On positive solutions of semi-linear elliptic inequalities on Riemannian manifolds

Alexander Grigor'yan
University of Bielefeld

University of Minnesota, February 2018

Setup and problem statement

Let (M, g) be a connected Riemannian manifold, Δ – the Laplace-Beltrami operator on M. In local coordinates $x_1, ..., x_n$ it has the form

$$\Delta = \frac{1}{D} \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij} (x) \, \partial_{x_j} \right) ,$$

where $D = \sqrt{\det (g_{ij})}$ and $(a_{ij}) = D \, (g_{ij})^{-1}$.

Consider the equation (or inequality)

$$-\Delta u + \Phi (x) \, u^{\sigma} = f \qquad \qquad (\text{or} \quad -\Delta u + \Phi (x) \, u^{\sigma} \geq f \ )$$

where $\Phi, f \in C (M)$, $f \geq 0$, $\sigma > 0$. Solution u should be *non-negative* and in $C^2 (M)$.

Our goal is obtaining pointwise estimates of u.

Assume that Δ has a positive Green function $G (x, y)$ on M. Set $Gf(x) = \int\limits_M G(x, y) f(y) d\mu(y)$.

It is known that if Gf is finite then $\Delta (Gf) = -f$.

Linear case $\sigma = 1$

W. Hansen–Z.Ma 1990, AG–W.Hansen 2008: if

$$-\Delta u + \Phi(x)\, u \geq f \quad \text{on } M$$

where $\Phi \geq 0$ and the function $h = Gf$ is positive and finite, then

$$u \geq h \exp\left(-\frac{1}{h} G(h\Phi)\right).$$

This implies the lower bound for the Green function G_Φ of $-\Delta + \Phi$:

$$G_\Phi(x,y) \geq G(x,y) \exp\left(-\frac{\int_M G(x,z)\, G(z,y)\, \Phi(z)\, d\mu(z)}{G(x,y)}\right).$$

In the case $\Phi \leq 0$ a similar estimate under additional assumptions was obtained by N.Kalton–I.Verbitsky 1999.

Main result

Theorem 1 (AG–I.Verbitsky, 2015) *Let $u \geq 0$ solve $-\Delta u + \Phi u^\sigma \geq f$ in M. Set $h = Gf$ and assume that $0 < h < \infty$. Assume also that $G(h^\sigma \Phi)$ be well defined.*

(i) *If $\sigma = 1$ then*

$$u \geq h \exp\left(-\frac{1}{h} G(h\Phi)\right). \tag{1}$$

(ii) *If $\sigma > 1$ then*

$$u \geq \frac{h}{\left[1 + (\sigma - 1)\frac{1}{h} G(h^\sigma \Phi)\right]^{\frac{1}{\sigma-1}}}, \tag{2}$$

where the expression in square brackets is necessarily positive, that is,

$$-(\sigma - 1)G(h^\sigma \Phi) < h. \tag{3}$$

(iii) *If $0 < \sigma < 1$ then*

$$u \geq h \left[1 - (1 - \sigma)\frac{1}{h} G\left(1_{\{u>0\}} h^\sigma \Phi\right)\right]_+^{\frac{1}{1-\sigma}}. \tag{4}$$

Estimates with boundary condition

Let Ω be a relatively compact domain in M with smooth boundary. Let $G_\Omega(x,y)$ be the Green function of Δ in Ω with the Dirichlet boundary condition.

It suffices to prove (1)-(4) in Ω with G_Ω instead of G and with $h = G_\Omega f$ instead of Gf.

Consider the following problem. Let $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ be positive and superharmonic in Ω. Set $f := -\Delta h \geq 0$. Assume that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, $u \geq 0$, satisfies

$$\begin{cases} -\Delta u + \Phi u^\sigma \geq f & \text{in } \Omega \\ u \geq h & \text{on } \partial\Omega \end{cases} \tag{5}$$

Theorem 2 *(i) If* $\sigma = 1$ *then* $u \geq h \exp\left(-\frac{1}{h} G_\Omega(h\Phi)\right)$.

