraph G is *contractible* if it is homotopy equivalent to the one-vertex digraph.

Thus we obtain a well defined category \mathcal{D}' of digraphs with the classes of homotopic maps as digraph maps in \mathcal{D}' .

A homotopy between two based digraph maps $f, g: G^* \rightarrow H^*$ is defined as in Definition 2.6 with additional requirement that $F|_{\{*\} \square I_n} = *$. Then we obtain a homotopy category \mathcal{D}^{*} of based digraphs.

For any $I_n \in \mathcal{I}_n$ define the line digraph $\hat{I}_n \in \mathcal{I}_n$ as follows:

$$i \rightarrow j \text{ in } \hat{I}_n \Leftrightarrow (n-i) \rightarrow (n-j) \text{ in } I_n.$$

For any two line digraphs I_n and I_m , define the line digraph $I_{n+m} = I_n \vee I_m \in \mathcal{I}_{n+m}$ by identification of the vertices $n \in I_n$ and $0 \in I_m$ and keeping the arrows in I_n, I_m .

Definition 2.7 i) A *path-map* in a digraph G is any digraph map $\phi: I_n \rightarrow G$, where $I_n \in \mathcal{I}_n$. A *based path-map* on a based digraph G^* is a based digraph map $\phi: I_n^* \rightarrow G^*$. A *loop* on a based digraph G^* is a based path-map $\phi: I_n^* \rightarrow G^*$ such that $\phi(n) = *$.

ii) For a path-map $\phi: I_n \rightarrow G$ define the *inverse path-map* $\hat{\phi}: \hat{I}_n \rightarrow G$ by $\hat{\phi}(i) = \phi(n-i)$.

iii) For two path-maps $\phi: I_n \rightarrow G$ and $\psi: I_m \rightarrow G$ with $\phi(n) = \psi(0)$ define the *concatenation path-map* $\phi \vee \psi: I_{n+m} \rightarrow G$ by

$$\phi \vee \psi(i) = \begin{cases} \phi(i), & 0 \leq i \leq n \\ \psi(i-n), & n \leq i \leq n+m. \end{cases}$$

Definition 2.8 A digraph map $h: I_n \rightarrow I_m$ is called a *shrinking map* if $h(0) = 0$, $h(n) = m$, and $h(i) \leq h(j)$ whenever $i \leq j$.

Definition 2.9 Consider two path-maps

$$\phi: I_n \rightarrow G, \psi: I_m \rightarrow G \text{ such that } \phi(0) = \psi(0), \phi(n) = \psi(m). \quad (2.1)$$

A *one-step direct C_∂ -homotopy* from ϕ to ψ is a pair (h, F) , where $h: I_n \rightarrow I_m$ is a shrinking map and $F: C_h \rightarrow G$ is a digraph map such that

$$F|_{I_n} = \phi \text{ and } F|_{I_m} = \psi. \quad (2.2)$$

If the same is true with C_h replaced everywhere by C_h^- then we refer to an one-step *inverse C_∂ -homotopy*.

Now we define an equivalence relation on the set of path-maps of a digraph G .

Definition 2.10 Let ϕ, ψ be path-maps as in (2.1). We call these path-maps *C_∂ -homotopic* and write $\phi \stackrel{C_\partial}{\simeq} \psi$ if there exists a finite sequence $\{\phi_k\}_{k=0}^m$ of path-maps such that $\phi_0 = \phi$, $\phi_m = \psi$ and, for any $k = 0, \dots, m-1$, ϕ_k is one-step C_∂ -homotopic to ϕ_{k+1} or inverse ϕ_{k+1} is one-step C_∂ -homotopic to ϕ_k .

As follows from Definition 2.10, the relation $\phi \stackrel{C_\partial}{\simeq} \psi$ is an equivalence relation. Note, that for the based loops in a based digraph G^* , our notion of C_∂ -homotopy from Definition 2.10 coincides with the notion of C -homotopy of [8, Definition 4.10].

Theorem 2.11 [8] *Let $\pi_1(G^*)$ be the set of equivalence classes under C_∂ -homotopy of based loops of a digraph G^* . The C_∂ -homotopy class of a based loop ϕ will be denoted by $[\phi]$. Then $\pi_1(G^*)$ is a group with the neutral element $[e]$, where $e: I_0^* \rightarrow G^*$ is the trivial loop, the inverse element of $[\phi]$ is $[\hat{\phi}]$, and the product is given by concatenation of the loops $[\phi][\psi] = [\phi \vee \psi]$.*

Now we discuss properties of the \times -product of digraphs.

Proposition 2.12 *For any line digraph I_n and any digraph G*

$$G \times I_n \simeq G.$$

Proof. Consider the case $n = 1$ and the digraph $G \times I$. We have a natural inclusion

$$j: G \rightarrow G \times I, \quad j(v) = v \times \{0\}, \quad v \in V_G$$

and a natural projection $p: G \times I \rightarrow G$ such that the composition $p \circ j: G \rightarrow G$ is the identity map. Now we prove that the composition $j \circ p$ is homotopic to the identity map $\text{Id}_{G \times I}$. Define a homotopy

$$H: (G \times I) \square I^- \rightarrow G \times I$$

as follows:

$$H_0 = \text{Id}_{G \times I}: (G \times I) \square \{0\} \rightarrow G \times I, \quad H_0(v, t, 0) = (v, t, 0), \quad v \in V_G, \quad t \in V_I,$$

on the bottom, and the composition

$$j \circ p: (G \times I) \square \{1\} \rightarrow G \times \{0\}, \quad H_1(v, t, 1) = (v, 0), \quad v \in V_G, \quad t \in V_I, \quad 1 \in V_{I^-}$$

on the top. The map H is a well defined digraph map of digraphs.

The case of $G \times I^-$ is similar, which settles the claim for $n = 1$. The claim for general n is proved by induction in n . ■

Let D, G, H be arbitrary digraphs. For a given a digraph map

$$f: D \rightarrow G \rtimes H,$$

consider the digraph maps

$$f_1 = p \circ f: D \rightarrow G, \quad f_2 = q \circ f: D \rightarrow H,$$

where $p: G \rtimes H \rightarrow G$, $q: G \rtimes H \rightarrow H$ are natural projections.

Proposition 2.13 *There exists a one to one correspondence between the sets $\text{Hom}(D, G \rtimes H)$ and $\text{Hom}(D, G) \times \text{Hom}(D, H)$, given by the rule*

$$f \leftrightarrow (f_1, f_2).$$

Proof. Let r be the map of sets

$$\text{Hom}(D, G \rtimes H) \rightarrow \text{Hom}(D, G) \times \text{Hom}(D, H), \quad r(f) = (f_1, f_2). \quad (2.3)$$

Let $f \neq g \in \text{Hom}(D, G \rtimes H)$. Since the digraph maps f and g are defined on the set of vertices, there exists a vertex $v \in V_D$, such that

$$f(v) = (v_1, v_2) \neq g(v) = (w_1, w_2) \text{ where } (v_1, v_2), (w_1, w_2) \in V_{G \rtimes H}.$$

Hence, at least one inequality $v_1 \neq w_1$, $v_2 \neq w_2$ is true. Hence $(f_1, f_2)(v) \neq (g_1, g_2)(v)$, since $f_i(v) = v_i, g_i(v) = w_i$. Thus the map r is a one-to-one inclusion. The map r is a surjection, since any two maps $f_1 \in \text{Hom}(D, G)$, $f_2 \in \text{Hom}(D, H)$ are defined by the maps of vertices

$$f_1: V_D \rightarrow V_G, \quad f_2: V_D \rightarrow V_H$$

which define a map of vertices

$$f = (f_1, f_2): V_D \rightarrow V_{G \rtimes H} = V_G \times V_H$$

that is a well defined digraph map of digraphs $f: D \rightarrow G \rtimes H$, for which $r(f) = (f_1, f_2)$. ■

Lemma 2.14 *Consider a path map $\phi: I_s \rightarrow I_n \rtimes I_n$ such that $\phi(0) = (0, 0)$ and $\phi(k) = (n, n)$. Then ϕ is C_∂ -homotopic to the diagonal path map $\Delta: I_n \rightarrow I_n \rtimes I_n$ given by $\Delta(i) = (i, i)$.*

Proof. Using [8, Prop. 3.6] it is easy to construct a deformation retraction r from $I_n \rtimes I_n$ onto its diagonal diag, which leads to a homotopy

$$F: (I_n \rtimes I_n) \square I_k \rightarrow I_n \rtimes I_n,$$

such that $F|_{(I_n \rtimes I_n) \square \{0\}} = \text{id}$, $F|_{(I_n \rtimes I_n) \square \{k\}} = r$ and additionally

$$F|_{\text{diag} \square \{i\}} = \text{id}_{\text{diag}} \quad (2.4)$$

for any $i \in I_k$. For any path-map $\phi: I_s \rightarrow I_n \rtimes I_n$ define a digraph map

$$\phi \square \text{id}_{I_k}: I_s \square I_k \rightarrow (I_n \rtimes I_n) \square I_k.$$

Then the composition

$$\Phi := F \circ (\phi \square \text{id}_{I_k}): I_s \square I_k \rightarrow I_n \rtimes I_n$$

has the following properties: $\Phi|_{I_s \square \{0\}} = \phi$ and $\Phi|_{I_s \square \{k\}}$ is the digraph map onto diag such that $\Phi(0, k) = (0, 0)$, $\Phi(s, k) = (n, n)$. Now by (2.4) $\Phi|_{I_s \square \{0\}} = \phi$ and $\Phi|_{I_s \square \{k\}}$ are homotopic and, hence, C_∂ -homotopic. It remains to observe that the path-maps

$$\Phi|_{I_s \square \{k\}}: I_s \rightarrow \text{diag}$$

and $\Delta: I_n \rightarrow \text{diag}$ are C_∂ -homotopic (for example, using [8, Thm. 4.13]). ■

Definition 2.15 Let $\phi: I_m \rightarrow G$ be a path-map. An extension ϕ^E of ϕ is any path-map

$$\phi^E: I_n \rightarrow G, \quad I_n \in \mathcal{I}.$$

that is given by the composition $\phi \circ h$, where $h: I_n \rightarrow I_m$ is a shrinking map.

