Hardy inequality and heat semigroup estimates for Riemannian manifolds with singular data

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Abstract
Upper bounds are obtained for the heat content of an open set $D$ in a geodesically complete Riemannian manifold $M$ with Dirichlet boundary condition on $\partial D$, and non-negative initial condition. We show that these upper bounds are close to being sharp if (i) the Dirichlet-Laplace-Beltrami operator acting in $L^2(D)$ satisfies a strong Hardy inequality with weight $\delta^{-2}$, (ii) the initial temperature distribution, and the specific heat of $D$ are given by $\delta^{-\alpha}$ and $\delta^{-\beta}$ respectively, where $\delta$ is the distance to $\partial D$, and $1 < \alpha < 2, 1 < \beta < 2$.


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1 Introduction

Let $D$ be a smooth, connected, $m$-dimensional Riemannian manifold and let $\Delta$ be the Laplace-Beltrami operator on $D$. It is well known (see [11], [14]) that the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in D, \quad t > 0,$$

(1)

has a unique minimal positive fundamental solution $p(x, y; t)$ where $x \in D$, $y \in D$, $t > 0$. This solution, the Dirichlet heat kernel for $D$, is symmetric in $x, y$, strictly positive, jointly smooth in $x, y \in D$ and $t > 0$, and it satisfies the semigroup property

$$p(x, y; s + t) = \int_D p(x, z; s)p(z, y; t)dz,$$

(2)

for all $x, y \in D$ and $t, s > 0$, where $dz$ is the Riemannian measure on $D$.

Equation (1) with the initial condition

$$u(x; 0^+) = \psi(x), \quad x \in D,$$

(3)

has a solution

$$u_\psi(x; t) = \int_D p(x, y; t)\psi(y)dy,$$

(4)

for any function $\psi$ on $D$ from a variety of function spaces like $C_b(D)$ or $L^p(D)$, $1 \leq p < \infty$. Note that $u_\psi \in C_b(D)$ if $\psi \in C_b(D)$ or that $u_\psi \in L^p(D)$ if $\psi \in L^p(D)$. Initial condition (3) is understood in the sense that $u_\psi(\cdot; t) \rightarrow \psi(\cdot)$ as $t \rightarrow 0^+$, where the convergence is appropriate for the function space of initial conditions. For example, if $\psi \in C_b(D)$ then the convergence is locally uniform, or if $\psi \in L^p(D)$, $1 \leq p < \infty$ then the convergence is in the norm of $L^p(D)$. In general, (4) is not the unique solution of (1)- (3). However, it has the following distinguished property: if $\psi \geq 0$ then $u_\psi$ is the minimal non-negative solution of that problem (and if $\psi$ is signed then $u_\psi = u_\psi_+ - u_\psi_-$. If $D$ is an open subset of another Riemannian manifold $M$ and if the boundary $\partial D$ of $D$ in $M$ is smooth then the minimality property of $u_\psi$ implies that, for any $t > 0$,

$$\lim_{x \rightarrow \partial D} u_\psi(x; t) = 0.$$  

(5)

If $\partial D$ is non-smooth then (5) can still be understood in a weak sense. Expression (4) makes sense for any non-negative measurable function $\psi$ on $D$, provided the value $+\infty$ is allowed for $u_\psi$. It is known that if $u_\psi \in L^1_{\text{loc}}(D \times \mathbb{R}^+)$ then $u_\psi$ is a smooth function in $D \times \mathbb{R}^+$ and it solves (1) (see p. 201 in [14]). For any two non-negative measurable functions $\psi_1, \psi_2$ on $D$, we define for $t > 0$

$$Q_{\psi_1, \psi_2}(t) = \int_{D \times D} p(x, y; t)\psi_1(x)\psi_2(y)dxdy.$$  

(6)

Using the properties of the Dirichlet heat kernel we have for $0 < s < t$

$$Q_{\psi_1, \psi_2}(t) = \int_D u_{\psi_1}(x; s)u_{\psi_2}(x; t-s)dx.$$  

(7)

Assuming that $D$ is an open subset of a complete Riemannian manifold $M$, $Q_{\psi_1, \psi_2}(t)$ has the following physical interpretation: it is the amount of heat in
$D$ at time $t$ if $D$ has initial temperature distribution $\psi_1$, and a specific heat $\psi_2$, while the $\partial D$ is kept at fixed temperature 0.

This function has been subject of a thorough investigation. Its asymptotic behavior for small $t$ is well understood if $D$ has compact closure with $C^\infty$ boundary, and both $\psi_1$ and $\psi_2$ are $C^\infty$ on the closure $\overline{D}$ of $D$. In that case $Q_{\psi_1,\psi_2}(t)$ has an asymptotic series in $t^{1/2}$, and its coefficients are computable in terms of local geometric invariants [2, 12]. No such series are known if $D$ is unbounded, or if either the initial data or $\partial D$ are non-smooth.

In this paper we will obtain upper bounds for the heat content $Q_{\psi_1,\psi_2}(t)$ under quite general assumptions on $D$ and on $\psi_1$ and $\psi_2$.

We are particularly interested in the situation where $D$ is a open subset of another manifold $M$, and where $\psi_1(x)$ and $\psi_2(x)$ blow up as $x \to \partial D$. In order to guarantee finite heat content for $t > 0$, sufficient cooling at $\partial D$ needs to take place. This will be guaranteed by a condition on $D$, that is formulated in terms of a Hardy inequality. Note that in this setting $Q_{\psi_1,\psi_2}(t)$ may be unbounded as $t \to 0^+$, and one of the interesting points of this study is to obtain the rate of convergence of $Q_{\psi_1,\psi_2}(t)$ to $+\infty$ as $t \to 0^+$.