(ii) If $\sigma > 1$ *then*

$$u \geq \frac{h}{\left[1 + (\sigma - 1)\frac{1}{h} G_\Omega(h^\sigma \Phi)\right]^{\frac{1}{\sigma-1}}},$$

where necessarily $-(\sigma-1)G_\Omega(h^\sigma\Phi) < h$.

(iii) If $0 < \sigma < 1$ *then* $u \geq h\left[1 - (1-\sigma)\frac{1}{h} G_\Omega\left(1_{\{u>0\}} h^\sigma \Phi\right)\right]_+^{\frac{1}{1-\sigma}}$.

Approach to the proof of Theorem 2

Assume for simplicity that $u > 0$ and $h > 0$ in $\overline{\Omega}$. Assume first $h \equiv 1$. Then $f = -\Delta h = 0$ and (5) becomes

$$\begin{cases} -\Delta u + \Phi u^{\sigma} \geq 0 & \text{in } \Omega \\ u \geq 1 & \text{on } \partial\Omega \end{cases}$$

Fix a C^2 function φ on (a interval of) $\mathbb{R}$ with $\varphi' > 0$ and make the following change:

$$v = \varphi^{-1}(u).$$

By the chain rule we have

$$\Delta u = \Delta \varphi(v) = \varphi'(v) \Delta v + \varphi''(v) |\nabla v|^2,$$

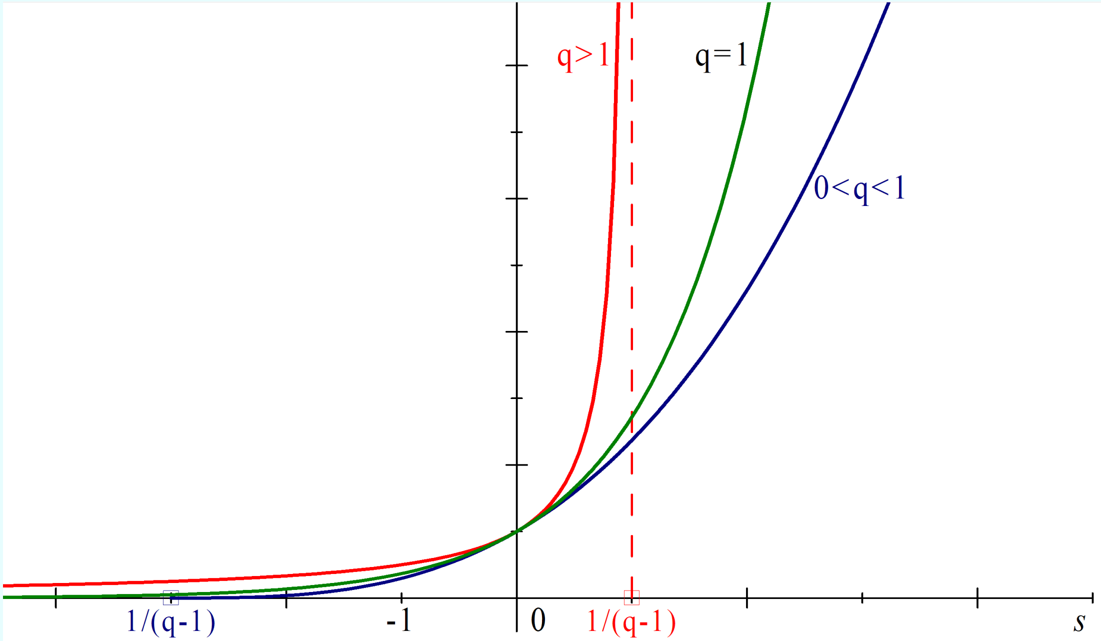
whence

$$-\Delta v = \frac{\varphi'' |\nabla v|^2}{\varphi'} - \frac{\Delta u}{\varphi'} \geq \frac{\varphi''}{\varphi'} |\nabla v|^2 - \Phi \frac{\varphi(v)^{\sigma}}{\varphi'(v)}. \tag{6}$$

Choose φ to solve the following initial value problem

$$\varphi'(s) = \varphi^{\sigma}(s), \quad \varphi(0) = 1.$$

If $\sigma = 1$ then $\varphi(s) = e^s$. If $\sigma \neq 1$ then $\varphi(s) = [(1-\sigma)s + 1]^{\frac{1}{1-\sigma}}$.