Note that any extension ϕ^E of ϕ by means of shrinking map $h: I_n \rightarrow I_m$ satisfies the conditions

$$\phi^E(0) = \phi(0), \quad \phi^E(n) = \phi(m).$$

The following technical result will be used in Section 3 to describe the fundamental groupoid of the product of digraphs.

Proposition 2.16 Let $\psi = (\psi_1, \psi_2): I_m \rightarrow G \times H$ be a path-map and $h_1, h_2: I_n \rightarrow I_m$ be shrinking maps that induce extensions $\psi_1^E: I_n \rightarrow G$ and $\psi_2^E: I_n \rightarrow H$. Consider the digraph map

$$\psi' := (\psi_1^E, \psi_2^E): I_n \rightarrow G \times H.$$

Then $\psi \stackrel{C_\partial}{\simeq} \psi'$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} I_n & = & I_n & & \\ \downarrow \Delta_n & & \downarrow \delta & \searrow \psi' & \\ I_n \times I_n & \xrightarrow{h_1 \times h_2} & I_m \times I_m & \xrightarrow{\psi_1^E \times \psi_2^E} & G \times H \\ & & \uparrow \Delta_m & \nearrow \psi & \\ & & I_m & & \end{array}$$

in which Δ_i denotes the natural diagonal inclusions, and $\delta = (h_1 \times h_2) \circ \Delta_n$. By Lemma 2.14 the path-maps $\delta: I_n \rightarrow I_m \times I_m$ and $\Delta_m: I_m \rightarrow I_m \times I_m$ are C_∂ -homotopic. From commutativity of the diagram, it follows that $\psi \stackrel{C_\partial}{\simeq} \psi'$. ■

For two digraphs G, H define a digraph $\text{Dhom}(G, H)$ with set of vertices

$$V_{\text{Dhom}(G, H)} = \text{Hom}(G, H).$$

and $f \rightarrow g$ in $\text{Dhom}(G, H)$ if there is an *one-step homotopy* as follows:

$$F: G \square I \rightarrow H, \quad F|_{G \square \{0\}} = f, \quad F|_{G \square \{1\}} = g. \quad (2.5)$$

Theorem 2.17 For digraphs D, G, H , there is a natural isomorphism of digraphs

$$\text{Dhom}(D, G \times H) \cong \text{Dhom}(D, G) \times \text{Dhom}(D, H).$$

Proof. By the proof of Proposition 2.13, the map r from (2.3) defines a bijective map of vertices

$$V_{\text{Dhom}(D, G \times H)} \rightarrow V_{\text{Dhom}(D, G)} \times V_{\text{Dhom}(D, H)}.$$

Let F be the homotopy of (2.5) that gives an arrow $f \rightarrow g$ in $\text{Dhom}(G, H)$. Then

$$p \circ F: D \square I \rightarrow G$$

provides a homotopy between $p \circ f$ and $p \circ g$, and

$$q \circ F: D \square I \rightarrow H$$

provides a homotopy between $q \circ f$ and $q \circ g$. Hence, the map r maps arrows to arrows, and it is an injective map on arrows.

Now we prove that it is surjective. Consider a part of the digraph $\text{Dhom}(D, G) \times \text{Dhom}(D, H)$

$$\begin{array}{ccccc}
 & f_1 & \rightarrow & g_1 & \\
 & & & & \\
 g_2 & (f_1, g_2) & \xrightarrow{a_0} & (g_1, g_2) & \\
 \uparrow & \uparrow a_1 & \nearrow a_2 & \uparrow a_3 & \\
 f_2 & (f_1, f_2) & \xrightarrow{a_4} & (g_1, f_2) &
 \end{array} \tag{2.6}$$

which is obtained by the \times -product of the arrows $f_1 \rightarrow g_1$ and $f_2 \rightarrow g_2$. Let us show that a_i belongs to the image of the map r on arrow, for any $i = 0, \dots, 4$.

Consider, at first, the case $i = 0$. By definition of the digraphs $\text{Dhom}(D, G)$ and $\text{Dhom}(D, H)$ it follows from (2.6), that there is a homotopy $F_1: D \square I \rightarrow G$ between f_1 and g_1 , and there is a homotopy $F_2: D \square I \rightarrow H$ between f_2 and g_2 . Let $d: D \square I \rightarrow D$ be the natural projection. The map

$$(F_1, g_2 \circ d): D \square I \rightarrow G \times H$$

gives a homotopy between the map $(f_1, g_2): D \rightarrow G \times H$ and the map $(g_1, g_2): D \rightarrow G \times H$. The homotopy $(F, g_2 \circ d)$ represents an arrow in $\text{Dhom}(D, G \times H)$ that maps by means r to the arrow a_0 in (2.6).

The cases $i = 1, 3, 4$ are similar. Consider the case $i = 2$. Let Ψ denote the composition

$$D \square I \xrightarrow{F_1 \square F_2} G \square H \rightarrow G \times H$$

where the second map is the natural inclusion. This map gives a homotopy between the map $(f_1, f_2): D \rightarrow G \times H$ and the map $(g_1, g_2): D \rightarrow G \times H$, and hence represents an arrow in $\text{Dhom}(D, G \times H)$ that maps by means r to the arrow a_2 in (2.6). The theorem is proved. ■

For two based digraphs G^*, H^* define a based digraph $\text{Dhom}(G^*, H^*)$ with the set of vertices

$$V_{\text{Dhom}(G^*, H^*)} = \text{Hom}(G^*, H^*)$$

consisting of based maps and the base point is given by the trivial map $*$: $G \rightarrow * \in H$. There is an arrow $f \rightarrow g$ in $\text{Dhom}(G^*, H^*)$ if there is a one-step homotopy

$$F: G^* \square I^* \rightarrow H^*, * = 0 \in I, \quad F|_{G \square \{0\}} = f, \quad F|_{G \square \{1\}} = g. \tag{2.7}$$

The products \square and \times of digraphs are defined naturally in the category \mathcal{D}^* of based digraphs, where $* = * \square * \in G \square H$ is a based vertex and, similarly, $* = * \times * \in G \times H$.

Corollary 2.18 *For based digraphs D^*, G^*, H^* , there is a natural isomorphism of based digraphs*

$$\text{Dhom}(D^*, G^* \times H^*) \cong \text{Dhom}(D^*, G^*) \times \text{Dhom}(D^*, H^*).$$

Proof. The result follows from Theorem 2.17 since the correspondence $f \leftrightarrow (f_1, f_2)$ given in Proposition 2.13 preserves the based maps. ■

3 Fundamental groupoids of digraphs

In this section we define a notion of the fundamental groupoid of a digraph and describe its basic properties. We prove the theorem about fundamental groupoid of the products of digraphs. As a corollary we obtain the corresponding results for the fundamental groups of digraphs. Our definition is motivated by the classical definition of groupoid from [13, 1 §7 and 3 §§6,7,8].

A *groupoid* is a small category in which every morphism is an equivalence.

Definition 3.1 i) An *edge* of a digraph $G = (V, E)$ is an ordered pair (v, w) of vertices such that either $v = w$ or there is at least one of the arrows $v \rightarrow w$ or $v \leftarrow w$.

ii) An *edge-path* ξ of a digraph G is a finite nonempty sequence

$$(v_0, v_1)(v_1, v_2) \dots (v_{n-2}, v_{n-1})(v_{n-1}, v_n) \quad (3.1)$$

of edges of the digraph G , where n is any natural number. The vertex v_0 is called the *tail* of the edge-path ξ and v_n the *head* of ξ . We shall write $v_0 = t(\xi)$ and $v_n = h(\xi)$.

iii) A *closed edge-path* at the vertex $v_0 \in V_G$ is an edge-path ξ such that $t(\xi) = h(\xi) = v_0$.

iv) If ξ_1 and ξ_2 are two edge paths with $h(\xi_1) = t(\xi_2)$, then we define the *product edge-path* $\xi_1\xi_2$ consisting of the sequence of edges ξ_1 followed by the edges of ξ_2 .

v) For any edge-path ξ from (3.1) define the inverse edge-path ξ^{-1} by

$$\xi^{-1} := (v_n, v_{n-1})(v_{n-1}, v_{n-2}) \dots (v_1, v_0).$$

We collect some obvious properties of edge-paths in the next statement.