Given a positive measurable function $h$ on a manifold $D$, we say that the Dirichlet Laplacian acting in $L^2(D)$ satisfies a strong Hardy inequality with weight $h$ if, for all $w \in C^\infty_c(D)$,

$$\int_D |\nabla w|^2 \geq \int_D \frac{w^2}{h}.$$  \hfill (8)

Here, and in what follows, we put $\int_D f = \int_D f(x)dx$ if this does not cause confusion. We also put $|D| = \int_D 1$, and $\|f\|_p = (\int_D |f|^p)^{1/p}$. A typical example of a Hardy inequality is when $D$ is an open subset of another manifold $M$, and $h(x) = c^2 \delta(x)^2$,

$$\delta(x) = \min\{d(x, y) : y \in \partial D\},$$  \hfill (9)

where $c \geq 2$ is a constant, $\delta$ is the distance to the boundary, and $d(x, y)$ is the geodesic distance from $x$ to $y$ on $M$. Both the validity and applications of Hardy inequalities with weight (9) have been investigated extensively [1], [7], [9], [10], [11], [4]. For example, inequality (8) holds with weight (9) with $c = 4$ if $D$ is simply connected with non-empty boundary in $\mathbb{R}^2$, with $c = 2$ if $D$ is convex in $\mathbb{R}^m$, and for some $c \geq 2$ if $D$ is bounded with smooth boundary in $\mathbb{R}^m$.

In [3] it was shown that if $D$ has finite measure and satisfies the Hardy inequality with weight $h$, and if $\psi$ is a non-negative measurable function on $D$, such that, for some $q > 1$,

$$\|\psi h^{1/q}\|_{q/(q-1)} < \infty,$$  \hfill (10)

then, for all $t > 0$,

$$Q_{\psi,1}(t) \leq \left(\frac{q^2}{4(q-1)}\right)^{1/q} \|\psi h^{1/q}\|_{q/(q-1)} \|1 - u_1 (\cdot ; t)\|^1_{1/q} t^{-1/q},$$  \hfill (11)
where $Q_{1,1}$ is defined by (6) for $\psi_1 = \psi_2 = 1$, that is,
\[
Q_{1,1}(t) = \int_D u_1(x; t) \, dx = \int_{D \times D} p(x, y; t) \, dx \, dy.
\]

A similar estimate holds for arbitrary open sets $D \subset \mathbb{R}^m$, satisfying the Hardy inequality with weight $h$. If $\psi$ is a non-negative measurable function on $D$ such that, for some $q > 1$,
\[
\|\max\{\psi, 1\} h^{1/q}\|_{q/(q-1)} < \infty, \tag{12}
\]
then, for all $t > 0$,
\[
Q_{\psi,1}(t) \leq a(q) \|\psi h^{1/q}\|_{q/(q-1)} \|h^{1/(q(q-1))}\|_{q^{-1}/(q-1)}, \tag{13}
\]
where
\[
a(q) = 4^{-1/q} \left( \frac{q}{q-1} \right)^{(2q-1)/(q(q-1))}. \tag{14}
\]

Below we give a sufficient condition for the finiteness of $Q_{\psi_1, \psi_2}(t)$ for all $t > 0$, and reduce the problem of finding upper bounds for $Q_{\psi_1, \psi_2}(t)$ to the case $\psi_1 = \psi_2$.

**Theorem 1.** Let $\psi_1$ and $\psi_2$ be non-negative and Borel measurable on a manifold $D$.

(i) If $Q_{\psi_i, \psi_i}(t) < \infty$, $i = 1, 2$, for all $t > 0$, then $Q_{\psi_1, \psi_2}(t) < \infty$ for all $t > 0$, and
\[
Q_{\psi_1, \psi_2}(t) \leq (Q_{\psi_i, \psi_i}(t) Q_{\psi_2, \psi_2}(t))^{1/2}, \quad t > 0. \tag{15}
\]

(ii) If $Q_{\psi_i, 1}(t) < \infty$, $i = 1, 2$, for all $t > 0$, and if
\[
c_t := \sup_{x \in D} p(x, x; t) < \infty, \quad t > 0, \tag{16}
\]
then
\[
Q_{\psi_1, \psi_2}(t) \leq c_{t/3} Q_{\psi_1, 1}(t/3) Q_{\psi_2, 1}(t/3) < \infty, \quad t > 0. \tag{17}
\]

Our main results are the following three theorems, in which we assume that $D$ is a Riemannian manifold that satisfies the Hardy inequality with some weight $h$, and $\psi$ is a non-negative measurable function on $D$. In particular we do not assume any smoothness conditions on $\partial D$, nor do we assume that $D$ has finite measure or that $D$ is bounded.

