Yamabe type equations on finite graphs

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Abstract

Let $G = (V, E)$ be a locally finite graph, $\Omega \subset V$ be a bounded open domain, Δ be the usual graph Laplacian, and $\lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ with respect to Dirichlet boundary condition. Using the mountain pass theorem of Ambrosette and Rabinowitz, We prove that if $\alpha < \lambda_1(\Omega)$, then for any $p > 2$, there exists a positive solution to

$$\begin{cases} -\Delta u - \alpha u = |u|^{p-2}u \ \text{ in } \ \Omega^\circ, \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where Ω° and $\partial\Omega$ denote the interior and the boundary of Ω respectively. Also we consider similar problems involving the p-Laplacian and poly-Laplacian by the same method. Such problems can be viewed as discrete versions of the Yamabe type equation on Euclidean space or compact Riemannian manifolds.

Key words: Sobolev embedding, Yamabe type equation, locally finite graph
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1. Introduction

Let Ω be a domain of $\mathbb{R}^n$ and $W_0^{1,q}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,q}(\Omega)} = \left(\int_\Omega (|\nabla u|^q + |u|^q)\, dx \right)^{1/q}.$$

Then the Sobolev embedding theorem reads

$$W_0^{1,q}(\Omega) \hookrightarrow \begin{cases} L^p(\Omega) \ \text{ for } \ q \le p \le q^* = \frac{nq}{n-q}, & \text{when } \ n > q \\ L^p(\Omega) \ \text{ for } \ q \le p < +\infty, & \text{when } \ q = n \\ C^{1-n/q}(\overline{\Omega}), & \text{when } \ q > n. \end{cases}$$

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The model problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1}$$

or its variants has been extensively studied since 1960s. Let $J : W_0^{1,2}(\Omega) \to \mathbb{R}$ be a functional defined by

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)dx - \frac{1}{p}\int_\Omega |u|^p dx.$$

Clearly the critical points of φ are weak solutions to the problem (1). In the case $2 < p < +\infty$ when $n = 1, 2$, or $2 < p \leq 2^* = 2n/(n-2)$ when $n \geq 3$, one can check that $\sup_{W_0^{1,2}(\Omega)} J = +\infty$ and $\inf_{W_0^{1,2}(\Omega)} J = -\infty$. In [12], Nehari obtained a nontrivial solution of (1) when $\lambda \geq 0$ and $\Omega = (a, b)$, by minimizing J in the manifold

$$\mathcal{N} = \{u \in W_0^{1,2}(\Omega) : \langle J'(u), u\rangle = 0,\ u \neq 0\}.$$

In [13], he proved the existence of infinitely many solutions and in [14], he solved the case where $\Omega = \mathbb{R}^3$, $\lambda > 0$ and $2 < p < 6$ after reduction to an ordinary differential equation. When Ω is unbounded or when $p = 2^*$, there is a lack of compactness in Sobolev spaces because of invariance by translation or by dilation. Some nonexistence results follow from the Pohozaev identity. General existence theorems were first obtained by Strauss [18] when $\Omega = \mathbb{R}^n$ and by Brezis-Nirenberg [5] when $p = 2^*$. The Brezis-Lieb lemma and Lions' concentration compactness principle are important tools in solving those problems. For other existence results for variants of (1), we refer the reader to [21].

Analogous to (1), one can consider the problem

$$\begin{cases} -\Delta_n u = f(x, u) \quad \text{in} \quad \Omega \\ u \in W_0^{1,n}(\Omega), \end{cases} \tag{2}$$

where Δ_n is the n-Laplace operator and $f(x, s)$ has exponential growth as $s \to +\infty$. Instead of the Sobolev embedding theorem, the key tool in solving the problem (2) is the Trudinger-Moser embedding contributed by Yudovich [27], Pohozaev [16], Peetre [15], Trudinger [19] and Moser [11]. In [1], Adimurthi proved an existence of positive solution to (2) by using a method of Nehari manifold. In [6], de Figueiredo, Miyagaki and Ruf considered (2) in the case that Ω is a bounded domain in $\mathbb{R}^2$, by using the critical point theory. In [7], by using the mountain pass theorem without the Palais-Smale condition, do Ó improve the results of [1, 6]. In [8], using the same method, he extended these results to the case that Ω is the whole Euclidean space $\mathbb{R}^n$. For related problems, we refer the reader to [9, 2, 23, 24] and the references there in.

On Riemannian manifolds, an analog of the model problem (1) arises from the Yamabe problem: Let (M, g) be a compact n (≥ 3) dimensional Riemannian manifold without boundary. Does there exist a good metric $\tilde{g}$ in the conformal class of g such that the scalar curvature $R_{\tilde{g}}$ is a constant? This problem was studied by Yamabe [22], Trudinger [20], Aubin [4], and completely solved by Schoen [17]. Though there is no background of geometry or physics, there are still some works concerning the problem (2) on Riemannian manifolds, see for examples [26, 10, 28, 25].

Our goal is to consider problems (1) and (2) when an Euclidean domain Ω is replaced by a graph. Such problems can be viewed as discrete versions of (1) and (2). In this paper, we only concern finite graph, which is one of the simplest graphs. The key point is an observation of pre-compactness of the Sobolev space in our setting. Using the mountain pass theorem due to Ambrossete and Rabinowich [3], we prove the existence of nontrivial solutions to Yamabe type equations on finite graphs.

This paper is organized as follows: In Section 2, we give some notations on graph and state main results. In Section 3, we establish Sobolev embedding such that the mountain pass theorem can be applied to our problems. Local existence results (Theorems 1-3) are proved in Section 4, and global existence results (Theorems 4-6) are proved in Section 5.