The inverse function φ^{-1} is always defined on $(0, +\infty)$.

The function φ is convex, and we obtain from (6)

$$-\Delta v \geq -\Phi \quad \text{in } \Omega. \tag{7}$$

Since on $\partial\Omega$ we have $v = \varphi^{-1}(u) \geq \varphi^{-1}(1) = 0$, it follows that

$$v \geq -G_\Omega \Phi \quad \text{in } \Omega$$

and, hence,

$$u \geq \varphi(-G_\Omega \Phi) \quad \text{in } \Omega.$$

This yields the cases $(i) - (iii)$ of Theorem 2 in the case $h = 1$.

Indeed, in the case $\sigma = 1$ we have $\varphi(s) = e^s$ and, hence,

$$u \geq \exp(-G_\Omega \Phi).$$

In the case $\sigma > 1$ we have $\varphi(s) = [(1-\sigma)s + 1]^{-\frac{1}{\sigma-1}}$, which gives the estimate of (ii)

$$u \geq \frac{1}{[1 + (\sigma - 1)G_\Omega \Phi]^{\frac{1}{\sigma-1}}}.$$

Similarly one treats the case $0 < \sigma < 1$.

For a general $h>0$, we use the *h-transform* of Δ in Ω: $\Delta^h := \frac{1}{h}\circ\Delta\circ h$. That is,

$$\Delta^h u = \frac{1}{h}\left(\Delta\left(hu\right)\right) = \frac{1}{h}\left(h\Delta u + 2\nabla h\cdot\nabla h + \left(\Delta h\right)u\right) = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla u\right) + \frac{\Delta h}{h}u = Lu + \frac{\Delta h}{h}u$$

where

$$L = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right)$$

is the *weighted* Laplacian associated with measure $d\tilde{\mu} = h^2 d\mu$.

For function $\tilde{u} = \frac{u}{h}$ we have

$$-\Delta^h\tilde{u} = -\frac{1}{h}\Delta u \geq \frac{1}{h}\left(-\Phi u^\sigma + f\right) = -h^{\sigma-1}\Phi\tilde{u}^\sigma - \frac{\Delta h}{h}.$$

Setting $\tilde{\Phi} = h^{\sigma-1}\Phi$, we obtain that $\tilde{u}$ satisfies

$$-\Delta^h\tilde{u} + \tilde{\Phi}\tilde{u}^\sigma \geq -\frac{\Delta h}{h} \text{ in } \Omega, \qquad \tilde{u}\geq 1 \text{ on } \partial\Omega.$$

Now we use the same approach as in the case $h=1$, but for operator Δ^h in place of Δ.

Set $v = \varphi^{-1}\left(\tilde{u}\right) = \varphi^{-1}\left(u/h\right)$ and compute $\Delta^h v$ as in (6). For the part $L = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right)$ of the operator Δ^h, computation is the same as for Δ.

The part $\frac{\Delta h}{h}$ gives in the end an additional term so that instead of (7) we obtain

$$-\Delta^h v \geq -\tilde{\Phi} + \left(\frac{\varphi(v) - 1}{\varphi'(v)} - v \right) \frac{\Delta h}{h}.$$

Multiplying by h, we obtain

$$-\Delta(hv) \geq -h^\sigma \Phi + \left(\frac{\varphi(v) - 1}{\varphi'(v)} - v \right) \Delta h. \tag{8}$$