Lemma 3.2 *The edge-paths of a digraph G satisfy the following properties:*

$$\begin{aligned} (\xi_1\xi_2)\xi_3 &= \xi_1(\xi_2\xi_3), \\ (\xi^{-1})^{-1} &= \xi, \\ t(\xi_1\xi_2) &= t(\xi_1), \quad h(\xi_1\xi_2) = h(\xi_2), \\ t(\xi) &= h(\xi^{-1}), \quad h(\xi) = t(\xi^{-1}), \end{aligned}$$

where we assume that all the products are well-defined.

Definition 3.3 i) We shall say that the sequence of three vertices (v_0, v_1, v_2) of a digraph G forms a *triangle* if there is a permutation π of (v_0, v_1, v_2) such that the map $i \mapsto \pi(v_i)$, $i = 0, 1, 2$, provides the isomorphism from the following *triangle*

$$\begin{array}{ccc} 0 & \rightarrow & 1 \\ & \searrow & \downarrow \\ & & 2 \end{array}$$

to the subdigraph of G with the vertices $\pi(v_0), \pi(v_1), \pi(v_2)$.

ii) We shall say that the sequence of four vertices (v_0, v_1, v_2, v_3) of a digraph G forms a *square* if there is a cyclic permutation π of (v_0, v_1, v_2, v_3) such that the map $i \mapsto \pi(v_i)$, $i = 0, 1, 2, 3$, provides the isomorphism of the following *square*

$$\begin{array}{ccc} 1 & \rightarrow & 3 \\ \uparrow & & \uparrow \\ 0 & \rightarrow & 2 \end{array}$$

to the subdigraph of G with the vertices $\pi(v_0), \pi(v_1), \pi(v_2), \pi(v_3)$.

Now we introduce the *edge-path groupoid of a digraph*.

Definition 3.4 Two edge-paths ξ_1 and ξ_2 are called *equivalent* (and we write $\xi_1 \sim \xi_2$) if ξ_1 can be obtained from ξ_2 by a finite sequence of the *local transformations* of the following types or their inverses (where the dots “...” denote the unchanged parts of the edge-paths):

- i) $\dots(v_0, v_1)(v_1, v_2)\dots \mapsto \dots(v_0, v_2)\dots$, provided (v_0, v_1, v_2) forms a triangle in G .
- ii) $\dots(v_0, v_1)(v_1, v_3)\dots \mapsto \dots(v_0, v_2)(v_2, v_3)\dots$ provided (v_0, v_1, v_2, v_3) forms a square in G .
- iii) $\dots(v_0, v_1)(v_1, v_3)(v_3, v_2)\dots \mapsto \dots(v_0, v_2)\dots$ provided (v_0, v_1, v_2, v_3) forms a square in G .
- iv) $\dots(v_0, v_1)(v_1, v_0)\dots \mapsto \dots(v_0, v_0)\dots$ provided $v_0 \rightarrow v_1$ or $v_1 \rightarrow v_0$, or $v_0 = v_1$.
- v) $\dots(v_0, v_0)(v_0, v_1)\dots \mapsto \dots(v_0, v_1)\dots$

Using transformation iv) and v), we obtain also that

- vi) $\dots(v_0, v_1)(v_1, v_1)\dots \mapsto \dots(v_0, v_1)\dots$

It follows directly from definition that the relation \sim has the following properties.

Proposition 3.5 *The relation “ \sim ” is an equivalence relation on the set of edge-paths of the digraph G . It has the following properties:*

- i) If $\xi_1 \sim \xi_2$ then $t(\xi_1) = t(\xi_2)$, $h(\xi_1) = h(\xi_2)$.
- ii) If $\xi_1 \sim \xi'_1$, $\xi_2 \sim \xi'_2$ and $t(\xi_2) = h(\xi_1)$, then $\xi_1\xi_2 \sim \xi'_1\xi'_2$.
- iii) Let $t(\xi) = v_0, h(\xi) = v_1$, then $(v_0, v_0)\xi \sim \xi \sim \xi(v_1, v_1)$.
- iv) If $\xi_1 \sim \xi_2$ then $\xi_1^{-1} \sim \xi_2^{-1}$.

For a path-map $\phi: I_n \rightarrow G$ with $v_i = \phi(i) \in V$, we have, for any $i = 0, \dots, n-1$, at least one of the following relations:

$$v_i = v_{i+1}, \quad v_i \rightarrow v_{i+1}, \quad v_{i+1} \rightarrow v_i.$$

Hence, the path-map ϕ determines the following edge-path in G :

$$\xi_\phi = (\phi(0), \phi(0))(\phi(0), \phi(1)) \dots (\phi(n-1), \phi(n)).$$

Theorem 3.6 *Two path-maps $\phi: I_n \rightarrow G$ and $\psi: I_m \rightarrow G$ with $\phi(0) = \psi(0), \phi(n) = \psi(m)$ are C_∂ -homotopic if and only if $\xi_\psi \sim \xi_\phi$.*

Proof. The proof is similar to [8, Theorem 4.13] where the case of loops was treated. ■

Proposition 3.7 *The following identities are true for path-maps ϕ and ψ on G*

$$(\xi_\phi)^{-1} \sim \xi_{\hat{\phi}}, \quad \xi_{\phi \vee \psi} \sim \xi_\phi \xi_\psi,$$

where $\hat{\phi}$ is the inverse path-map and $\phi \vee \psi$ is the concatenation of ϕ and ψ assuming that it is well-defined.

The proof is trivial.

Denote by $[\xi]$ the equivalence class of the edge-path ξ under the relation “ \sim ”. As follows from Proposition 3.5 the following notations make sense:

$$t([\xi]) := t(\xi), \quad h([\xi]) := h(\xi)$$

and

$$[\xi]^{-1} := [\xi^{-1}], \quad [\xi_1] \circ [\xi_2] := [\xi_1 \xi_2],$$

provided $\xi_1 \xi_2$ is well-defined. The following statement follows from Lemma 3.2 and Proposition 3.5.

Theorem 3.8 For any digraph G , the vertex set of G as the set of objects and the set of the equivalence classes of edge-paths ξ as morphisms from $t(\xi)$ to $h(\xi)$, form a category $\mathcal{E}(G)$ that is a groupoid. The composition of two morphisms $[\xi_1]$ and $[\xi_2]$ is given by $[\xi_1] \circ [\xi_2]$, and the inverse morphism of $[\xi]$ is $[\xi]^{-1}$.

The groupoid $\mathcal{E}(G)$ is called *the fundamental groupoid of the digraph G* . We shall denote by $\text{Hom}_{\mathcal{E}(G)}(v, w)$ the set of morphisms from $v \in V$ to $w \in V$ in the category $\mathcal{E}(G)$, or simply $\text{Hom}(v, w)$ if the digraph G is clear from the context.

Let $v \in V_G$ be a vertex in a digraph G . Consider the edge-paths ξ in G with $t(\xi) = h(\xi) = v$. These edges-paths form a group with the neutral element (v, v) and with the product of edge-paths. Denote this group by $E(G, v)$.

Proposition 3.9 We have an isomorphism

$$E(G, v) \cong \pi_1(G^v).$$

Proof. For any path-map $\phi: I_n \rightarrow G$ with $\phi(0) = \phi(n) = v$ we already define an edge-path ξ_ϕ with $t(\xi_\phi) = h(\xi_\phi) = v$. By Theorem 3.6, the map

$$\begin{aligned} \Theta: \pi_1(G, v) &\rightarrow E(G, v) \\ \Theta([\phi]) &= [\xi_\phi] \end{aligned}$$

is well-defined and preserves the group operations by Proposition 3.7. The map Θ is an epimorphism and a monomorphism as follows from Theorem 3.6. ■

Let \mathcal{G} and \mathcal{H} be groupoids. We shall consider a functor $\mathcal{F}: \mathcal{G} \rightarrow \mathcal{H}$ as a morphism of groupoids. Thus we obtain the category \mathcal{Grpd} of groupoids and morphisms of groupoids.