2. Settings and main results

Let $G = (V, E)$ be a finite or locally finite graph, where V denotes the vertex set and E denotes the edge set. For any edge $xy \in E$, we assume that its weight $w_{xy} > 0$ and that $w_{xy} = w_{yx}$. The degree of $x \in V$ is defined as $\deg(x) = \sum_{y\sim x} w_{xy}$, where we write $y \sim x$ if $xy \in E$. Let $\mu : V \to \mathbb{R}$ be a finite measure. Then the μ-Laplacian (or Laplacian for short) on G is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x)). \tag{3}$$

The associated gradient form reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{4}$$

Write $\Gamma(u) = \Gamma(u, u)$. For any function $u : V \to \mathbb{R}$, we denote the length of its gradient by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} w_{xy}(u(y) - u(x))^2 \right)^{1/2}. \tag{5}$$

For any positive integer m, we define the length of m-order gradient of u by

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{when } m \text{ is odd} \\ |\Delta^{\frac{m}{2}} u|, & \text{when } m \text{ is even,} \end{cases} \tag{6}$$

where $|\nabla \Delta^{\frac{m-1}{2}} u|$ is defined as in (5) with u is replaced by $\Delta^{\frac{m-1}{2}} u$, and $|\Delta^{\frac{m}{2}} u|$ denotes the usual absolute of the function $\Delta^{\frac{m}{2}} u$. Let Ω be a domain in V. To compare with the Euclidean setting, we denote, for any function $u : V \to \mathbb{R}$,

$$\int_\Omega u d\mu = \sum_{x\in\Omega} \mu(x)u(x). \tag{7}$$

The first eigenvalue of the Laplacian with respect to the Dirichlet boundary condition reads

$$\lambda_1(\Omega) = \inf_{u\not\equiv 0,\, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu}, \tag{8}$$

where $\partial\Omega$ is the boundary of Ω, namely $\partial\Omega = \{x \in \Omega : \exists y \notin \Omega \text{ such that } xy \in E\}$. Moreover, we denote the interior of Ω by $\Omega^\circ = \Omega \setminus \partial\Omega$. Our first result is the following:

Theorem 1. *Let $G = (V, E)$ be a locally finite graph, $\Omega \subset V$ be a bounded open domain with $\Omega^\circ \neq \varnothing$, and $\lambda_1(\Omega)$ be defined as in (8). Then for any $p > 2$ and any $\alpha < \lambda_1(\Omega)$, there exists a solution to the equation*

$$\begin{cases} -\Delta u - \alpha u = |u|^{p-2}u & \text{in} \quad \Omega^\circ \\ u > 0 \ \text{ in } \ \Omega^\circ, \ \ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{9}$$

The p-Laplacian of $u : V \to \mathbb{R}$, namely $\Delta_p u$, is defined in the distributional sense by

$$\int_V (\Delta_p u)\phi d\mu = -\int_V |\nabla u|^{p-2}\Gamma(u, \phi) d\mu, \quad \forall \phi \in C_c(V),$$

where $\gamma(u, \phi)$ is defined as in (4) and $C_c(V)$ denotes the set of all functions with compact support. Point-wisely, $\Delta_p u$ can be written as

$$\Delta_p u(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \left(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x)\right) w_{xy}(u(y) - u(x)).$$

When $p = 2$, Δ_p is the standard graph Laplacian Δ (see (3) above). The first eigenvalue of the p-Laplacian with respect to Dirichlet boundary condition reads

$$\lambda_p(\Omega) = \inf_{u\not\equiv 0,\, u|_{\partial\Omega}=0} \frac{\int_\Omega |\nabla u|^p d\mu}{\int_\Omega |u|^p d\mu}. \tag{10}$$

Our second result can be stated as follows:

Theorem 2. *Let $G = (V, E)$ be a locally finite graph, $\Omega \subset V$ be a bounded open domain with $\Omega^\circ \neq \varnothing$. Let $\lambda_p(\Omega)$ be defined as in (10) for some $p > 1$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:*
(H_1) For any $x \in \Omega$, $f(x, t)$ is continuous in $t \in \mathbb{R}$;
(H_2) For all $(x, t) \in \Omega \times [0, +\infty)$, $f(x, t) \geq 0$, and $f(x, 0) = 0$ for all $x \in \Omega$;
(H_3) There exists some $q > p$ and $s_0 > 0$ such that if $s \geq s_0$, then there holds

$$F(x, s) = \int_0^s f(x, t) dt \leq \frac{1}{q} s f(x, s), \quad \forall x \in \Omega;$$

(H_4) For any $x \in \Omega$, there holds

$$\limsup_{t\to 0+} \frac{f(x, t)}{t^{p-1}} < \lambda_p(\Omega).$$

Then there exists a nontrivial solution to the equation

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in} \quad \Omega^\circ \\ u \geq 0 \ \text{ in } \ \Omega^\circ, \ \ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{11}$$

In Theorem 2, if $p = 2$, then $|s|^{q-2}s$ ($q > 2$) satisfies $(H_1) - (H_4)$. Moreover, the nonlinearities in Theorem 2 include the case of exponential growth as in the problem (2). For further extension, we define an analog of $\lambda_p(\Omega)$ by

$$\lambda_{mp}(\Omega) = \inf_{u\in\mathcal{H}} \frac{\int_\Omega |\nabla^m u|^p d\mu}{\int_\Omega |u|^p d\mu}, \tag{12}$$

where m is any positive integer and $\mathcal{H}$ denotes a set of all functions $u \not\equiv 0$ with $u = |\nabla u| = \cdots = |\nabla^{m-1} u| = 0$ on $\partial\Omega$. Then we have the following:

Theorem 3. *Let $G = (V, E)$ be a locally finite graph and $\Omega \subset V$ be a bounded open domain with $\Omega^\circ \neq \varnothing$. Let $m \geq 2$ be an integer and $p > 1$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:*
(A_1) $f(x, 0) = 0$, $f(x, t)$ is continuous with respect to $t \in \mathbb{R}$;
(A_2) $\limsup_{t\to 0} \frac{|f(x,t)|}{|t|^{p-1}} < \lambda_{mp}(\Omega)$;
(A_3) there exists some $q > p$ and $M > 0$ such that if $|s| \geq M$, then

$$0 < qF(x, s) \leq sf(x, s), \quad \forall x \in \Omega.$$

Then there exists a nontrivial solution to the equation

$$\begin{cases} \mathcal{L}_{m,p}\, u = f(x, u) \quad \text{in} \quad \Omega^\circ \\ |\nabla^j u| = 0 \text{ on } \partial\Omega,\ 0 \leq j \leq m - 1, \end{cases} \tag{13}$$

where $\mathcal{L}_{m,p}\, u$ is defined as follows: for any ϕ with $\phi = |\nabla\phi| = \cdots = |\nabla^{m-1}\phi| = 0$ on $\partial\Omega$, there holds

$$\int_\Omega (\mathcal{L}_{m,p}\, u)\phi d\mu = \begin{cases} \int_\Omega |\nabla^m u|^{p-2}\Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \phi) d\mu, & \text{when } m \text{ is odd}, \\ \int_\Omega |\nabla^m u|^{p-2}\Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}} \phi d\mu, & \text{when } m \text{ is even}. \end{cases}$$

In particular, if $p = 2$, then $\mathcal{L}_{m,p}\, u = (-\Delta)^m u$.