**Theorem 2.** If $|D| < \infty$, and if there exists $1 < q \leq 2$ such that
\[
\|\psi h^{1/q}\|_{q/(q-1)} < \infty, \tag{17}
\]
then, for all $t > 0$,
\[
Q_{\psi, \psi}(t) \leq \frac{q^{(q-4)/2}}{(2(q-1))^{2/q}} \|\psi h^{1/q}\|_{q/(q-1)}^2 \|1 - u_1(\cdot; t)\|_{1}^{(2-q)/2} t^{-2/q}. \tag{18}
\]
Theorem 3. If $1 < q \leq 2$ is such that (17) holds and that
\[ \|h^{1/q}\|_{q/(q-1)} < \infty, \]
then
\[ Q_{\psi,\psi}(t) \leq b(q) \|\psi h^{1/q}\|_{q/(q-1)}^2 \|h^{1/q}\|_{q/(q-1)}^{(2-q)/(q-1)} t^{-(q-1)/(q-1)}, \quad t > 0, \quad (19) \]
where
\[ b(q) = 2^{(4q-3)/(q(q-1))} \left(\frac{q}{q-1}\right)^{(q^2-4q+2)/(q(q-1))}. \quad (20) \]

Theorem 4. If $0 \leq r \leq 2$, and $1 < q \leq 2$ are such that
\[ \|\psi^r\|_q < \infty, \]
and
\[ \|\psi^{2-r} h^{1/q}\|_{(q-1)/q} < \infty, \]
then
\[ Q_{\psi,\psi}(t) \leq \left(\frac{q}{4(q-1)}\right)^{1/q} \|\psi^r\|_q \|\psi^{2-r} h^{1/q}\|_{q/(q-1)} t^{-1}/q, \quad t > 0. \quad (21) \]

In Theorem 5 in Section 3 we use the bounds of Theorems 2 and 4 together with (15) to obtain an upper bound for the heat content of $D$, when $D$ satisfies a Hardy inequality with weight (9), and $\psi_1(x) = \delta(x)^{-\alpha}$ and $\psi_2(x) = \delta(x)^{-\beta}$, where $\alpha, \beta \in (1, 2)$. Even though the bounds in e.g. 2 and 4 look very different, both of them are needed to cover the maximal range of $\alpha$ and $\beta$ in Theorem 5.

Theorem 2 has a curious consequence. We claim that if a manifold $D$ has finite measure $|D|$, and is stochastically complete then no Hardy inequality holds on $D$ (which confirms the philosophy that the Hardy inequality corresponds to cooling that comes from the boundary). Indeed, stochastic completeness means that $u_1 \equiv 1$. In this case, $\|1 - u_1(\cdot,t)\|_1 = 0$ so that we obtain from (18) that $Q_{\psi,\psi}(t) = 0$ whenever function $\psi$ satisfies the condition (17) for some $q \in (1, 2)$. However, if $h$ is finite then it is easy to construct a non-trivial function $\psi$ that satisfies (17): choose any measurable set $S$ with finite positive measure such that $h$ is bounded on $S$, and let $\psi = 1_S$. Then (17) holds with any $q > 1$ while $Q_{\psi,\psi}(t) > 0$ so that we obtain contradiction. Of course, without the finiteness of $|D|$, the Hardy inequality may hold on stochastically complete manifolds like $\mathbb{R}^m \setminus \{0\}$.

This paper is organized as follows. In Section 2 we will prove Theorems 1, 2, 3 and 4. In Section 3 we will state and prove Theorem 5. Finally in Section 4 we obtain very refined asymptotics in the special case of the ball in $\mathbb{R}^3$ with $\psi_1(x) = \delta(x)^{-\alpha}, \alpha < 2, \psi_2(x) = \delta(x)^{-\beta}, \beta < 2$, and $\alpha + \beta > 3$ (Theorem 7). This special case shows that the bound obtained in Theorem 5 is close to being sharp. Moreover it suggests formulae for the first few terms in the asymptotic series of a compact Riemannian manifold $D$ with the singular data above.
2 Proofs of Theorems 1, 2, 3 and 4

Proof of Theorem 1. In both parts, it suffices to prove the claims for non-negative functions \( \psi_1, \psi_2 \) from \( L^2(D) \). Arbitrary non-negative measurable functions \( \psi_1, \psi_2 \) can be approximated by monotone increasing sequences of non-negative functions from \( L^2(D) \), whence the both claims follow by the monotone convergence theorem.

To prove part (i) we use symmetry and the semigroup property, and obtain by (7) for \( s = t/2 \) that
\[
Q_{\psi_1, \psi_2}(t) = \int_D u_{\psi_1}(x; t/2)u_{\psi_2}(x; t/2)dx \\
\leq \left( \int_D u^2_{\psi_1}(x; t/2)dx \right)^{1/2} \left( \int_D u^2_{\psi_2}(x; t/2)dx \right)^{1/2} \\
= (Q_{\psi_1, \psi_1}(t)Q_{\psi_2, \psi_2}(t))^{1/2}.
\]
It follows from (2) and (16) that
\[
p(x, y; t) \leq (p(x, x; t)p(y, y; t))^{1/2} \leq c_t. \tag{22}
\]

To prove part (ii) we have by (22) that
\[
p(x, y; t) = \int_D p(x, z_1; t/3)p(z_1, z_2; t/3)p(z_2, y; t/3)dz_1dz_2 \\
\leq c_{t/3}u_1(x; t/3)u_1(y; t/3). \tag{23}
\]
This together with definition (6) completes the proof.

For the proofs of Theorems 2, 3, 4, choose a sequence \( \{D_n\} \) that consists of precompact open subsets of \( D \) with smooth boundaries such that \( \overline{D}_n \subset D_{n+1} \) and \( \bigcup_n D_n = D \). Obviously, Hardy inequality (8) remains true in any \( D_n \) with the same weight \( h \), because \( C^\infty_c(D_n) \subset C^\infty(D) \). Moreover, we claim that (8) holds for any function \( w \in C(\overline{D}_n) \cap C^1(D_n) \) that satisfies the boundary condition \( w|_{\partial D_n} = 0 \). Indeed, if \( \int_{D_n} |\nabla w|^2 = \infty \) then (8) is trivially satisfied.