Proposition 3.10 The fundamental groupoid is a functor

$$\mathcal{E}: \mathcal{D} \rightarrow \mathcal{Grpd}.$$

Proof. Let $f: G \rightarrow H$ be a digraph map. For any edge-path

$$\xi = (v_0, v_1) \dots (v_{n-1}, v_n)$$

of the digraph G define an edge-path $f_*(\xi)$ of the digraph H by the rule

$$f_*(\xi) = (f(v_0), f(v_1)) \dots (f(v_{n-1}), f(v_n)).$$

By Definitions 2.3 and 3.1 the edge-path $f_*(\xi)$ is well defined. Using Definition 3.4 it is an easy exercise to check that $\xi_1 \sim \xi_2$ implies $f_*(\xi_1) \sim f_*(\xi_2)$. Thus we obtain a well defined function $f_\# : \mathcal{E}(G) \rightarrow \mathcal{E}(H)$ that satisfies the relations $f_\#(1_v) = 1_{f(v)}$ ($v \in V_G$) and $f_\#(\xi_1 \circ \xi_2) = f_\#(\xi_1) \circ f_\#(\xi_2)$. ■

Now we recall the definition of product of groupoids [5, §6.4]. The product $\mathcal{C}_1 \times \mathcal{C}_2$ of two groupoids \mathcal{C}_1 and \mathcal{C}_2 is a groupoid with the set of objects $Ob(\mathcal{C}_1 \times \mathcal{C}_2)$ consisting of all ordered pairs (A_1, A_2) where $A_1 \in Ob(\mathcal{C}_1), A_2 \in Ob(\mathcal{C}_2)$. The set $Mor((A_1, A_2), (B_1, B_2))$ consists of ordered pairs of morphisms (f_1, f_2) where $f_1: A_1 \rightarrow B_1, f_2: A_2 \rightarrow B_2$ are the morphisms of the categories \mathcal{C}_1 and \mathcal{C}_2 , respectively. The composition of morphisms and the inverse morphism in $\mathcal{C}_1 \times \mathcal{C}_2$ are defined by a natural way

$$(g_1, g_2) \circ (f_1, f_2) = (g_1 f_1, g_2 f_2), \quad (f_1, f_2)^{-1} = (f_1^{-1}, f_2^{-1}).$$

We have the natural projection functors

$$\pi_1: \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1, \quad \pi_2: \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_2$$

such that for any functors $f_1: \mathcal{B} \rightarrow \mathcal{C}_1, f_2: \mathcal{B} \rightarrow \mathcal{C}_2$ there is a unique functor $f: \mathcal{B} \rightarrow \mathcal{C}_1 \times \mathcal{C}_2$ such that $\pi_1 f = f_1, \pi_2 f = f_2$.

Theorem 3.11 *Let G, H be digraphs. Then the groupoid $\mathcal{E}(G \rtimes H)$ is isomorphic to $\mathcal{E}(G) \times \mathcal{E}(H)$.*

Proof. The natural projections of digraphs $p: G \rtimes H \rightarrow G, q: G \rtimes H \rightarrow H$ induce morphisms of groupoids

$$\mathcal{E}(p): \mathcal{E}(G \rtimes H) \rightarrow \mathcal{E}(G), \quad \mathcal{E}(q): \mathcal{E}(G \rtimes H) \rightarrow \mathcal{E}(H)$$

which determines a morphism of groupoids

$$f: \mathcal{E}(G \rtimes H) \rightarrow \mathcal{E}(G) \times \mathcal{E}(H).$$

Recall that $Ob(\mathcal{E}(G \rtimes H)) = Ob(\mathcal{E}(G) \times \mathcal{E}(H)) = V_G \times V_H$. The morphism f is the identity map on the set of objects $V_G \times V_H$, and for any morphism in $\mathcal{E}(G \rtimes H)$ that is given by a class $[\xi]$ of an edge-path ξ we have

$$f([\xi]) = ([p_{\#}(\xi)], [q_{\#}(\xi)]).$$

We prove, at first, that the map f is surjective. For an edge-path $\xi_1 = (v_0, v_1) \dots (v_{n-1}, v_n)$ in G and an edge-path $\xi_2 = (w_0, w_1) \dots (w_{m-1}, w_m)$ in H , we define an edge-path ξ in $G \rtimes H$ such that

$$f([\xi]) = ([p_{\#}(\xi)], [q_{\#}(\xi)]) = ([\xi_1], [\xi_2]). \quad (3.2)$$

Without restriction of generality we can suppose that $n \geq m$. By Definition 3.4, we have

$$\xi_2 \sim \xi_2' := (w_0, w_1) \dots (w_{m-1}, w_m) \underbrace{(w_m, w_m), \dots (w_m, w_m)}_{n-m \text{ times}}.$$

Define an edge-path ξ in $G \rtimes H$ as

$$\xi = ((v_0, w_0), (v_1, w_1)) \dots ((v_{m-1}, w_{m-1}), (v_m, w_m)) \dots ((v_{n-1}, w_m), (v_n, w_m)).$$

By definition of \rtimes -product this is, indeed, an edge-path and the condition (3.2) is satisfied. Hence the map f is surjective.

Now we prove that the map f is injective. Let ξ_1 and ξ_2 be two edge-paths in $G \rtimes H$ such that

$$t(\xi_1) = t(\xi_2) = (v_0, w_0), \quad h(\xi_1) = h(\xi_2) = (v_n, w_n)$$

and

$$p_{\#}(\xi_1) \sim p_{\#}(\xi_2), \quad q_{\#}(\xi_1) \sim q_{\#}(\xi_2). \quad (3.3)$$

Define path-maps $\phi: I_n \rightarrow G \rtimes H$ and $\psi: I_m \rightarrow G \rtimes H$ in such a way that $\xi_1 = \xi_{\phi}, \xi_2 = \xi_{\psi}$. Note, that these path-maps can be not unique. From definition of the projections p and q and Theorem 3.6 we obtain

$$p_{\#}(\xi_1) = \xi_{p\phi}, \quad p_{\#}(\xi_2) = \xi_{p\psi}, \quad q_{\#}(\xi_1) = \xi_{q\phi}, \quad q_{\#}(\xi_2) = \xi_{q\psi}$$

and

$$p\phi \stackrel{C_{\partial}}{\cong} p\psi \text{ in } G, \quad (3.4)$$

$$q\phi \stackrel{C_{\partial}}{\cong} q\psi \text{ in } H. \quad (3.5)$$

We would like to conclude from (3.4)-(3.5) that $\phi \stackrel{C_{\partial}}{\cong} \psi$. Then, by Theorem 3.6, $\xi_1 = \xi_{\phi} \sim \xi_{\psi} = \xi_2$, and hence the map f in (3.2) is injective. It is sufficient to prove $\phi \stackrel{C_{\partial}}{\cong} \psi$ in the following cases:

- 1) in (3.4) we have an one-step direct C_∂ -homotopy and in (3.5) equality;
- 2) in (3.4) we have an one-step inverse C_∂ -homotopy and in (3.5) equality;
- 3) in (3.4) and in (3.5) we have an one-step direct C_∂ -homotopy;
- 4) in (3.4) and in (3.5) we have an one-step inverse C_∂ -homotopy;
- 5) in (3.4) we have an one-step direct C_∂ -homotopy and in (3.5) an one-step inverse C_∂ -homotopy.

From these cases the general case follows. Note that the cases 1 and 2 follows directly from the cases 3 and 4. We consider only the case 5. In other cases the argument is similar and simpler.

Let (h_1, F_1) be a one-step direct C_∂ -homotopy from $p\phi$ to $p\psi$ that is given by a shrinking map $h_1: I_n \rightarrow I_m$ and the commutative diagram

$$\begin{array}{ccccc}
I_n & \rightarrow & C_{h_1} & \leftarrow & I_m \\
\downarrow p\phi & & \downarrow F_1 & & \downarrow p\psi \\
G & = & G & = & G.
\end{array} \tag{3.6}$$

This diagram extends to the commutative diagram

$$\begin{array}{ccccccc}
I_n & \xrightarrow{=} & I_n & \xrightarrow{p\phi} & G & & \\
\downarrow & & \downarrow & & \parallel & & \\
I_n \square I^- & \xrightarrow{S_1} & C_{h_1} & \xrightarrow{F_1} & G & & \\
\uparrow & & \uparrow & & \parallel & & \\
I_n & \xrightarrow{h_1} & I_m & \xrightarrow{p\psi} & G. & &
\end{array} \tag{3.7}$$

It follows from (3.7) that

$$p\phi \simeq p\psi h_1 \text{ in } G, \tag{3.8}$$

which implies

$$(p\phi, q\psi h_2) \simeq (p\psi h_1, q\psi h_2) \text{ in } G \times H. \tag{3.9}$$

Let (h_2, F_2) be a one-step inverse C_∂ -homotopy from $q\phi$ to $q\psi$ that is given by a shrinking map $h_2: I_n \rightarrow I_m$ and the commutative diagram

$$\begin{array}{ccccc}
I_n & \rightarrow & C_{h_2}^- & \leftarrow & I_m \\
\downarrow q\phi & & \downarrow F_1 & & \downarrow q\psi \\
H & = & H & = & H.
\end{array} \tag{3.10}$$

This diagram extends to the commutative diagram

$$\begin{array}{ccccccc}
I_n & \xrightarrow{=} & I_n & \xrightarrow{q\phi} & H & & \\
\downarrow & & \downarrow & & \parallel & & \\
I_n \square I & \xrightarrow{S_2} & C_{h_2}^- & \xrightarrow{F_2} & H & & \\
\uparrow & & \uparrow & & \parallel & & \\
I_n & \xrightarrow{h_2} & I_m & \xrightarrow{q\psi} & H. & &
\end{array} \tag{3.11}$$

It follows that

$$q\phi \simeq q\psi h_2 \text{ in } H$$

and, hence,

$$(p\phi, q\phi) \simeq (p\phi, q\psi h_2) \text{ in } G \times H.$$

Together with (3.9) this yields

$$(p\phi, q\phi) \simeq (p\psi h_1, q\psi h_2) \text{ in } G \times H.$$

By Proposition 2.16 we have

$$(p\psi h_1, q\psi h_2) \stackrel{C_\partial}{\simeq} (p\psi, q\psi) \text{ in } G \times H,$$

which implies

$$\phi = (p\phi, q\phi) \stackrel{C_\partial}{\simeq} (p\psi, q\psi) = \psi \text{ in } G \times H,$$

which was to be proved. ■

Corollary 3.12 *For based digraphs G^* and H^* there is a natural isomorphism*

$$\pi_1(G^* \times H^*) \cong \pi_1(G^*) \times \pi_1(H^*)$$

where $\pi_1(G^*) \times \pi_1(H^*)$ is the direct product of fundamental groups.