If $G = (V, E)$ is a finite graph, we also have existence results similar to the above theorems. Analogous to Theorem 1, we state the following:

Theorem 4. *Let $G = (V, E)$ be a finite graph. Suppose that $p > 2$ and $h(x) > 0$ for all $x \in V$. Then there exists a solution to the equation*

$$\begin{cases} -\Delta u + hu = |u|^{p-2}u \quad \text{in} \quad V \\ u > 0 \text{ in } V. \end{cases} \tag{14}$$

Similar to Theorem 2, we have

Theorem 5. *Let $G = (V, E)$ be a finite graph. Suppose that $h(x) > 0$ for all $x \in V$. Suppose that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:*
(H_V^1) For any $x \in V$, $f(x, t)$ is continuous in $t \in \mathbb{R}$;
(H_V^2) For all $(x, t) \in V \times [0, +\infty)$, $f(x, t) \geq 0$, and $f(x, 0) = 0$ for all $x \in V$;
(H_V^3) There exists some $q > p > 1$ and $s_0 > 0$ such that if $s \geq s_0$, then there holds

$$F(x, s) = \int_0^s f(x, t)dt \leq \frac{1}{q} sf(x, s), \quad \forall x \in V;$$

(H_V^4) For any $x \in V$, there holds

$$\limsup_{t\to 0+} \frac{f(x, t)}{t^{p-1}} < \lambda_p(V) = \inf_{u\not\equiv 0} \frac{\int_V (|\nabla u|^p + h|u|^p)d\mu}{\int_V |u|^p d\mu}.$$

Then there exists a nontrivial solution to the equation

$$\begin{cases} -\Delta_p u + h|u|^{p-2}u = f(x,u) \quad \text{in} \quad V \\ u \geq 0 \ \text{ in } \ V, \end{cases} \tag{15}$$

where $\Delta_p u$ denotes the p-Laplacian of u.

Finally we have an analog of Theorem 3, namely

Theorem 6. *Let $G = (V, E)$ be a finite graph. Let $m \geq 2$ be an integer and $p > 1$. Suppose that $h(x) > 0$ for all $x \in V$. Assume $f(x,u)$ satisfies the following assumptions:*
(A_V^1) For any $x \in V$, $f(x,0) = 0$, $f(x,t)$ is continuous with respect to $t \in \mathbb{R}$;
(A_V^2) $\limsup_{t\to 0} \frac{|f(x,t)|}{|t|^{p-1}} < \lambda_{mp}(V) = \inf_{u\not\equiv 0} \frac{\int_V (|\nabla^m u|^p + h|u|^p)d\mu}{\int_V |u|^p d\mu}$;
(A_V^3) there exists some $q > p$ and $M > 0$ such that if $|s| \geq M$, then

$$0 < qF(x,s) \leq sf(x,s), \quad \forall x \in V.$$

Then there exists a nontrivial solution to

$$\mathcal{L}_{m,p}\, u + h|u|^{p-2}u = f(x,u) \quad \text{in} \quad V, \tag{16}$$

where $\mathcal{L}_{m,p}\, u$ is defined in the distributional sense: for any function ϕ, there holds

$$\int_V (\mathcal{L}_{m,p}\, u)\phi d\mu = \begin{cases} \int_V |\nabla^m u|^{p-2}\Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}}\phi)d\mu, & \text{when } m \text{ is odd,} \\ \int_V |\nabla^m u|^{p-2}\Delta^{\frac{m}{2}} u \Delta^{\frac{m}{2}}\phi d\mu, & \text{when } m \text{ is even.} \end{cases}$$

3. Preliminary analysis

Let $G = (V, E)$ be a locally finite graph, $\Omega \subset V$ be an open domain, $\partial\Omega$ be its boundary and Ω° be its interior. For any $p > 1$, $W^{m,p}(\Omega)$ is defined as a space of all functions $u : V \to \mathbb{R}$ satisfying

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{k=0}^{m} \int_\Omega |\nabla^k u|^p d\mu \right)^{1/p} < \infty. \tag{17}$$

Denote $C_0^m(\Omega)$ be a set of all functions $u : \Omega \to \mathbb{R}$ with $u = |\nabla u| = \cdots = |\nabla^{m-1} u| = 0$ on $\partial\Omega$. We denote $W_0^{m,p}(\Omega)$ be the completion of $C_0^m(\Omega)$ under the norm (17). If we further assume that Ω is a bounded domain, then Ω is a finite set. Observing that the dimension of $W_0^{m,p}(\Omega)$ is finite when Ω is bounded, we have the following Sobolev embedding:

Theorem 7. *Let $G = (V, E)$ be a locally finite graph, Ω be a bounded open domain of V such that $\Omega^\circ \neq \varnothing$. Let m be any positive integer and $p > 1$. Then $W_0^{m,p}(\Omega)$ is embedded in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$. In particular, there exists a constant C depending only on m, p and Ω such that*

$$\left(\int_\Omega |u|^q d\mu \right)^{1/q} \leq C \left(\int_\Omega |\nabla^m u|^p d\mu \right)^{1/p} \tag{18}$$

for all $1 \leq q \leq +\infty$ and for all $u \in W_0^{m,p}(\Omega)$. Moreover, $W_0^{m,p}(\Omega)$ is pre-compact, namely, if u_k is bounded in $W_0^{m,p}(\Omega)$, then up to a subsequence, there exists some $u \in W_0^{m,p}(\Omega)$ such that $u_k \to u$ in $W_0^{m,p}(\Omega)$.