The convexity of φ implies

$$\frac{\varphi(s) - 1}{\varphi'(s)} - s \leq 0, \tag{9}$$

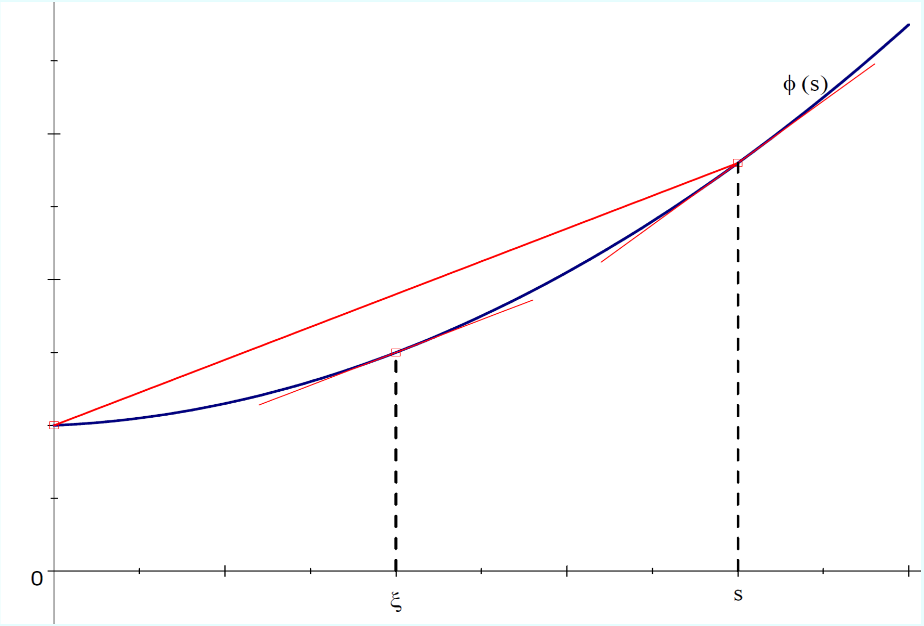
for any s in the domain of φ. Indeed, if $s > 0$ then $\exists \xi \in [0, s]$ such that

$$\varphi'(\xi) = \frac{\varphi(s) - \varphi(0)}{s}.$$

It follows that

$$\varphi'(s) \geq \varphi'(\xi) = \frac{\varphi(s) - 1}{s},$$

which yields (9).



If $s < 0$ then $\exists \xi \in [s, 0]$ such that

$$\frac{\varphi(s) - 1}{s} = \frac{\varphi(s) - \varphi(0)}{s} = \varphi'(\xi) \geq \varphi'(s),$$

which again implies (9) since $s < 0$.

Since $\Delta h \leq 0$ and

$$\frac{\varphi(v)-1}{\varphi'(v)} - v \leq 0,$$

we obtain

$$\left(\frac{\varphi(v)-1}{\varphi'(v)} - v\right)\Delta h \geq 0$$

and therefore by (8)

$$-\Delta\left(hv\right) \geq -h^{\sigma}\Phi \ \text{ in } \Omega.$$

On $\partial\Omega$ we have $v = \varphi^{-1}\left(u/h\right) \geq \varphi^{-1}\left(1\right) = 0$, which implies

$$hv \geq -G_{\Omega}\left(h^{\sigma}\Phi\right) \quad \text{in } \Omega.$$

Dividing by h and applying φ, we conclude that

$$\frac{u}{h} \geq \varphi\left(-\frac{1}{h}G_{\Omega}\left(h^{\sigma}\Phi\right)\right) \ \text{ in } \Omega.$$

Existence of positive solutions

Let us ask for which values $\sigma > 1$ the inequality

$$\Delta u + u^\sigma \leq 0 \tag{10}$$

has a positive solution u on M (the case of $\Phi \equiv -1$). For example, in $\mathbb{R}^n$ with $n \leq 2$ any non-negative solution of (10) is 0 while in the case $n > 2$, the inequality (10) has a positive solution in $\mathbb{R}^n$ if and only if

$$\sigma > \frac{n}{n-2}$$

(Mitidieri and Pohozaev, 1998). Hence, $\sigma_{crit} = \frac{n}{n-2}$.