Proof. Follows from Proposition 3.9 and Theorem 3.11. ■

Theorem 3.13 *For digraphs G and H , the natural inclusion $\sigma: G \square H \rightarrow G \times H$ induces an isomorphism of fundamental groupoids*

$$\mathcal{E}(G \square H) \cong \mathcal{E}(G \times H).$$

In particular, for the based digraphs, this map induces an isomorphism

$$\pi_1(G^* \square H^*) \cong \pi_1(G^* \times H^*) \cong \pi_1(G^*) \times \pi_1(H^*)$$

of fundamental groups.

Proof. The inclusion σ induces a morphism $\sigma_\# : \mathcal{E}(G \square H) \rightarrow \mathcal{E}(G \times H)$ of groupoids. We will prove that it is surjective and injective.

For any edge-path ξ in $G \times H$, define an edge-path ξ^\square in $G \square H$, such that $\xi \sim \xi^\square$ in $G \times H$. To that end we transform any diagonal edge $((v_1, w_1), (v_2, w_2))$ of ξ in $G \times H$ to the edge path

$$((v_1, w_1), (v_1, w_2))((v_1, w_2), (v_2, w_2)), \quad (3.12)$$

that lies in $G \square H$, using transformation (i) of Definition 3.4 in $G \times H$. Doing that to all diagonal edges of ξ , we obtain an edge-path ξ^\square as was claimed above. This implies immediately, that $\sigma_\#$ is surjective, since $\sigma_\#([\xi^\square]) = [\xi]$.

Now let us prove the following claim: if ξ, η are edge-paths in $G \times H$ which are equivalent in $G \times H$, then $\xi^\square \sim \eta^\square$ in $G \square H$. If this is already known, then, for any two edge-paths ξ and η in $G \square H$, such that $\xi \sim \eta$ in $G \times H$, we have $\xi^\square = \xi$, $\eta^\square = \eta$, and, hence, $\xi \sim \eta$ in $G \square H$, which implies the injectivity of $\sigma_\#$.

To prove the above claim, it suffices to assume, that η is obtained from ξ by one elementary transformation in $G \times H$. By Definition 3.4 any elementary transformation is done along an embedded digraph $S \subset G \times H$, where S is isomorphic to one of the following digraphs: a single vertex digraph, $0 \rightarrow 1$, $0 \xleftrightarrow{\leftarrow} 1$, the triangle, the square. Let P be projection of S onto G and Q be projection of S onto H . Then $P \square Q$ is a subgraph of $G \square H$. By inspecting all the above cases of S , one sees that $P \square Q$ is always contractible. By the assumption that η is obtained from ξ by one elementary transformation, we have

$$\xi = \gamma_1 \alpha \gamma_2, \quad \eta = \gamma_1 \beta \gamma_2,$$

where γ_1, γ_2 are edge-paths in $G \times H$, α, β are edge-paths in $P \times Q$, and $t(\alpha) = t(\beta), h(\alpha) = h(\beta)$, where α is transformed to β along S . By definition of the operation $\xi \rightarrow \xi^\square$, we obtain

$$\xi^\square = \gamma_1^\square \alpha^\square \gamma_2^\square, \quad \eta^\square = \gamma_1^\square \beta^\square \gamma_2^\square,$$

where

$$t(\alpha^\square) = t(\beta^\square), \quad h(\alpha^\square) = h(\beta^\square).$$

By the contractibility of $P \square Q$, the edge-paths $\alpha^\square, \beta^\square$ are equivalent in $P \square Q$. Hence $\xi^\square \sim \eta^\square$ in $G \square H$, which finishes the proof. ■

The notion of homology groups $H_p(G, \mathbb{Z})$ of digraphs was introduced in [10] (see also [6], [7], [9]). The physical applications of homology (cohomology) theory of digraphs requires development of effective methods of computing of these groups.

Using isomorphism between the first homology group and the abelization of the fundamental group for digraphs [8, Thm. 4.23] and applying Theorem 3.13, we obtain the following result.

Theorem 3.14 *For any two connected digraphs G, H we have*

$$H_1(G \square H, \mathbb{Z}) \cong H_1(G \times H, \mathbb{Z}) \cong H_1(G, \mathbb{Z}) \oplus H_1(H, \mathbb{Z}).$$

4 Geometric realization and Van Kampen theorem

In this section for any finite digraph $G = (V, E)$ we construct a 2-dimensional finite CW-complex $K = \Delta(G)$ (with topological space $|K|$) for which the set of 0-dimensional cells coincides with the set of vertices V . We prove the functoriality of Δ and obtain an isomorphism

$$\text{Hom}_{\mathcal{E}(G)}(v, w) \cong \text{Hom}_{\mathcal{P}(|K|)}(v, w), \quad v, w \in V$$

where $\text{Hom}_{\mathcal{E}(G)}(v, w)$ is the set of morphisms from v to w of the groupoid $\mathcal{E}(G)$ and $\text{Hom}_{\mathcal{P}(|K|)}(v, w)$ is the set of morphisms from v to w of the fundamental groupoid $\mathcal{P}(|K|)$ of the topological space $|K|$ [13, Chapter 1, §7]. This implies, in particular, that for any vertex $v \in V$

$$\pi_1(G, v) \cong \pi_1(|K|, v).$$

Then we obtain a Van Kampen theorem for the fundamental groupoids of digraphs and provide several examples which illustrate this theorem.

At first we need several technical definitions and lemmas.

Definition 4.1 Let $\{G_i\}_{i \in A}$ be a family of subdigraphs of one digraph, where A is any index set.

i) The *union* $G = \bigcup_{i \in A} G_i$ of digraphs G_i is a digraph G such that

$$V_G = \bigcup_{i \in A} V_{G_i}, \quad E_G = \bigcup_{i \in A} E_{G_i}$$

ii) The *intersection* $G = \bigcap_{i \in A} G_i$ is a digraph G such that

$$V_G = \bigcap_{i \in A} V_{G_i}, \quad E_G = \bigcap_{i \in A} E_{G_i}.$$

Now, for any finite digraph $G = (V, E)$, we construct functorially a 2-dimensional cell complex $K = \Delta(G)$.

The 0-dimensional skeleton K^0 of K consists of the set of vertices V . Let $D^1 = [0, 1]$ denote the standard closed unit interval which is a closed 1-cell with the boundary $\partial D^1 = \{0, 1\}$.

Let P be the set of all ordered pairs (v, w) , where $v, w \in V$, such that $v \rightarrow w$; if also $w \rightarrow v$ then we choose in P only one of the pairs (v, w) , (w, v) . For any pair $(v, w) \in P$, we attach a one-dimensional cell D^1 to K^0 , using attaching map

$$\phi_{v,w}: \partial D^1 \rightarrow K^0, \quad \phi_{v,w}(0) = v, \quad \phi_{v,w}(1) = w.$$

Now, we define 1-dimensional skeleton K^1 of K by attaching to K^0 1-dimensional cells D^1 according to the maps $\phi_{v,w}$ for all $(v, w) \in P$.

Let T be the set of subdigraphs of G that are isomorphic to the triangle from Definition 3.3 i). For any subdigraph

$$\tau = \begin{array}{ccc} v_0 & \rightarrow & v_1 \\ & \searrow & \downarrow \\ & & v_2, \end{array} \quad (4.1)$$

from T , we attach to K^1 a standard triangle $D^2 \subset \mathbb{R}^2$ with the vertices $\{a_0, a_1, a_2\}$ and with boundary $\partial D^2 = [a_0, a_1] \cup [a_1, a_2] \cup [a_0, a_2]$ using attaching map

$$\phi_\tau: \partial D^2 \rightarrow K^1, \quad \phi_\tau([a_i, a_j]) = [v_i, v_j].$$

Let S be the set of all subdigraphs of G that are isomorphic to the square. For any subdigraph

$$\sigma = \begin{array}{ccc} v_0 & \rightarrow & v_1 \\ \downarrow & & \downarrow \\ v_2 & \rightarrow & v_3 \end{array} \quad (4.2)$$

from S , we attach to K^1 a standard square $D^2 \subset \mathbb{R}^2$ with the vertices $\{a_0, a_1, a_2, a_3\}$ with boundary $\partial D^2 = [a_0, a_1] \cup [a_0, a_2] \cup [a_1, a_3] \cup [a_2, a_3]$ using attaching map

$$\phi_\sigma: \partial D^2 \rightarrow K^1, \quad \phi_\sigma([a_i, a_j]) = [v_i, v_j]. \quad (4.3)$$

Now we define $\Delta(G) = K = K^2$ as the cell complex that is obtained from K^1 by attaching all the triangles and squares as above.