If \( \int_{D_n} |\nabla w|^2 < \infty \) then \( w \) belongs to the Sobolev space \( W^{1,2}(D_n) \). Extend function \( w \) to \( D_{n+1} \) by setting \( w = 0 \) in \( D_{n+1} \setminus D_n \). Due to the boundary condition \( w|_{\partial D_n} = 0 \), we obtain that \( w_n \in W^{1,2}(D_{n+1}) \). Since \( w \) is compactly supported in \( D_{n+1} \), it follows that \( w \in W^{1,2}_0(D_{n+1}) \) where \( W^{1,2}_0(\Omega) \) is the closure of \( C^\infty_c(\Omega) \) in \( W^{1,2}(\Omega) \). Since the Hardy inequality (8) holds for functions from \( C^\infty_c(D_{n+1}) \), passing to the limit in \( W^{1,2}(D_{n+1}) \) and using Fatou’s lemma, we obtain that \( w \) also satisfies (8).

Assume for a moment that the statements of the theorems have been proved in each domain \( D_n \). Then one can take the limit in (18), (19), (21) as \( n \to \infty \), and obtain the statements for \( D \). Indeed, the left hand side of these inequalities is \( Q_{\psi_1, \psi_2}^{D_n}(t) = \int_{D_n} p_{D_n}(x, y; t)\psi(x)\psi(y)dxdy \), where \( p_{D_n} \) is the Dirichlet heat kernel for \( D_n \). This converges to \( Q_{\psi_1, \psi_2}^{D}(t) \) as \( n \to \infty \). The right hand sides of (18), (19), (21) contain various \( L^p(D_n) \)-norms that can be estimated from above by the \( L^p(D) \)-norms. The only exception is the term \( \|1 - \int_{D_n} p_{D_n}(\cdot, y; t)dy\|_1 \) in (18) that is decreasing as \( n \to \infty \). If \( |D| < \infty \) then \( 1 \in L^1(D) \) so that the passage to the limit is justified by the dominated convergence theorem.
Hence, it suffices to prove each of the statements for $D_n$ instead of $D$. Renaming $D_n$ back to $D$, we assume in all three proofs that $D$ is a precompact open domain with smooth boundary in $M$.

Another observation is that all inequalities (18), (19), (21) survive the increasing monotone limits in $\psi$. So it suffices to prove them when $\psi$ is bounded and has a compact support in $D$, which will be assumed below. Furthermore, since all the statements of Theorems 2, 3, 4 are homogeneous with respect to $\psi$, we can assume that $0 \leq \psi \leq 1$. If $\psi \equiv 0$ then there is nothing to prove; hence, we assume that $\psi$ is non-trivial. Then $u_\psi(x; t)$ is smooth and bounded in $\overline{D} \times (0, +\infty)$ and positive in $D \times (0, +\infty)$.

**Proof of Theorem 2.** Let $\nu$ be the outwards normal vector field on $\partial D$. Using the Green’s formula, we obtain

$$
- \frac{d}{dt} \int_D u_\psi^q = -q \int_D u_\psi^{q-1} \frac{\partial u_\psi}{\partial t} = -q \int_D u_\psi^{q-1} \Delta u_\psi
$$

$$
= -q \int_{\partial D} u_\psi^{q-1} \frac{\partial u_\psi}{\partial \nu} + q \int_D \left( \nabla u_\psi^{q-1}, \nabla u_\psi \right)
$$

$$
= q (q - 1) \int_D u_\psi^{q-2} |\nabla u_\psi|^2 ,
$$

where we have used that $q > 1$ and, hence $u_\psi^{q-1} = 0$ on $\partial D$. Observing that $u_\psi^{q/2} \in C(D) \cap C^1(D)$,

$$
|\nabla u_\psi^{q/2}|^2 = \frac{q^2}{4} u_\psi^{q-2} |\nabla u_\psi|^2 ,
$$

and applying the Hardy inequality (8) to $u^{q/2}$, we obtain that

$$
- \frac{d}{dt} \int_D u_\psi^q = \frac{4(q - 1)}{q} \int_D |\nabla (u_\psi^{q/2})|^2 \geq \frac{4(q - 1)}{q} \int_D \frac{u_\psi^q}{h} .
$$

(25)

By Hölder’s inequality we have that

$$
Q_{\psi,\psi}(t) = \int_D u_\psi \psi
$$

$$
\leq \left( \int_D \left( \frac{u_\psi}{h^{1/q}} \right)^q \right)^{1/q} \left( \int \left( \psi h^{1/q} \right)^{q-1} \right)^{2/q - 1} = \left( \int_D \frac{u_\psi^q}{h} \right)^{1/q} \left\| \psi h^{1/q} \right\|_{q/(q-1)}. \tag{26}
$$

By (25) and (26) we conclude that

$$
- \frac{d}{dt} \int_D u_\psi^q \geq \frac{4(q - 1)}{q} \left\| \psi h^{1/q} \right\|_{q/(q-1)} (Q_{\psi,\psi}(t))^q
$$

(27)

By (25) and (26) we conclude that

$$
- \frac{d}{dt} \int_D u_\psi^q \geq \frac{4(q - 1)}{q} \left\| \psi h^{1/q} \right\|_{q/(q-1)} (Q_{\psi,\psi}(t))^q
$$

(27)