Proof. Since Ω is a finite set, $W_0^{m,p}(\Omega)$ is a finite dimensional space. Hence $W_0^{m,p}(\Omega)$ is pre-compact. We are left to show (18). It is not difficult to see that

$$\|u\|_{W_0^{m,p}(\Omega)} = \left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1/p} \tag{19}$$

is a norm equivalent to (17) on $W_0^{m,p}(\Omega)$. Hence for any $u \in W_0^{m,p}(\Omega)$, there exists some constant C depending only on m, p and Ω such that

$$\left(\int_\Omega |u|^p d\mu\right)^{1/p} = \left(\sum_{x\in\Omega} \mu(x)|u(x)|^p\right)^{1/p} \leq C\left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1/p}, \tag{20}$$

since $W_0^{m,p}(\Omega)$ is a finite dimensional space. Denote $\mu_{\min} = \min_{x\in\Omega}\mu(x)$. Then (20) leads to

$$\|u\|_{L^\infty(\Omega)} \leq \frac{C}{\mu_{\min}}\|u\|_{W_0^{m,p}(\Omega)},$$

and thus for any $1 \leq q < +\infty$,

$$\left(\int_\Omega |u|^q d\mu\right)^{1/q} \leq \frac{C}{\mu_{\min}}|\Omega|^{1/q}\|u\|_{W_0^{m,p}(\Omega)} \leq \frac{C}{\mu_{\min}}(1+|\Omega|)\|u\|_{W_0^{m,p}(\Omega)},$$

where $|\Omega| = \sum_{x\in\Omega}\mu(x)$ denotes the volume of Ω. Therefore (18) holds. $\square$

If V is a finite graph, then $W^{m,p}(V)$ can be defined as the set of all functions $u : V \to \mathbb{R}$ under the norm

$$\|u\|_{W^{m,p}(V)} = \left(\int_V (|\nabla^m u|^p + h|u|^p)d\mu\right)^{1/p}, \tag{21}$$

where $h(x) > 0$ for all $x \in V$. Then we have an obvious analog of Theorem 7 as follows:

Theorem 8. *Let V be a finite graph. Then $W^{m,p}(V)$ is embedded in $L^q(V)$ for all $1 \leq q \leq +\infty$. Moreover, $W^{m,p}(V)$ is pre-compact.*

Obviously, both $W_0^{k,p}(\Omega)$ and $W^{k,p}(V)$ with norms (19) and (21) respectively are Banach spaces. Let $(X, \|\cdot\|)$ be a Banach space, $J : X \to \mathbb{R}$ be a functional. We say that J satisfies the $(PS)_c$ condition for some real number c, if for any sequence of functions $u_k : X \to \mathbb{R}$ such that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$, there holds up to a subsequence, $u_k \to u$ in X. To prove Theorems 1-6, we need the following mountain pass theorem.

Theorem 9. *(Ambrosetti-Rabinowitz, [3]). Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e).$$

If J satisfies the $(PS)_c$ condition with $c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$, where

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of J.

4. Local existence

In this section, we prove Theorems 1-3 by applying Theorem 9.

Proof of Theorem 1. Let $p > 2$ and $\alpha < \lambda_1(\Omega)$ be fixed. For any $u \in W_0^{1,2}(\Omega)$, we let

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \alpha u^2)d\mu - \frac{1}{p}\int_\Omega (u^+)^p d\mu,$$

where $u^+(x) = \max\{u(x), 0\}$. It is clear that $J \in C^1(W_0^{1,2}(\Omega), \mathbb{R})$. We claim that J satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To see this, we take a sequence of functions $u_k \in W_0^{1,2}(\Omega)$ such that $J(u_k) \to c$, $J'(u_k) \to 0$ as $k \to +\infty$. This leads to

$$\frac{1}{2}\int_\Omega (|\nabla u_k|^2 - \alpha u_k^2)d\mu - \frac{1}{p}\int_\Omega (u_k^+)^p d\mu = c + o_k(1), \tag{22}$$

$$\left|\int_\Omega (|\nabla u_k|^2 - \alpha u_k^2)d\mu - \int_\Omega (u_k^+)^p d\mu\right| \leq o_k(1)\|u_k\|_{W_0^{1,2}(\Omega)}. \tag{23}$$

Noting that $p > 2$, we conclude from (22) and (23) that u_k is bounded in $W_0^{1,2}(\Omega)$. Then the Sobolev embedding (Theorem 7) implies that up to a subsequence, u_k converges to some function u in $W_0^{1,2}(\Omega)$. Hence the $(PS)_c$ condition holds.

To proceed, we need to check that J satisfies all conditions in the mountain pass theorem (Theorem 9). Note that

$$J(0) = 0. \tag{24}$$

By Theorem 7, there exists some constant C depending only on p and Ω such that

$$\left(\int_\Omega (u^+)^p d\mu\right)^{1/p} \leq C\left(\int_\Omega |\nabla u|^2 d\mu\right)^{1/2}.$$

Hence there holds for all $u \in W_0^{1,2}(\Omega)$

$$J(u) \geq \frac{1}{2}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{C^p}{p}\|u\|^p_{W_0^{1,2}(\Omega)}.$$

One can find some sufficiently small $r > 0$ such that

$$\inf_{\|u\|_{W_0^{1,2}(\Omega)}=r} J(u) > 0. \tag{25}$$

Take a function $u^* \in W_0^{1,2}(\Omega)$ satisfying $u^* > 0$ in Ω°. Passing to the limit $t \to +\infty$, we have

$$J(tu^*) = \frac{t^2}{2}\int_\Omega (|\nabla u^*|^2 - \alpha(u^*)^2)d\mu - \frac{t^p}{p}\int_\Omega (u^{*+})^p d\mu \to -\infty.$$

Hence there exists some $u_0 \in W_0^{1,2}(\Omega)$ such that

$$J(u_0) < 0, \quad \|u_0\|_{W_0^{1,2}(\Omega)} > r. \tag{26}$$

Combining (24), (25) and (26), we conclude by Theorem 9 that $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$ is a critical value of J, where $\Gamma = \{\gamma \in C([0,1], W_0^{1,2}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_0\}$. In particular, there exists some function $u \in W_0^{1,2}(\Omega)$ such that

$$-\Delta u - \alpha u = (u^+)^{p-1} \quad \text{in} \quad \Omega^\circ. \tag{27}$$

Testing this equation by $u^- = \min\{u, 0\}$ and noting that $u^-u^+ = u^-(u^+)^{p-1} = 0$, we have

$$-\int_\Omega u^-\Delta u d\mu - \alpha\int_\Omega (u^-)^2 d\mu = 0.$$

Since $u = u^+ + u^-$, the above equation leads to

$$\frac{\alpha}{\lambda_1(\Omega)}\int_\Omega |\nabla u^-|^2 d\mu \geq \alpha\int_\Omega (u^-)^2 d\mu = \int_\Omega |\nabla u^-|^2 d\mu - \int_\Omega u^-\Delta u^+ d\mu. \tag{28}$$