Proposition 4.2 *For any digraph map $f: G \rightarrow H$ we can define a cellular map*

$$\Delta_f: \Delta(G) \rightarrow \Delta(H)$$

which coincides with f on the set of 0-dimensional cells (that is, with V_G) in such a way that we obtain a functor Δ from the category of digraphs \mathcal{D} to the category of CW-complexes (with cellular maps).

Proof. By definition of a digraph map, f can map triangles to triangles or edges or vertices, and squares to squares or triangles, or edges, or vertices. Now, it follows that the map Δ_f , that is firstly defined on vertices as f , extends uniquely to a cellular map $\Delta(G) \rightarrow \Delta(H)$. ■

For a digraph $G = (V, E)$, let $\mathcal{P}(|K|)$ denote the fundamental groupoid of the topological space $|K|$, where $K = \Delta(G)$. The class of the path $\phi: [0, 1] \rightarrow |K|$ in $\mathcal{P}(|K|)$ we denote by $[\phi]$. For the points $v, w \in |K|$ we denote by $\text{Hom}_{\mathcal{P}(|K|)}(v, w)$ the set of morphisms from v to

w in $\mathcal{P}(|K|)$. Any vertex $v \in V$ determines a 0-dimensional cell in K and a point in $|K|$, which we continue to denote by v .

Let J_n be the CW-complex that is the subdivision of the closed unit interval $[0, 1]$ in n equal parts (1-cells) by 0-cells $i_0 = 0, \dots, i_n = 1$. For any edge-path $\xi = (v_0, v_1) \dots (v_{n-1}, v_n)$ in the digraph G , define a cellular map $\phi_\xi: J_n \rightarrow K$ by $\phi_\xi(i_k) = v_k$ on the set of 0-cells, and

$$\phi_\xi[i_k, i_{k+1}] = [v_k, v_{k+1}]$$

on the 1-cells. This map defines a path in $|K|$ by

$$|\phi_\xi|: [0, 1] \rightarrow |K|, \quad |\phi_\xi|(0) = v_0 = t(\xi), \quad |\phi_\xi|(1) = v_n = h(\xi).$$

Lemma 4.3 *i) Let $\xi_1 \sim \xi_2$ be edge-paths in a digraph G . Then the maps $|\phi_{\xi_1}|$ and $|\phi_{\xi_2}|$ are homotopic relative to the boundary.*

*ii) If $h(\xi_1) = t(\xi_2)$, then $|\phi_{\xi_1 \xi_2}|$ is homotopic to the path $|\phi_{\xi_1}| * |\phi_{\xi_2}|$ relative to the boundary*

Proof. Follows from Definition 3.4 and the construction of $\Delta(G)$ (this result is a cellular analog of the simplicial case [13, Chpt. 3, §6, 9-11]). ■

It follows from Lemma 4.3 that for any two vertices $v, w \in V$ the map

$$\rho: \text{Hom}_{\mathcal{E}(G)}(v, w) \rightarrow \text{Hom}_{\mathcal{P}(K)}(v, w), \quad \rho([\xi]) = [|\phi_\xi|] \quad (4.4)$$

is well-defined.

Proposition 4.4 *For any two vertices $v, w \in V$ the map ρ in (4.4) is a bijection.*

Proof. This is a cellular version of the simplicial theorem [13, Chapter 3, §6, Theorem 16]. The proof is standard using cellular approximation theorem [11, Chpt. 4.1, Th. 4.8]. ■

Corollary 4.5 *For any digraph $G = (V, E)$ and $v \in V$ we have isomorphisms*

$$\pi_1(G, v) \cong E(G, v) \cong \pi_1(|\Delta(G)|, v).$$

Now we recall several notions from the category theory (see, for example, [5, 6.6]). Let \mathcal{C} be a category. A commutative square **C**

$$\begin{array}{ccc} C_0 & \xrightarrow{i_1} & C_1 \\ \downarrow i_2 & & \downarrow u_1 \\ C_2 & \xrightarrow{u_2} & C \end{array} \quad (4.5)$$

in the category \mathcal{C} is called a *pushout*, if for any commutative diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{i_1} & C_1 \\ \downarrow i_2 & & \downarrow u'_1 \\ C_2 & \xrightarrow{u'_2} & C' \end{array}$$

in the category \mathcal{C} there is a unique morphism $c: C \rightarrow C'$ such that $cu_i = u'_i (i = 1, 2)$.

Now let X be a CW-complex with CW-subcomplexes X_1, X_2 , such that $X = X_1 \cup X_2$, and set $X_0 = X_1 \cap X_2$. Then we obtain a pushout **X** of natural inclusions

$$\begin{array}{ccc} |X_0| & \xrightarrow{i_1} & |X_1| \\ \downarrow i_2 & & \downarrow u_1 \\ |X_2| & \xrightarrow{u_2} & |X| \end{array} \quad (4.6)$$

in the category of topological spaces [5, Chpt. 4 and 6].

Let $A \subset X$ be a subset of a topological space X . Then we can define a full subgroupoid $\mathcal{P}_A(X)$ of the fundamental groupoid $\mathcal{P}(X)$ by the following way [5, Chpt. 6.3]. The elements of $\mathcal{P}_A(X)$ are all classes of homotopy paths relative to the boundary in the space X , joining points of A . Thus, for example, $\mathcal{P}_*(X) = \pi_1(X, *)$ where $*$ is a base point. Any inclusion of topological spaces $j: Y \rightarrow X$ induces a morphism of groupoids $j_*: \mathcal{P}_A(Y) \rightarrow \mathcal{P}_A(X)$. A set A is called *representative* in X if A meets each path-component of the space X . We need the following result.

Theorem 4.6 (Van Kampen Theorem [5, Chpt. 6.7.2]) *Let X be a path-connected space and let \mathbf{X} be the pushout (4.6). If the set $A \subset X$ is representative in X_0, X_1, X_2 then the square*

$$\begin{array}{ccc} \mathcal{P}_A(X_0) & \xrightarrow{i_{1*}} & \mathcal{P}_A(X_1) \\ \downarrow i_{2*} & & \downarrow u_{1*} \\ \mathcal{P}_A(X_2) & \xrightarrow{u_{2*}} & \mathcal{P}_A(X) \end{array} \quad (4.7)$$

is a pushout square in the category of groupoids.

Definition 4.7 We shall call by a *cell* of a digraph G any of the following subdigraphs:

- i) any subdigraph that consists of two adjacent vertices of G with all arrows between them;
- ii) any subdigraph that is a triangle;
- iii) any subdigraphs that is a square (see Definition 3.3).

Theorem 4.8 (Van Kampen Theorem for digraphs) *Let a connected digraph G be the union of two subdigraphs $G = G_1 \cup G_2$ such that any cell of the digraph G lies at least in one of the subdigraphs $G_i (i = 1, 2)$, and let $G_0 = G_1 \cap G_2$. Then the square*

$$\begin{array}{ccc} \mathcal{E}(G_0) & \rightarrow & \mathcal{E}(G_1) \\ \downarrow & & \downarrow \\ \mathcal{E}(G_2) & \rightarrow & \mathcal{E}(G), \end{array} \quad (4.8)$$

in which all morphisms are induced by natural inclusions, is a pushout in the category of groupoids.

Proof. The inclusions of digraphs induce the inclusion of topological spaces of their CW-complexes

$$\begin{array}{ccc} |\Delta(G_0)| & \rightarrow & |\Delta(G_1)| \\ \downarrow & & \downarrow \\ |\Delta(G_2)| & \rightarrow & |\Delta(G)|. \end{array} \quad (4.9)$$

Now the result follows from Theorem 4.6 and Proposition 4.4, since the set of vertices V is representative in $|\Delta(G_0)|, |\Delta(G_1)|$, and $|\Delta(G_2)|$. ■

Corollary 4.9 *Let G^* be a based connected digraph with connected based subdigraphs $G_i^* (i = 1, 2)$, such that $G = G_1 \cup G_2$ and $G_0 = G_1 \cap G_2$ is connected. Under the assumptions of Theorem 4.8*

$$\pi_1(G^*) = \pi_1(G_1^*) * \pi_1(G_2^*) / N, \quad (4.10)$$

*where N is the normal subgroup of the free product generated by all the elements of the form $[x] * [x]^{-1}$ where x is a based loop in G_0 .*

In the following examples we show, that the conditions of Theorem 4.8 cannot be relaxed.