Note that the function $t \mapsto Q_{\psi,\psi}(t) = \|u_\psi(\cdot; t/2)\|^2_2$ is decreasing in $t$, which, for example, follows from (24) with $q = 2$. Integrating differential inequality
(27) with respect to $t$ over the interval $[t, 2t]$ gives that

$$\int_{D} u_{\psi}^{q} \geq \frac{4(q - 1)}{q} \|\psi_{h}^{1/q} \|_{q/(q-1)}^{q} (Q_{\psi, \psi}(2t))^{q} t. \quad (28)$$

On the other hand, using $1 < q < 2$ and the Hölder inequality, we obtain

$$\int_{D} u_{\psi}^{2} = \int_{D} u_{\psi}^{2-q} u_{\psi}^{q-2} \leq \left( \int_{D} u_{\psi}^{2} \right)^{2-q} \left( \int_{D} u_{\psi}^{q-2} \right)^{q-1}$$

that is,

$$\int_{D} u_{\psi}^{q} \leq (Q_{\psi, 1}(t))^{2-q} (Q_{\psi, \psi}(2t))^{q-1}. \quad (29)$$

Combining (28) and (29) yields

$$Q_{\psi, \psi}(2t) \leq \frac{q}{4(q - 1)} \|\psi_{h}^{1/q} \|_{q/(q-1)}^{q} (Q_{\psi, 1}(t))^{2-q} t^{-1}. \quad (30)$$

Estimating $Q_{\psi, 1}$ by (11), we obtain

$$Q_{\psi, \psi}(2t) \leq \frac{q}{4(q - 1)} a(q)^{2-q} \|\psi_{h}^{1/q} \|_{q/(q-1)}^{q} (Q_{\psi, 1}(t))^{2-q} t^{-1}.$$ 

which completes the proof.

Proof of Theorem 3. Since $\psi \leq 1$ we have that (12) is satisfied. We obtain by (13) and (30) that

$$Q_{\psi, \psi}(2t) \leq \frac{q}{4(q - 1)} a(q)^{2-q} \|\psi_{h}^{1/q} \|_{q/(q-1)}^{q} (Q_{\psi, 1}(t))^{2-q} t^{-1}.$$ 

This completes the proof of Theorem 3 since, by (14) and (20),

$$2^{1/(q-1)} \frac{q}{4(q - 1)} a(q)^{2-q} = b(q). \quad \square$$

Proof of Theorem 4. By the arithmetic-geometric mean inequality, we have

$$\psi(x)\psi(y) \leq \frac{1}{2} (\psi(x)^{r} \psi(y)^{2-r} + \psi(x)^{2-r} \psi(y)^{r}).$$

By non-negativity and symmetry of the Dirichlet heat kernel

$$Q_{\psi, \psi}(t) \leq \int_{D} u_{\psi} \psi^{2-r}. \quad (31)$$

Next, Hölder’s inequality yields

$$\int_{D} u_{\psi} \psi^{2-r} \leq \left( \int_{D} u_{\psi}^{q} \frac{1}{h} \right)^{1/q} \|\psi^{2-r} h^{1/q} \|_{q/(q-1)}. \quad (32)$$

By (25) we have

$$- \frac{d}{dt} \int_{D} u_{\psi}^{q} \geq \frac{4(q - 1)}{q} \int_{D} u_{\psi}^{q} \frac{1}{h}. \quad (33)$$
Combining (31), (32), (33) we obtain that

\[
(Q_{\psi,\psi}(t))^q \leq -\frac{q}{4(q-1)} \frac{d}{dt} \left( \int_D u^{q}_{\psi^q} \right) \|\psi^{2-r} h^{1/q}\|_q^q.
\]

Since the function \( t \mapsto Q_{\psi,\psi}(t) \) is decreasing in \( t \), we obtain by integrating the differential inequality (33) with respect to \( t \) over the interval \([0, t]\) that

\[
t^{Q_{\psi,\psi}(t))^q} \leq -\frac{q}{4(q-1)} \left( \int_D \psi^{q} \right) \|\psi^{2-r} h^{1/q}\|_q^q \frac{d}{dt} \left( \int_D u^{q}_{\psi^q} \right) \|\psi^{2-r} h^{1/q}\|_q^q,
\]

and (21) follows.

\[\square\]

3 Singular initial temperature and singular specific heat

Below we make some further hypothesis on the geometry of \( D \), and obtain an upper bound for the heat content for a wide class of geometries using Theorems 2 and 4, and (15), if the initial temperature distribution and specific heat are given by \( \delta - \alpha \), \( 1 < \alpha < 2 \), and \( \delta - \beta \), \( 1 < \beta < 2 \) respectively.

**Theorem 5.** Let \( D \) be an open set in a smooth complete \( m \)-dimensional Riemannian manifold \( M \), and suppose that

i. The Ricci curvature on \( M \) is non-negative.

ii. For \( x \in D \),

\[\psi_{\alpha}(x) = \delta(x)^{-\alpha}.\]

iii. \( D \) has finite inradius i.e. \( \rho_D = \sup \{\delta(x) : x \in D\} < \infty \).

iv. There exist constants \( \kappa_D < \infty, d \in [m-1, m) \) such that

\[
\int_{\{x \in D : \delta(x) < \rho\}} 1 \leq \kappa_D \rho^{m-d}, \quad 0 < \rho \leq \rho_D.
\]

v. The strong Hardy inequality (8) holds with (9) for some \( c \geq 2 \).