Note that

$$\begin{aligned} -\int_\Omega u^-\Delta u^+ d\mu &= -\sum_{x\in\Omega^\circ} u^-(x)\sum_{y\sim x} w_{xy}(u^+(y) - u^+(x)) \\ &= -\sum_{x\in\Omega^\circ}\sum_{y\sim x} w_{xy}u^-(x)u^+(y) \geq 0. \end{aligned} \tag{29}$$

Inserting (29) into (28) and recalling that $\alpha < \lambda_1(\Omega)$, we obtain $\int_\Omega |\nabla u^-|^2 d\mu = 0$, which implies that $u^- \equiv 0$ in Ω. Whence $u \geq 0$ and (27) becomes

$$\begin{cases} -\Delta u - \alpha u = u^{p-1} \quad \text{in} \quad \Omega^\circ \\ u \geq 0 \ \text{ in } \ \Omega^\circ, \ \ u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{30}$$

Suppose $u(x) = 0 = \min_{x\in\Omega} u(x)$ for some $x \in \Omega^\circ$. If y is adjacent to x, then we know from (30) that $\Delta u(x) = 0$, and thus by the definition of Δ (see (3) above), $u(y) = 0$. Therefore we conclude that $u \equiv 0$ in Ω, which contradicts (25). Hence $u > 0$ in Ω° and this completes the proof of the theorem. □

Proof of Theorem 2. Let $p > 1$ be fixed. For any $u \in W_0^{1,p}(\Omega)$, we let

$$J_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p d\mu - \int_\Omega F(x, u^+) d\mu,$$

where $u^+(x) = \max\{u(x), 0\}$. In view of (H_4), there exist two constants λ and $\delta > 0$ such that $\lambda < \lambda_p(\Omega)$ and

$$F(x, u^+) \leq \frac{\lambda}{p}(u^+)^p + \frac{(u^+)^{p+1}}{\delta^{p+1}} F(x, u^+).$$

For any $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)} \leq 1$, we have by the Sobolev embedding (Theorem 7) that $\|u\|_\infty \leq C$ for some constant C depending only on p and Ω, and that

$$F(x, u^+) \leq \frac{\lambda}{p}(u^+)^p + C(u^+)^{p+1}.$$

This together with (10), the definition of $\lambda_p(\Omega)$, and Theorem 7 leads to

$$\int_\Omega F(x,u^+)d\mu \le \frac{\lambda}{p\lambda_p(\Omega)}\int_\Omega |\nabla u^+|^p d\mu + C\left(\int_\Omega |\nabla u^+|^p d\mu\right)^{1+1/p}.$$

Here and throughout this paper, we often denote various constant by the same C. Noting that

$$\begin{aligned}|\nabla u|^2(x) &= \frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}(u(y)-u(x))^2\\ &= \frac{1}{2\mu(x)}\sum_{y\sim x} w_{xy}\Big\{(u^+(y)-u^+(x))^2+(u^-(y)-u^-(x))^2\\ &\qquad -2u^+(y)u^-(x)-2u^+(x)u^-(y)\}\\ &\ge |\nabla u^+|^2(x),\end{aligned}$$

we have $|\nabla u^+|^p \le |\nabla u|^p$. Hence if $\|u\|_{W_0^{1,p}(\Omega)} \le 1$, then we obtain

$$J_p(u) \ge \frac{\lambda_p(\Omega)-\lambda}{p\lambda_p(\Omega)}\int_\Omega |\nabla u|^p d\mu - C\left(\int_\Omega |\nabla u|^p d\mu\right)^{1+1/p}.$$

Therefore

$$\inf_{\|u\|_{W_0^{1,p}(\Omega)}=r} J_p(u) > 0 \tag{31}$$

provided that $r > 0$ is sufficiently small.

By (H_3), there exist two positive constants c_1 and c_2 such that

$$F(x,u^+) \ge c_1(u^+)^q - c_2.$$

Take $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0 \ge 0$ and $u_0 \not\equiv 0$. For any $t > 0$, we have

$$J_p(tu_0) \le \frac{t^p}{p}\int_\Omega |\nabla u_0|^p d\mu - c_1 t^q \int_\Omega u_0^q d\mu - c_2|\Omega|.$$

Since $q > p$, we conclude $J_p(tu_0) \to -\infty$ as $t \to +\infty$. Hence there exists some $u_1 \in W_0^{1,p}(\Omega)$ satisfying

$$J_p(u_1) < 0, \quad \|u_1\|_{W_0^{1,p}(\Omega)} > r. \tag{32}$$

We claim that J_p satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To see this, we assume $J_p(u_k) \to c$ and $J_p'(u_k) \to 0$ as $k \to +\infty$. It follows that

$$\frac{1}{p}\int_\Omega |\nabla u_k|^p d\mu - \int_\Omega F(x,u_k^+)d\mu = c + o_k(1)$$

$$\int_\Omega |\nabla u_k|^p d\mu - \int_\Omega u_k f(x,u_k^+)d\mu = o_k(1)\|u_k\|_{W_0^{1,p}(\Omega)}.$$

In view of (H_3), we obtain from the above two equations that u_k is bounded in $W_0^{1,p}(\Omega)$. Then the $(PS)_c$ condition follows from Theorem 7.

Combining (31), (32) and the obvious fact $J_p(0) = 0$, we conclude from Theorem 9 that there exists a function $u \in W_0^{1,p}(\Omega)$ such that $J_p(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_p(\gamma(t)) > 0$ and $J_p'(u) = 0$, where $\Gamma = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = u_1\}$. Hence there exists a nontrivial solution $u \in W_0^{1,p}(\Omega)$ to the equation

$$-\Delta_p u = f(x, u^+) \quad \text{in} \quad \Omega^\circ.$$

Noting that $\Gamma(u^-, u) = \Gamma(u^-) + \Gamma(u^-, u^+) \geq |\nabla u^-|^2$, we obtain

$$\int_\Omega |\nabla u^-|^p d\mu \leq \int_\Omega |\nabla u^-|^{p-2}\Gamma(u^-, u)d\mu = -\int_\Omega u^-\Delta_p u d\mu = \int_\Omega u^- f(x, u^+)dx = 0.$$