Example 4.10 i) Consider the following based digraph G^*

$$\begin{array}{ccc}
 1 & \longrightarrow & 2 \\
 & \swarrow & \searrow \\
 \downarrow & & * & & \downarrow \\
 & \swarrow & & \swarrow & \\
 3 & \longrightarrow & 4
 \end{array} \tag{4.11}$$

and let G_1 be the subdigraph that is obtained from G by removing the vertex 4 with adjacent arrows, and let G_2 – by similarly removing the vertex 1. Clearly, $G = G_1 \cup G_2$ and the intersection $G_0 = G_1 \cap G_2$ is the following line digraph

$$2 \rightarrow * \rightarrow 3.$$

There are deformation retractions of G_1^* , G_2^* , and G^* to the following cyclic subdigraphs respectively

$$\begin{array}{ccc}
 1 & \longrightarrow & 2 \\
 & \swarrow & \searrow \\
 & * &
 \end{array}
 \quad
 \begin{array}{ccc}
 * & & * \\
 \swarrow & & \swarrow \\
 3 & \longrightarrow & 4
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 1 & \longrightarrow & 2 \\
 & \swarrow & \searrow \\
 & * &
 \end{array}$$

(see [8, Example 3.14]). Hence, by [8]

$$\pi_1(G_1^*) \cong \pi_1(G_2^*) \cong \pi_1(G^*) \cong \mathbb{Z}, \quad \pi_1(G_0^*) = \{e\}, \tag{4.12}$$

which implies that (4.10) is not satisfied. In this case Corollary 4.9 does not apply since the cell given by the square $\{1, 2, 3, 4\}$ lies neither in G_1 nor in G_2 .

ii) For the based digraph G^*

$$\begin{array}{ccc}
 3 & \rightarrow & 2 \\
 \downarrow & \swarrow & \downarrow \\
 1 & \rightleftarrows & *
 \end{array}$$

there is a deformation retraction of G^* onto $(* \rightleftarrows 1)$. Hence, by [8], $\pi_1(G^*) = \{e\}$. The digraph G is the union of two digraphs

$$G_1 = \begin{array}{ccc} 3 & \rightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \rightleftarrows & * \end{array} \quad \text{and} \quad G_2 = \begin{array}{ccc} 3 & \rightarrow & 2 \\ \swarrow & & \downarrow \\ & & * \end{array} \tag{4.13}$$

Then $G_0 = G_1 \cap G_2 = (3 \rightarrow 2 \rightarrow *)$, and

$$\pi_1(G^*) = \pi_1(G_1^*) = \pi_1(G_2^*) = \{e\}, \quad \pi_1(G_0^*) \cong \mathbb{Z},$$

so that (4.10) fails. In this case the cell

$$\begin{array}{ccc}
 3 & & \\
 \downarrow & \swarrow & \\
 1 & \leftarrow & *
 \end{array}$$

does not lie in G_1 or in G_2 and Corollary 4.9 is not applicable.

iii) Consider a digraph G^*

$$\begin{array}{ccc}
 & 1 & \\
 \nearrow & \downarrow \uparrow & \searrow \\
 2 & \rightarrow * & \leftarrow 3
 \end{array} \tag{4.14}$$

There is an evident deformation retraction of G^* onto $(* \rightrightarrows 1)$, hence $\pi_1(G^*) = 0$. We can present G^* as the union of two digraphs

$$G_1 = \begin{array}{ccc} & 1 & \\ \nearrow & \downarrow & \\ 2 & \rightarrow & * \end{array} \quad \text{and} \quad G_2 = \begin{array}{ccccc} & & 1 & & \\ \nearrow & & \uparrow & \searrow & \\ 2 & \rightarrow & * & \leftarrow & 3 \end{array} \quad (4.15)$$

Then $G_0 = G_1 \cap G_2 = (1 \leftarrow 2 \rightarrow *)$, and

$$\pi_1(G^*) = \pi_1(G_1^*) = \pi_1(G_2^*) = \{e\}, \quad \pi_1(G_0^*) = \mathbb{Z}.$$

In this case the cell $(* \rightrightarrows 1)$ does not lie in G_1 or in G_2 , Corollary 4.9 is not applicable, and (4.10) fails.

5 Fundamental groupoids of graphs

The deep connections between Atkin homotopy theory and a homotopy theory for graphs was exhibited in [3] and [4]. In particular, the new notion of the fundamental group for undirected graphs was introduced there. In [8] the notion of the fundamental group for digraphs was introduced, and it was transferred to the category of graphs, using isomorphism between the category of graphs and the full subcategory of symmetric digraphs. The so obtained fundamental group is isomorphic to the fundamental group from [3].

In this section we transfer the results about fundamental groupoids of digraphs to that of undirected graphs, similarly to [8].

We recall shortly notation from [8, §6], that we shall use in this section with minimal changes. To denote graphs and the graph maps, we shall use a bold font, for example, $\mathbf{G} = (\mathbf{V}_{\mathbf{G}}, \mathbf{E}_{\mathbf{G}})$, $\mathbf{f}: \mathbf{G} \rightarrow \mathbf{H}$.

Definition 5.1 A *graph* $\mathbf{G} = (\mathbf{V}_{\mathbf{G}}, \mathbf{E}_{\mathbf{G}})$ is a couple of a set $\mathbf{V}_{\mathbf{G}}$ of *vertices* and a subset $\mathbf{E}_{\mathbf{G}} \subset \{\mathbf{V}_{\mathbf{G}} \times \mathbf{V}_{\mathbf{G}} \setminus \text{diag}\}$ of non-ordered pairs of vertices that are called *edges*. We shall write $v \sim w$ for $(v, w) \in \mathbf{E}_{\mathbf{G}}$.

A *graph map* from a graph $\mathbf{G} = (\mathbf{V}_{\mathbf{G}}, \mathbf{E}_{\mathbf{G}})$ to a graph $\mathbf{H} = (\mathbf{V}_{\mathbf{H}}, \mathbf{E}_{\mathbf{H}})$ is a map

$$\mathbf{f}: \mathbf{V}_{\mathbf{G}} \rightarrow \mathbf{V}_{\mathbf{H}}$$

such that for any edge $v \sim w$ on \mathbf{G} we have either $\mathbf{f}(v) = \mathbf{f}(w)$ or $\mathbf{f}(v) \sim \mathbf{f}(w)$.

As usually, a *based graph* \mathbf{G}^* is the graph \mathbf{G} with the fixed vertex $*$ and a based graph map preserves base vertexes.

The set of all graphs with graph maps forms a category \mathcal{G} . Let us associate to each graph $\mathbf{G} = (\mathbf{V}_{\mathbf{G}}, \mathbf{E}_{\mathbf{G}})$ a symmetric digraph $\mathcal{O}(\mathbf{G}) = G = (V_G, E_G)$ where $V_G = \mathbf{V}_{\mathbf{G}}$ and E_G is defined by the condition $v \rightarrow w \Leftrightarrow v \sim w$. Thus, we obtain a functor \mathcal{O} that provides an isomorphism of the category \mathcal{G} and the full subcategory of symmetric digraphs of the category \mathcal{D} .

The functor \mathcal{O} allows us to transfer the notions and results obtained in category \mathcal{D} to the category \mathcal{G} . In particular, we obtain in this way the definition of the fundamental groupoid of a graph as below.

Definition 5.2 i) A *formal edge* of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is an ordered pair (v, w) of vertices such that $v \sim w$ or $v = w$.

ii) An *edge-path* ξ of a graph \mathbf{G} is a finite nonempty sequence

$$(v_0 v_1)(v_1, v_2) \dots (v_{n-1}, v_n) \quad (5.1)$$

of formal edges of the graph \mathbf{G} . The vertex v_0 is called by the *tail* of the edge-path ξ and v_n the *head* of the edge-path ξ . We write $v_0 = t(\xi), v_n = h(\xi)$.

iii) A *closed edge-path* at the vertex $v_0 \in \mathbf{V}$ is an edge-path ξ such that $t(\xi) = h(\xi) = v_0$.

iv) For two edge paths ξ_1 and ξ_2 with $h(\xi_1) = t(\xi_2)$, we define a *product edge-path* $\xi_1\xi_2$ consisting of the sequence of formal edges ξ_1 followed by the formal edges of ξ_2 .

v) For any edge-path ξ from (5.1) define the inverse edge-path ξ^{-1} by

$$\xi^{-1} := (v_n, v_{n-1})(v_{n-1}, v_{n-2}) \dots (v_1, v_0).$$

It follows directly from the Definition 5.2 that the edge-paths of a graph \mathbf{G} satisfy the following properties:

$$t(\xi_1\xi_2) = t(\xi_1), h(\xi_1\xi_2) = h(\xi_2),$$

$$t(\xi) = h(\xi^{-1}), h(\xi) = t(\xi^{-1}),$$

$$(\xi_1\xi_2)\xi_3 = \xi_1(\xi_2\xi_3),$$

$$(\xi^{-1})^{-1} = \xi,$$

were we suppose that all products are defined.

Define an *edge-path groupoid of a graph* similarly to Section 3.