If \( 1 < \alpha < 2, 1 < \beta < 2, \) and if \( \epsilon > 0 \) then

\[
Q_{\psi_{\alpha},\psi_{\beta}}(t) = O(t^{-\epsilon+(m-d-\alpha-\beta)/2}), \quad t \to 0.
\]

**Proof.** Note that (iii) and (iv) in Theorem 5 imply that \( |D| \leq \kappa_D \rho_D^{m-d} < \infty \). By (15) it suffices to prove (35) in the special case \( \alpha = \beta \) with \( 1 < \alpha < 2 \). In order to estimate \( \|1 - u_1(.; t)\|_1 \) in Theorem 2 we rely on the following lower bound for \( u_1 \) (Lemma 5 in [5]).

**Lemma 6.** Let \( M \) be a smooth, geodesically complete Riemannian manifold with non-negative Ricci curvature, and let \( D \) be an open subset of \( M \) with boundary \( \partial D \). Then for \( x \in D, t > 0 \)

\[
u_1(x; t) \geq 1 - 2^{(2+m)/2} e^{-\delta(x)^2/(8t)}.
\]

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To prove (35) we first consider the case
\[(2 + m - d)/2 < \alpha < 2.\]  
(36)

This set of \(\alpha\)'s is non-empty since \(d \in [m - 1, m)\). By (9) we have that
\[\|\psi_\alpha \|^{1/q}_{q/(q-1)} = c^{2/q} \left( \int_D \delta^{(2-q\alpha)/(q-1)} \right)^{(q-1)/q}.\]  
(37)

Denote the left hand side of (34) by \(\omega_D(\rho)\). Then we can write the right hand side of (37) as
\[c^{2/q} \left( \int_{\mathbb{R}^+} \rho^{(2-q\alpha)/(q-1)} \omega_D(d\rho) \right)^{(q-1)/q}.\]  
(38)

An integration by parts, using (36) shows that (38) is finite for
\[q < \frac{2 - m + d}{\alpha - m + d},\]  
(39)

Since \(\alpha\) satisfies (36), we have that the right hand side of (39) is in \((1, 2)\). We now choose \(\epsilon > 0\) such that
\[2 - m + d \alpha^{-1} \in (1, 2),\]  
(40)

and choose \(q\) equal to the left hand side of (40). By Lemma 6 and (34) we have that for \(t \to 0\)
\[\|1 - u_1(\cdot, t)\|_1 = \int_D (1 - u(x, t))dx \leq 2^{(m+2)/2} \int_D e^{-\rho^2/(8t)} \leq 2^{(m+2)/2} \int_{\mathbb{R}^+} e^{-\rho^2/(8t)} \omega_D(d\rho) = 2^{(m+2)/2} e^{-\rho_D^2/(8t)} [D] + 2^{(m+2)/2} 2^D t^{-1} \int_0^\rho D \rho^{m-d+1} e^{-\rho^2/(8t)} d\rho = O(t^{(m-d)/2}).\]  
(41)

By Theorem 2 and (37)-(41) we find that for all \(\alpha\) satisfying (36) and all \(\epsilon > 0\) satisfying (40)
\[Q_{\psi_\alpha, \psi_\alpha}(t) = O(t^{-\epsilon(\alpha - m + d) + (m - d - 2\alpha)/2}), \ t \to 0.\]  
(42)

We conclude that (35) holds for all \(\alpha = \beta\) satisfying (36), and all \(\epsilon > 0\).

Next consider the case
\[1 < \alpha < (2 + m - d)/2.\]  
(43)

This set of \(\alpha\)'s is again non-empty since \(d \in [m - 1, m)\). By (34) we have that
\[\|\psi_\alpha\|_{q^{1/q}} = \left( \int_{\mathbb{R}^+} \omega_D(d\rho) \rho^{-\alpha q} \right)^{1/q} < \infty\]  
(44)
for
\[ \alpha q < m - d, \]  
and
\[ \| \psi^{2-r} h^{1/q} \|_{q/(q-1)} = \left( \int_{\mathbb{R}^+} \omega_D(d\rho) \rho^{(2-\alpha)(2-r)/q(q-1)} \right)^{(q-1)/q} < \infty \]  
for
\[ \frac{\alpha q(2-r) - 2}{q-1} < m - d. \]

The optimal choice for \( r \) is henceforth given by
\[ r = 2\left( \alpha q - 1 \right) \frac{\alpha^{-1} q - 2}{q - 1}. \]

By (43) we also have that \( \alpha > 1 \). Hence \( r \in (0, 2) \). The requirements under (45) and (47) become with this choice of \( r \) that
\[ q < 2(2\alpha + d - m)^{-1}. \]

Since \( \alpha \) satisfies (43), the right hand side of (49) is in \((1,2)\). We now choose \( \epsilon > 0 \) such that
\[ 2((2\alpha + d - m)(1 + 2\epsilon))^{-1} \in (1,2), \]
and choose \( q \) equal to the left hand side of (50). By Theorem 4 and (44)-(49) we find that for all \( \alpha \) satisfying (43), and all \( \epsilon > 0 \) satisfying (50)
\[ Q_{\psi, \phi}(t) = O(t^{-(2\alpha - m + d + (m - d - 2\alpha)/2)}), \quad t \to 0. \]

We conclude that (35) holds for all \( \alpha = \beta \) satisfying (43), and all \( \epsilon > 0 \).