This implies that $u^- \equiv 0$ and thus $u \geq 0$. □

Proof of Theorem 3. Let $m \geq 2$ and $p > 1$ be fixed. For any $u \in W_0^{m,p}(\Omega)$, we write

$$J_{mp}(u) = \frac{1}{p}\int_\Omega |\nabla^m u|^p d\mu - \int_\Omega F(x, u)d\mu.$$

By (A_2), there exists some $\lambda < \lambda_{mp}(\Omega)$ and $\delta > 0$ such that for all $s \in \mathbb{R}$,

$$F(x, s) \leq \frac{\lambda}{p}|s|^p + \frac{|s|^{p+1}}{\delta^{p+1}}|F(x, s)|.$$

By Theorem 7, we have

$$\int_\Omega |u|^{p+1}d\mu \leq C\left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1+1/p}, \tag{33}$$

$$\|u\|_{L^\infty(\Omega)} \leq C\left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1/p}. \tag{34}$$

For any $u \in W_0^{m,p}(\Omega)$ with $\|u\|_{W_0^{m,p}(\Omega)} \leq 1$, in view of (34), there exists some constant C, depending only on m, p and Ω, such that $\|u\|_{L^\infty(\Omega)} \leq C$ and thus $\|F(x, u)\|_{L^\infty(\Omega)} \leq C$. This together with (33) gives

$$\begin{aligned}\int_\Omega F(x, u)d\mu &\leq \frac{\lambda}{p}\int_\Omega |u|^p d\mu + \frac{C}{\delta^{p+1}}\int_\Omega |u|^{p+1}|F(x, u)|d\mu \\ &\leq \frac{\lambda}{\lambda_{mp}(\Omega)p}\int_\Omega |\nabla^m u|^p d\mu + C\left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1+1/p}.\end{aligned}$$

Hence

$$J_{mp}(u) \geq \frac{\lambda_{mp}(\Omega) - \lambda}{\lambda_{mp}(\Omega)p}\int_\Omega |\nabla^m u|^p d\mu - C\left(\int_\Omega |\nabla^m u|^p d\mu\right)^{1+1/p},$$

and thus

$$\inf_{\|u\|_{W_0^{m,p}(\Omega)}=r} J_{mp}(u) > 0 \tag{35}$$

for sufficiently small $r > 0$.

By (A_3), there exist two positive constants c_1 and c_2 such that

$$F(x, s) \geq c_1|s|^q - c_2, \quad \forall s \in \mathbb{R}.$$

For any fixed $u \in W_0^{m,p}(\Omega)$ with $u \not\equiv 0$, we have

$$J_{mp}(tu) \leq \frac{|t|^p}{p} \int_\Omega |\nabla^m u|^p d\mu - c_1 |t|^q \int_\Omega |u|^q d\mu + c_2 |\Omega| \to -\infty$$

as $t \to \infty$. Hence there exists some $u_2 \in W_0^{m,p}(\Omega)$ such that

$$J_{mp}(u_2) < 0, \quad \|u_2\|_{W_0^{m,p}(\Omega)} > r. \tag{36}$$

Now we claim that J_{mp} satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. For this purpose, we take $u_k \in W_0^{m,p}(\Omega)$ such that $J_{mp}(u_k) \to c$ and $J'_{mp}(u_k) \to 0$ as $k \to +\infty$. In particular,

$$\frac{1}{p} \int_\Omega |\nabla^m u_k|^p d\mu - \int_\Omega F(x, u_k) d\mu = c + o_k(1), \tag{37}$$

$$\int_\Omega |\nabla^m u_k|^p d\mu - \int_\Omega u_k f(x, u_k) d\mu = o_k(1) \|u_k\|_{W_0^{m,p}}. \tag{38}$$

By (A_3), we have

$$\begin{aligned} q \int_\Omega F(x, u_k) d\mu &\leq q \int_{|u_k| \leq M} |F(x, u_k)| d\mu + q \int_{|u_k| > M} F(x, u_k) d\mu \\ &\leq \int_{|u_k| > M} u_k f(x, u_k) d\mu + C \\ &\leq \int_\Omega u_k f(x, u_k) d\mu + C. \end{aligned} \tag{39}$$

Combining (37), (38) and (39), we conclude that u_k is bounded in $W_0^{m,p}(\Omega)$. Then the $(PS)_c$ condition follows from Theorem 7.

In view of (35) and (36), applying the mountain pass theorem as before, we finish the proof of the theorem. □

5. Global existence

In this section, using Theorem 9, we prove global existence results (Theorems 4-6). These procedures are very similar to that of Theorems 1-3. We only give the outline of the proof.

Proof of Theorem 4. Let $p > 2$ be fixed. For $u \in W^{1,2}(V)$, we let

$$J_V(u) = \frac{1}{2} \int_V (|\nabla u|^2 + h u^2) d\mu - \frac{1}{p} \int_V (u^+)^p d\mu.$$

We first prove that J_V satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To see this, we assume $J_V(u_k) \to c$ and $J'_V(u_k) \to 0$ as $k \to \infty$, namely

$$\frac{1}{2} \int_V (|\nabla u_k|^2 + h u_k^2) d\mu - \frac{1}{p} \int_V (u^+)^p d\mu = c + o_k(1), \tag{40}$$

$$\left| \int_V (|\nabla u_k|^2 + h u_k^2) d\mu - \int_V (u_k^+)^p d\mu \right| \leq o_k(1) \|u_k\|_{W^{1,2}(V)}. \tag{41}$$

It follows from (40) and (41) that u_k is bounded in $W^{1,2}(V)$. By Theorem 8, $W^{1,2}(V)$ is pre-compact, we conclude up to a subsequence, $u_k \to u$ in $W^{1,2}(V)$. Thus J_V satisfies the $(PS)_c$ condition.