Let $d(x, y)$ be a distance function on M, not necessarily geodesic, but such that the metric balls

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

are precompact open subsets of M. Set

$$V(x,r) = \mu(B(x,r)).$$

Theorem 3 (AG – Yuhua Sun, 2017) *Assume that, for some* $x_0 \in M$*,*

$$V(x_0, r) \simeq r^\alpha \quad \textit{for large } r \tag{V}$$

and

$$G(x, y) \simeq d(x, y)^{-\gamma} \quad \textit{for large } d(x, y), \tag{G}$$

where $\alpha > \gamma > 0$*. Then, for any* σ *satisfying*

$$1 < \sigma \leq \frac{\alpha}{\gamma},$$

the inequality

$$\Delta u + u^\sigma \leq 0 \tag{11}$$

has no positive solution in any exterior domain of M*.*

If in addition d *is the geodesic distance,* M *has bounded geometry, and* (V) *holds for all* $x_0 \in M$*, then, for any*

$$\sigma > \frac{\alpha}{\gamma},$$

the inequality (11) *has a positive solution on* M*.*

Hence, $\sigma_{crit} = \frac{\alpha}{\gamma}$.

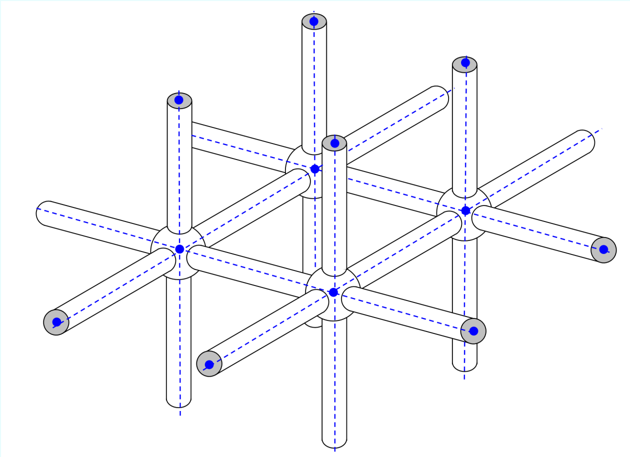
Example 1

Let Γ be an infinite connected graph with a uniformly bounded degree. Let $d(x,y)$ be the graph distance on Γ and $V(x,r)$ – the volume function.

Assume that the discrete Laplace operator on Γ has a positive Green function $G(x,y)$.

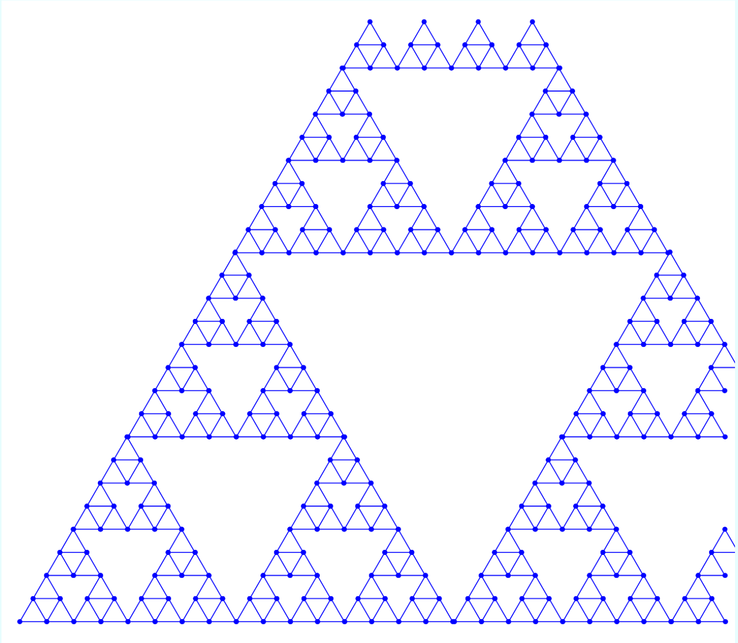
If Γ satisfies conditions (V) and (G) for some α and γ then we construct a manifold M satisfying (V) and (G) inflating the edges of Γ into 2-dim tubes.

Since M has bounded geometry, both parts of Theorem 3 apply in this case.



M.Barlow constructed in 2004 a family of fractal-like graphs such that each graph satsfies (V) with some α and has the *walk dimension* β, where α, β can be any real numbers satisfying $2 \leq \beta \leq \alpha + 1$.

For the *graphical Sierpinski gasket* we have $\alpha = \log_2 3$ and $\beta = \log_2 5$ so that $\beta > \alpha$.