To prove (35) for the limiting case \( \alpha = \beta = (2 + m - d)/2 := \alpha_c \) we note that \( Q_{\psi, \phi}(t) \) is monotone on the positive cone of non-negative and measurable \( \psi \) and \( \phi \). Let \( \alpha = \alpha_c + \epsilon \) where \( \epsilon \) is such that \( \alpha \in (\alpha_c, 2) \). Since
\[ \psi_{\alpha_c} \leq \psi_{\alpha_c}^{\alpha_c} \psi_{\alpha_c}. \]
we have by (42) that
\[ Q_{\psi_{\alpha_c}, \psi_{\alpha_c}}(t) \leq t^{2(\alpha - \alpha_c)} Q_{\psi_{\alpha_c}, \psi_{\alpha_c}}(t) \leq t^{2(\alpha - \alpha_c)} t^{(t^{-(2\alpha - m + d + (m - d - 2\alpha)/2)}/t^{-(2\alpha - (2 + \epsilon + (d - m)/2) + (m - d - 2\alpha_c)/2)})} \]
\[ = O(t^{-(2\alpha - m + d + (m - d - 2\alpha_c)/2)}). \]

We conclude that (35) holds for \( \alpha = \beta = \alpha_c \), and all \( \epsilon > 0 \).

4 The special case calculation for a ball in \( \mathbb{R}^3 \)

In this section we show by means of an example that the upper bound obtained in Theorem 5 is close to being sharp for \( \alpha < 2, \beta < 2, \alpha + \beta > 3 \).
Let $t \rightarrow 0$ and then there exist coefficients $b_0, b_1, \cdots$ depending on $\alpha$ and $\beta$ only such that for

$$Q_{\psi_0, \psi_1}(t) = 4\pi c_{\alpha, \beta}a^2(1-\alpha-\beta)/2 - 4\pi (c_{\alpha-1, \beta} + c_{\alpha, \beta-1})at^{(2-\alpha-\beta)/2}$$

$$+ 4\pi c_{\alpha-1, \beta-1}t^{(3-\alpha-\beta)/2} + \sum_{j=0}^{J} b_j a^3 j^{-\alpha-\beta}t^{j/2} + O(t^{(J+1)/2}), \quad (53)$$

where

$$c_{\alpha, \beta} = 2^{-\alpha-\beta}\pi^{-1/2}\Gamma((2-\alpha-\beta)/2)$$

$$\times \int_0^1 (\rho^{-\alpha} + \rho^{-\beta})(1-\rho)^{\alpha+\beta-2} - (1+\rho)^{\alpha+\beta-2})d\rho, \quad (54)$$

and

$$b_0 = -8\pi((\alpha + \beta - 1)(\alpha + \beta - 2)(\alpha + \beta - 3))^{-1},$$

$$b_1 = 0,$$

$$b_2 = 8\pi\alpha\beta((\alpha + \beta + 1)(\alpha + \beta)(\alpha + \beta - 1))^{-1},$$

$$b_3 = 0. \quad (55)$$

We see that the leading term in (53) jibes with (35) since (9) holds for some $c \geq 2$, and (34) holds with $d = m - 1$.

Theorem 7 suggests that for any precompact $D$ with smooth $\partial D$ in $M$, and for $\alpha < 2, \beta < 2, \alpha + \beta > 3$ and $t \rightarrow 0$

$$Q_{\psi_0, \psi_1}(t) = c_{\alpha, \beta} \int_{\partial D} f^{(1-\alpha-\beta)/2} - 2^{-1}(c_{\alpha-1, \beta} + c_{\alpha, \beta-1}) \int_{\partial D} L_{gg}t^{(2-\alpha-\beta)/2}$$

$$+ \int_{\partial D} (c_1 L_{gg}L_{hh} + c_2 L_{gh}L_{gh})t^{(3-\alpha-\beta)/2} + O(1), \quad (56)$$

where $c_1$ and $c_2$ are constants depending on $\alpha$ and $\beta$ only, and which satisfy

$$4c_1 + 2c_2 = c_{\alpha-1, \beta-1},$$

and where $L_{gg}$ is the trace of the second fundamental form on the boundary of $\partial D$ oriented by an inward unit vector field. Since $\int_{\partial B_a} 1 = 4\pi a^2$, $\int_{\partial B_a} L_{gg} = 8\pi a$ and $\int_{\partial B_a} (c_1 L_{gg}L_{hh} + c_2 L_{gh}L_{gh}) = 16\pi c_1 + 8\pi c_2$, we see that (56) holds for the ball in $\mathbb{R}^3$.

The proof of Theorem 7 rests on the following result (pp.237, 367-368 in [8]).

**Lemma 8.** Let $B_a$ as in Theorem 7, and let the initial datum be radially symmetric i.e. $\psi_1(x) = f(r)$, where $r = |x|$. Then the solution of (1), (3), (5) is given by