To apply the mountain pass theorem, we check the conditions of J_V. Obviously

$$J_V(0) = 0. \tag{42}$$

Note that $h(x) > 0$ for all $x \in V$. By (21),

$$\|u\|^2_{W^{1,2}(V)} = \int_V (|\nabla u|^2 + hu^2) d\mu, \quad \forall u \in W^{1,2}(V).$$

By the Sobolev embedding (Theorem 8), we have for any $p > 2$,

$$\int_V (u^+)^p d\mu \le C\|u\|^p_{W^{1,2}(V)}.$$

Hence

$$\inf_{\|u\|_{W^{1,2}(V)}=r} J_V(u) > 0 \tag{43}$$

for sufficiently small $r > 0$. For any fixed $\tilde{u} \ge 0$ with $\tilde{u} \not\equiv 0$, we have $J_V(t\tilde{u}) \to -\infty$ as $t \to -\infty$. Hence there exists some $e \in W^{1,2}(V)$ such that

$$J_V(e) < 0, \quad \|e\|_{W^{1,2}(V)} > r. \tag{44}$$

Combining (42), (43) and (44) and applying the mountain pass theorem (Theorem 9), we find some $u \in W^{1,2}(V) \setminus \{0\}$ satisfying

$$-\Delta u + hu = (u^+)^{p-1} \quad \text{in} \quad V.$$

Testing this equation by u^-, we have

$$\int_V (|\nabla u^-|^2 + h(u^-)^2) d\mu \le \int_V u^-(-\Delta u + hu) d\mu = 0,$$

which leads to $u^- \equiv 0$. Therefore $u \ge 0$ and

$$-\Delta u + hu = u^{p-1} \quad \text{in} \quad V.$$

If $u(x) = 0$ for some $x \in V$, then $\Delta u(x) = 0$, which together with $u \ge 0$ implies that $u \equiv 0$ in V. This contradicts $u \not\equiv 0$. Therefore $u > 0$ in V and this completes the proof of the theorem. □

Proof of Theorem 5. Let $p > 1$ be fixed. For $u \in W^{1,p}(V)$, we define

$$J_{V,p}(u) = \frac{1}{p}\int_V (|\nabla u|^p + h|u|^p) d\mu - \int_V F(x, u^+) d\mu.$$

Write

$$\|u\|_{W^{1,p}(V)} = \left(\int_V (|\nabla u|^p + h|u|^p) d\mu\right)^{1/p}.$$

By (H_V^4), there exist two constants λ and $\delta > 0$ such that $\lambda < \lambda_p^h(V)$ and

$$F(x, u^+) \le \frac{\lambda}{p}(u^+)^p + \frac{(u^+)^{p+1}}{\delta^{p+1}} F(x, u^+).$$

For any $u \in W^{1,p}(V)$ with $\|u\|_{W^{1,p}(V)} \le 1$, we have by Theorem 8 that $\|u\|_\infty \le C$, and that

$$F(x, u^+) \le \frac{\lambda}{p}(u^+)^p + C(u^+)^{p+1}.$$

This together with the definition of $\lambda_p^h(V)$ and the Sobolev embedding (Theorem 8) leads to

$$\int_V F(x, u^+) d\mu \le \frac{\lambda}{p\lambda_p^h(V)} \|u^+\|^p_{W^{1,p}(V)} + C\|u^+\|^{p+1}_{W^{1,p}(V)}.$$

Then we obtain for all u with $\|u\|_{W^{1,p}(V)} \le 1$,

$$J_{V,p}(u) \ge \frac{\lambda_p^h(V) - \lambda}{p\lambda_p^h(V)} \|u\|^p_{W^{1,p}(V)} - C\|u\|^{p+1}_{W^{1,p}(V)}.$$

Therefore

$$\inf_{\|u\|_{W^{1,p}(V)}=r} J_{V,p}(u) > 0 \tag{45}$$

provided that $r > 0$ is sufficiently small.

By the hypothesis (H_V^3), there exist two positive constants c_1 and c_2 such that

$$F(x, u^+) \ge c_1(u^+)^q - c_2.$$

For some $u^* \in W^{1,p}(V)$ with $u^* \ge 0$ and $u^* \not\equiv 0$, we have for any $t > 0$,

$$J_{V,p}(tu^*) \le \frac{t^p}{p} \|u^*\|^p_{W^{1,p}(V)} - c_1 t^q \int_V (u^*)^q d\mu - c_2|V|.$$

Hence $J_{V,p}(tu^*) \to -\infty$ as $t \to +\infty$, from which one can find some $e \in W^{1,p}(V)$ satisfying

$$J_{V,p}(e) < 0, \quad \|e\|_{W^{1,p}(V)} > r. \tag{46}$$

We claim that $J_{V,p}$ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To see this, we assume $J_{V,p}(u_k) \to c$ and $J'_{V,p}(u_k) \to 0$ as $k \to +\infty$. It follows that

$$\frac{1}{p}\|u_k\|^p_{W^{1,p}(V)} - \int_V F(x, u_k^+) d\mu = c + o_k(1)$$

$$\|u_k\|^p_{W^{1,p}(V)} - \int_V u_k^+ f(x, u_k^+) d\mu = o_k(1)\|u_k\|_{W^{1,p}(V)}.$$

In view of (H_V^3), we conclude from the above two equations that u_k is bounded in $W^{1,p}(V)$. Then the $(PS)_c$ condition follows from Theorem 8.

Combining (45), (46) and $J_{V,p}(0) = 0$, we conclude from Theorem 9 that there exists a nontrivial solution $u \in W^{1,p}(V)$ to the equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u^+) \quad \text{in} \quad V.$$

Noting that $\Gamma(u^-, u) = \Gamma(u^-) + \Gamma(u^-, u^+) \geq |\nabla u^-|^2$, we obtain

$$\int_V (|\nabla u^-|^p + h|u^-|^p)d\mu \leq -\int_V u^- \Delta_p u d\mu + \int_V h u^- |u|^{p-2} u d\mu = \int_V u^- f(x, u^+)dx = 0.$$

This implies that $u^- \equiv 0$ and thus $u \geq 0$ in V. $\square$

Proof of Theorem 6. Let $m \geq 2$ and $p > 1$ be fixed. We define

$$J_{mp}^V(u) = \frac{1}{p}\int_V (|\nabla^m u|^p + h|u|^p)d\mu - \int_V F(x, u)d\mu.$$

By (A_V^2), there exists some $\lambda < \lambda_{mp}(V)$ and $\delta > 0$ such that for all $x \in V$ and $s \in \mathbb{R}$,

$$F(x, s) \leq \frac{\lambda}{p}|s|^p + \frac{|s|^{p+1}}{\delta^{p+1}}|F(x, s)|.$$

For any $u \in W^{m,p}(V)$ with $\|u\|_{W^{m,p}(V)} \leq 1$, we have $\|u\|_\infty \leq C$ and thus $\|F(x, u)\|_\infty \leq C$. Hence

$$\begin{aligned}\int_V F(x, u)d\mu &\leq \frac{\lambda}{p}\int_V |u|^p d\mu + \frac{C}{\delta^{p+1}}\int_V |u|^{p+1}|F(x, u)|d\mu \\ &\leq \frac{\lambda}{\lambda_{mp}(V)p}\|u\|_{W^{m,p}(V)}^p + C\|u\|_{W^{m,p}(V)}^{p+1}.\end{aligned}$$

It follows that

$$J_{mp}^V(u) \geq \frac{\lambda_{mp}(V) - \lambda}{\lambda_{mp}(V)p}\|u\|_{W^{m,p}(V)}^p - C\|u\|_{W^{m,p}(V)}^{p+1},$$

and thus

$$\inf_{\|u\|_{W^{m,p}(V)}=r} J_{mp}^V(u) > 0 \tag{47}$$

for sufficiently small $r > 0$.