Definition 5.3 Two edge-paths ξ_1 and ξ_2 in $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ are called *equivalent* (and we write $\xi_1 \sim \xi_2$) if ξ_1 can be obtained from ξ_2 by a finite sequence of the following *local transformations* or their inverses (where the dots “...” denote the unchanged parts of the edge-paths):

i) $\dots(v_0, v_1)(v_1, v_2)\dots \mapsto \dots(v_0, v_2)\dots$ where $v_0 \sim v_2$ or $v_0 = v_2$.

ii) $\dots(v_0, v_1)(v_1, v_2)\dots \mapsto \dots(v_0, v_3)(v_3, v_2)\dots$ where the vertices v_0, v_1, v_2, v_3 are different and $v_0 \sim v_3$ and $v_3 \sim v_2$.

iii) $\dots(v_0, v_1)(v_1, v_2)(v_2, v_3)\dots \mapsto \dots(v_0, v_3)\dots$ where the vertices v_0, v_1, v_2, v_3 are different and $v_0 \sim v_3$.

Note, the list of local transformations in Definition 5.3 follows from Definition 3.4 using inverse functor \mathcal{O}^{-1} on the subcategory of symmetric digraphs, which allows to simplify this list.

The relation “ \sim ” on the set of edge paths of a graph \mathbf{G} is an equivalence relation. We shall denote by $[\xi]$ the equivalence class of the edge-path ξ . For equivalence classes the following notations and operations are well defined

$$t([\xi]) := t(\xi), \quad h([\xi]) := h(\xi), \quad \text{for } t(\xi_1) = h(\xi_1) \quad [\xi_1] \circ [\xi_2] := [\xi_1\xi_2], \quad [\xi]^{-1} = [\xi^{-1}].$$

Theorem 5.4 For any graph \mathbf{G} , the vertex set of \mathbf{G} as the set of objects and the set of the equivalence classes of edge-paths ξ as morphisms from $t(\xi)$ to $h(\xi)$, form a category $\mathcal{E}(\mathbf{G})$ that is a groupoid. The composition of two morphisms $[\xi_1]$ and $[\xi_2]$ is given by $[\xi_1] \circ [\xi_2]$, and the inverse morphism of $[\xi]$ is $[\xi]^{-1}$.

The groupoid $\mathcal{E}(\mathbf{G})$ is called *the fundamental groupoid* of the graph \mathbf{G} .

We denote by $\text{Hom}_{\mathcal{E}(\mathbf{G})}(v, w)$ the set of morphisms from $v \in \mathbf{V}$ to $w \in \mathbf{V}$ in the category $\mathcal{E}(\mathbf{G})$, or simply $\text{Hom}(v, w)$ if the graph \mathbf{G} follows from the context.

Let $v \in \mathbf{V}$ be a vertex in a graph \mathbf{G} . Consider the set of equivalence classes $[\xi]$ of edge-paths ξ of \mathbf{G} such that $t(\xi) = h(\xi) = v$. This set is a group with the neutral element $[(v, v)]$ and with the groupoid product of $\mathcal{E}(\mathbf{G})$. Denote this group by $\mathbf{E}(\mathbf{G}, v)$. Note, that the fundamental group $\pi_1(\mathbf{G}^*)$ of a based graph \mathbf{G}^* was introduced in [3], [4, Proposition 5.6]. It follows from [8] that

$$\mathbf{E}(\mathbf{G}, v) \cong \pi_1(\mathbf{G}^v).$$

Definition 5.5 Let $\mathbf{G} = (\mathbf{V}_{\mathbf{G}}, \mathbf{E}_{\mathbf{G}})$ and $\mathbf{H} = (\mathbf{V}_{\mathbf{H}}, \mathbf{E}_{\mathbf{H}})$ be two graphs.

i) Define the *Cartesian product* $\mathbf{\Pi} = \mathbf{G} \square \mathbf{H}$ as a graph with the set of vertices $\mathbf{V}_{\mathbf{\Pi}} = \mathbf{V}_{\mathbf{G}} \times \mathbf{V}_{\mathbf{H}}$ and with the set of edges $\mathbf{E}_{\mathbf{\Pi}}$ such that $(x, y) \sim (x', y')$ if and only if

$$\text{either } x' = x \text{ and } y \sim y', \text{ or } x \sim x' \text{ and } y = y'.$$

ii) Define a \times -product $\mathbf{P} = \mathbf{G} \times \mathbf{H}$ as a graph with the set of vertices $\mathbf{V}_{\mathbf{P}} = \mathbf{V}_{\mathbf{G}} \times \mathbf{V}_{\mathbf{H}}$ and there is an edge

$$(x, y) \sim (x', y') \text{ for } x, x' \in \mathbf{V}_{\mathbf{G}}; y, y' \in \mathbf{V}_{\mathbf{H}}$$

if one of the conditions is satisfied

$$x' = x, y \sim y'; \text{ or } y' = y, x \sim x'; \text{ or } x \sim x', y \sim y'.$$

Theorem 5.6 *We have an isomorphism of groupoids*

$$\mathcal{E}(\mathbf{G} \square \mathbf{H}) \cong \mathcal{E}(\mathbf{G} \times \mathbf{H}) \cong \mathcal{E}(\mathbf{G}) \times \mathcal{E}(\mathbf{H}).$$

In particular, for based graphs we have an isomorphism of fundamental groups

$$\pi_1(\mathbf{G}^* \square \mathbf{H}^*) \cong \pi_1(\mathbf{G}^* \times \mathbf{H}^*) \cong \pi_1(\mathbf{G}^*) \times \pi_1(\mathbf{H}^*)$$

This property of fundamental groups of graphs is new. We think, that the direct proof of this result, using the definition of π_1 from [3] and [4], can be very nontrivial.

Now we state the Van Kampen Theorem for the fundamental groupoids of graphs. For the fundamental group of graph it was proved in [4].

The union and intersection of subgraphs is defined in the same way as those for digraphs in Definition 4.1.

Definition 5.7 We shall call by *a cell* of a graph \mathbf{G} any of the following subgraphs:

- i) a full subgraph consisting of three vertices (triangle);
- ii) a subgraph consisting of four vertices forming with the edges a square.

Theorem 5.8 (*Van Kampen Theorem for graphs*) *Let $\mathbf{G} = \mathbf{G}_1 \cup \mathbf{G}_2$ be a connected graph, such that any cell of \mathbf{G} lies in one of the subgraphs $\mathbf{G}_1, \mathbf{G}_2$. Set $\mathbf{G}_0 = \mathbf{G}_1 \cap \mathbf{G}_2$. Then the square*

$$\begin{array}{ccc} \mathcal{E}(\mathbf{G}_0) & \rightarrow & \mathcal{E}(\mathbf{G}_1) \\ \downarrow & & \downarrow \\ \mathcal{E}(\mathbf{G}_2) & \rightarrow & \mathcal{E}(\mathbf{G}), \end{array} \quad (5.2)$$

in which all morphisms are induced by natural inclusions, is a pushout in the category of groupoids.

Corollary 5.9 [4] *Let \mathbf{G}^* be a based connected graph with connected based subdigraphs \mathbf{G}_i^* ($i = 1, 2$), such that $\mathbf{G} = \mathbf{G}_1 \cup \mathbf{G}_2$ and $\mathbf{G}_0 = \mathbf{G}_1 \cap \mathbf{G}_2$ is connected. Under the assumptions of Theorem 5.8 we have*

$$\pi_1(\mathbf{G}^*) = \pi_1(\mathbf{G}_1^*) * \pi_1(\mathbf{G}_2^*) / N,$$

*where N is the normal subgroup of the free product generated by all the elements of the form $[x] * [x]^{-1}$ where x is a based loop in \mathbf{G}_0 .*

6 An application to coloring

Now we formulate and prove a natural generalization of the classical Sperner Lemma, using the results of Section 3.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a planar (nondirected) finite connected graph which provides a simplicial triangulation of a simply-connected closed domain $D \subset \mathbb{R}^2$. Let \mathbf{H} be the subdigraph of \mathbf{G} that lies on the boundary ∂D . Let any vertex of \mathbf{G} be colored by one of three colors, say $\{0, 1, 2\}$. Define a digraph $G = (V, E)$ by putting $V = \mathbf{V}$ and defining the set E of arrows according to the colors of vertices as follows:

$$0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0, 0 \rightleftharpoons 0, 1 \rightleftharpoons 1, 2 \rightleftharpoons 2.$$

In particular, we obtain a subdigraph $H = (V_H, E_H)$ of G , that lies on ∂D . Let us fix a vertex $* \in V_H$ and set $n = |V_H|$. Going along ∂D clockwise, starting and ending at $*$, we obtain a loop $\phi: I_n^* \rightarrow H^* \subset G^*$.

Theorem 6.1 *i) If $[\phi] \neq [e]$ in $\pi_1(G^*)$ then there is at least one 3-color triangle in the triangulation of D .*

ii) If $\text{rank } \pi_1(G^) = r$ then there are at least r 3-color triangles in the triangulation of D .*

Proof. Follows from the description of local transformations in Section 3 and the method of [8, Theorem 4.20]. ■

Corollary 6.2 *The number of 3-color triangles in the triangulation of D is at least $\text{rank } H_1(G, \mathbb{Z})$.*

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