However, for higher dimensional constructions one can achieve $\beta < \alpha$. In this case the graph has a positive Green function satysying (G) with $\gamma = \alpha - \beta$.

Since γ can be arbitrarily small, the critical value $\sigma_{crit} = \frac{\alpha}{\gamma}$ can be arbitrarily large, unlike the Euclidean critical value $\frac{n}{n-2}$.

Example 2

Assume that $G(x,y)$ satisfies the following $3G$-inequality

$$\frac{1}{G(x,y)} \leq C\left(\frac{1}{G(x,z)} + \frac{1}{G(z,y)}\right)$$

for all $x, y, z \in M$ and some $C > 1$. Then the function $\rho(x,y) = \frac{1}{G(x,,y)}$ is a quasi-metric on M. For any quasi-metric ρ, there exists $\gamma > 0$ and a distance function $d(x,y)$ such that $\rho(x,y) \simeq d(x,y)^{\gamma}$.

Hence, $G(x,y) \simeq d(x,y)^{-\gamma}$, that is, M satisfies (G). Assume that (M,ρ) satisfies (V), that is, for ρ-balls centered at x_0,

$$\mu\left(B_{\rho}\left(x_0,r\right)\right) \simeq r^{\alpha}.$$

Then, for d-balls, we obtain

$$\mu(B_d(x_0,r)) \simeq r^{\alpha\gamma}.$$

Hence, (M,d) satisfies (V) with $\tilde{\alpha} := \alpha\gamma$. Assuming in addition that all balls are precompact, we obtain by Theorem 3 that, for any $\sigma \leq \frac{\tilde{\alpha}}{\gamma} = \alpha$, the inequality $\Delta u + u^{\sigma} \leq 0$ has no positive solution in any exterior domain of M.

Idea of the proof of Theorem 3

Assume that u is a positive solution in $M \setminus K$ of

$$\Delta u + u^{\sigma} \leq 0.$$

It follows that, for any precompact open set $U \supset K$, we have $u \geq G_{\overline{U}^c} u^{\sigma}$ in U^c.

Since $\inf_{\partial U} u > 0$ and u is superharmonic, it follows that

$$u(y) \geq cG(y, x_0) \quad \text{for } y \in U^c,$$

for some $c > 0$. Hence, for any $x \in U^c$,

$$u(x) \geq c^{\sigma} \int_{U^c} G_{\overline{U}^c}(x, y) \, G^{\sigma}(y, x_0) \, d\mu(y). \tag{12}$$

On the other hand, one can prove that, for any precompact open set $\Omega \subset M$,

$$\sup_{\Omega}(\Delta u + \lambda_1(\Omega) u) \geq 0,$$

where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω. It follows that

$$\lambda_1(\Omega) \geq \inf_{\Omega} u^{\sigma - 1}. \tag{13}$$

Assuming $\Omega \subset U^c$ and combining (13) with (12) we obtain

$$\lambda_1(\Omega)^{\frac{1}{\sigma-1}} \geq c^\sigma \inf_{x\in\Omega} \int_{U^c} G_{\overline{U}^c}(x,y)\, G^\sigma(y,x_0)\, d\mu(y)\,.$$

If $\sigma \leq \frac{\alpha}{\gamma}$ then we bring this inequality to contradiction by choosing Ω large enough and by applying the hypotheses (V), (G) to estimate all the quantities involved.

For the proof of the second part of Theorem 3, we construct a positive solution of the equation

$$\Delta u + u^\sigma + \lambda^\sigma f^\sigma = 0 \ \text{ in } M,$$

where f is a specifically chosen decreasing function and $\lambda > 0$ is small enough. This differential equation amounts to the integral equation

$$u(x) = \int_M G(x,y)\left(u^\sigma(y) + \lambda^\sigma f(y)^\sigma\right) d\mu(y),$$

and the latter is solved in a certain closed subset of $L^\infty(M)$ by observing that the operator in the right hand side is a contraction for small enough λ. Next, we improve the regularity properties of u in two steps: first show that u is Hölder and then that $u \in C^2$.