By (A_V^3), there exist two positive constants c_1 and c_2 such that

$$F(x, s) \geq c_1|s|^q - c_2, \quad \forall s \in \mathbb{R}.$$

For any fixed $u \in W^{m,p}(V)$ with $u \not\equiv 0$, we have

$$J_{mp}^V(tu) \leq \frac{|t|^p}{p}\int_V (|\nabla^m u|^p + h|u|^p)d\mu - c_1|t|^q \int_V |u|^q d\mu + c_2|V| \to -\infty$$

as $t \to \infty$. Hence there exists some $e \in W^{m,p}(V)$ such that

$$J_{mp}^V(e) < 0, \quad \|e\|_{W^{m,p}(V)} > r. \tag{48}$$

Now we claim that J_{mp}^V satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. For this purpose, we take $u_k \in W^{m,p}(V)$ such that $J_{mp}^V(u_k) \to c$ and $(J_{mp}^V)'(u_k) \to 0$ as $k \to +\infty$. In particular,

$$\frac{1}{p}\int_V (|\nabla^m u_k|^p + h|u_k|^p)d\mu - \int_V F(x, u_k)d\mu = c + o_k(1), \tag{49}$$

$$\int_V (|\nabla^m u_k|^p + h|u_k|^p)d\mu - \int_V u_k f(x, u_k)d\mu = o_k(1)\|u_k\|_{W^{m,p}(V)}. \tag{50}$$

By (A_V^3), we have

$$\begin{aligned}
q\int_V F(x,u_k)d\mu &\leq q\int_{|u_k|\leq M}|F(x,u_k)|d\mu + q\int_{|u_k|>M}F(x,u_k)d\mu\\
&\leq \int_{|u_k|>M}u_k f(x,u_k)d\mu + C\\
&\leq \int_V u_k f(x,u_k)d\mu + C. \qquad (51)
\end{aligned}$$

Combining (49), (50) and (51), we conclude that u_k is bounded in $W^{m,p}(V)$. Then the $(PS)_c$ condition follows from Theorem 8.

In view of (47), (48) and the fact $J_{mp}^V(0) = 0$, applying the mountain pass theorem, we finish the proof of the theorem. □

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References

[1] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian, Ann. Sc. Norm. Sup. Pisa 17 (1990) 393-413.

[2] Adimurthi, Y. Yang, An interpolation of Hardy inequality and Trudinger-Moser inequality in $\mathbb{R}^N$ and its applications, International Mathematics Research Notices 13 (2010) 2394-2426.

[3] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349-381.

[4] T. Aubin, The scalar curvature, Differential Geometry and Relativity, Holland, 1976, 5-18.

[5] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437-477.

[6] D. G. de Figueiredo, O. H. Miyagaki, B. Ruf, Elliptic equations in R^2 with nonlinearities in the critical growth range, Calc. Var. 3 (1995) 139-153.

[7] J. M. do Ó, Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nolinearities in the critical growth range, Differ. Integr. Equations 9 (1996) 967-979.

[8] J. M. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (1997) 301-315.

[9] J. M. do Ó, E. Medeiros, U. Severo, On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$, J. Differential Equations 246 (2009) 1363-1386.

[10] J. M. do Ó, Y. Yang, A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary, Ann. Glob. Anal. Geom. 38 (2010) 317-334.

[11] J. Moser, A sharp form of an inequality by N.Trudinger, Ind. Univ. Math. J. 20 (1971) 1077-1091.

[12] Z. Nehari, On a class of nonlinear second-order differential equations, Trans. Amer. Math. Soc. 95 (1960) 101-123.

[13] Z. Nehari, Characteristic values associated with a class of nonlinear second-order differential equations, Acta Math. 105 (1961) 141-175.

[14] Z. Nehari, On a nonlinear differential equation arising in nuclear physics, Proc. Roy. Irish Acad. Sect. A 62 (1963) 117-135.

[15] J. Peetre, Espaces d'interpolation et theoreme de Soboleff, Ann. Inst. Fourier (Grenoble) 16 (1966) 279-317.

[16] S. Pohozaev, The Sobolev embedding in the special case $pl = n$, Proceedings of the technical scientific conference on advances of scientific reseach 1964-1965, Mathematics sections, 158-170, Moscov. Energet. Inst., Moscow, 1965.

[17] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20 (1984) 479-495.

[18] W. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) 149-162.

[19] N. Trudinger, On embeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473-484.
[20] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa. 3 (1968) 265-274.
[21] M. Willem, Minimax theorems, Birkh$\ddot{a}$user, 1996.
[22] H. Yamabe, On a deformation of Riemannian structurs on compact manifolds, Osaka Math. J. 12 (1960) 21-37.
[23] Y. Yang, Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space, J. Funct. Anal. 262 (2012) 1679-1704.
[24] Y. Yang, Adams type inequalities and related elliptic partial differential equations in dimension four, J. Differential Equations 252 (2012) 2266-2295.
[25] Y. Yang, Trudinger-Moser inequalities on complete noncompact Riemannian manifolds, J. Funct. Anal. 263 (2012) 1894-1938.
[26] Y. Yang, L. Zhao, A class of Adams-Fontana type inequalities and related functionals on manifolds, Nonlinear Differ. Equ. Appl. 17 (2010) 119-135.
[27] V.I. Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, Sov. Math. Docl. 2 (1961) 746-749.
[28] L. Zhao, Exponential problem on a compact Riemannian manifold without boundary, Nonlinear Anal. 75 (2012) 433-443.