$$u(x; t) = (4\pi t)^{-1/2} \int_0^a r' f(r') \sum_{n \in \mathbb{Z}} (e^{-(2n\alpha-r')^2/(4t)} - e^{-(2n\alpha+r')^2/(4t)})dr'.$$
To prove Theorem 7 we have by Lemma 8 that
\[
Q_{\psi_\alpha, \psi_\beta}(t) = (4\pi/t)^{1/2} \int_{S_a} r r'(a - r)^{-\alpha}(a - r')^{-\beta} \times \sum_{n \in \mathbb{Z}} (e^{-(2na - r + r')^2/(4t)} - e^{-(2na + r + r')^2/(4t)}) dr dr',
\]  \hspace{1cm} (57)
where \( S_a = [0, a] \times [0, a] \). Substitution of \( a - r = p \) and \( a - r' = q \) in (57) gives that
\[
Q_{\psi_\alpha, \psi_\beta}(t) = A_0 + A_1 + A_2 + B,
\]
where
\[
A_0 = (4\pi/t)^{1/2} a^2 \int_{S_a} p^{-\alpha} q^{-\beta} (e^{-(p - q)^2/(4t)} - e^{-(p + q)^2/(4t)}) dp dq,
\]
\[
A_1 = -(4\pi/t)^{1/2} a \int_{S_a} (p + q) p^{-\alpha} q^{-\beta} (e^{-(p - q)^2/(4t)} - e^{-(p + q)^2/(4t)}) dp dq,
\]
\[
A_2 = (4\pi/t)^{1/2} a^2 \int_{S_a} \sum_{n \geq 1} p^{-\alpha} q^{-\beta} \left( e^{-(2na + p - q)^2/(4t)} - e^{-(2na + q + p)^2/(4t)} - e^{-(2na - q - p)^2/(4t)} \right) dp dq,
\]
and
\[
B = (4\pi/t)^{1/2} \int_{S_a} (a - p)(a - q) p^{-\alpha} q^{-\beta} \sum_{n \geq 1} (e^{-(2npa + p - q)^2/(4t)} - e^{-(2npa + q + p)^2/(4t)} - e^{-(2npa - q - p)^2/(4t)}) dp dq. \hspace{1cm} (58)
\]
We have the following.

**Lemma 9.** If \( 1 < \alpha < 2, 1 < \beta < 2 \) then for \( t \to 0 \)
\[
B = -8\pi^{1/2} 3^{-1} a^{-\alpha - \beta} t^{3/2} + O(t^2). \hspace{1cm} (59)
\]

**Proof.** The integrand in (58) can be rewritten as
\[
(a - p)(a - q) p^{-\alpha} q^{-\beta} \sum_{n \geq 1} \left( e^{-(2na - p - q)^2/(4t)} \right) \times \left( (e^{(p - 2na)q/t} + e^{(q - 2na)p/t})(1 - e^{-aq/t}) - (1 - e^{-2pna/t})(1 - e^{-2qna/t}) \right). \hspace{1cm} (60)
\]
The contribution from the terms with \( n \geq 2 \) in (60) is bounded in absolute value by
\[
2a^2 p^{1-\alpha} q^{1-\beta} t^{-1} \sum_{n \geq 2} e^{-a^2(n-1)^2/t} (1 + 2n^2 a^2 t^{-1}).
\]
After integrating with respect to \( p \) and \( q \) we see that this term contributes at most \( O(e^{-a^2/(2t)}) \) to \( B \). Next we will show that the main contribution from the term with \( n = 1 \) in (60) comes from a neighbourhood of the point \( (p, q) = (a, a) \). Let \( C_1(a) = \{(p, q) \in \mathbb{R}^2 : a/3 < p < a, a/3 < q < a\} \).
and

\[ C_2(a) = S_0 \setminus C_1(a). \]

On \( C_2(a) \) we have that \( 2a - p - q \geq 2a/3 \). Hence the term with \( n = 1 \) in (60) is bounded on \( C_2(a) \) in absolute value by

\[ 2(a - p)(a - q)p^{1-\alpha}q^{1-\beta}t^{-1}e^{-a^2/(9t)}(1 + 2a^2t^{-1}). \tag{61} \]

Integrating (61) over \( C \) and \( O \) is bounded on \( C_2(a) \) in absolute value by \( O(e^{-a^2/(18t)}) \). In order to calculate the contribution from the term with \( n = 1 \) on \( C_1(a) \) we use the expression under (58) instead. First we note that \( 2a + p - q \geq 2a/3, 2a + q - p \geq 2a/3, 2a + p + q \geq 8a/3 \). Hence the first three terms in the summand of (58) with \( n = 1 \) give after integration over \( C_1(a) \) a contribution \( O(e^{-a^2/(18t)}) \). Putting all this together gives that

\[ B = - (4\pi/t)^{1/2} \int_{C_1(a)} (a - p)(a - q)p^{-\alpha}q^{-\beta} \]

\[ \times e^{-2(a-q-p)^2/(4t)} \, dpdq + O(e^{-a^2/(18t)}). \]

Noting that

\[ p^{-\alpha}q^{-\beta} = a^{-\alpha-\beta} + O(a - p) + O(a - q) \tag{62} \]

uniformly in \( p \) and \( q \) yields after a change of variables that

\[ B = - (4\pi/t)^{1/2} a^{-\alpha-\beta} \int_{S_{a/3}} p q e^{-(p+q)^2/(4t)} \]

\[ \times (1 + O(p) + O(q)) \, dpdq + O(e^{-a^2/(18t)}), \]

which agrees with the right hand side of (59).

By taking higher order terms of the form \( (a - p)^{n_1}(a - q)^{n_2} \) in (62) into account one can determine the coefficient \( t^{(j+3)/2}, j = 0, 1, 2, \cdots \) in the expansion of \( B \).

To complete the proof of Theorem 7 we rewrite \( A_0, A_1 \) and \( A_2 \) respectively as follows.

\[ A_0 = (4\pi/t)^{1/2} a^2 \left( \int_0^a dp \int_0^p dq + \int_0^a dq \int_0^p dp \right) \]

\[ \times p^{-\alpha}q^{-\beta}(e^{-(p-q)^2/(4t)} - e^{-(p+q)^2/(4t)}) \]

\[ = (4\pi/t)^{1/2} a^2 \int_0^a p^{1-\alpha-\beta} dp \int_0^1 \rho^{-\alpha + \rho^{-\beta}} \]

\[ \times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) \, d\rho \]

\[ = 4\pi a^2 c_{\alpha,\beta} t^{(1-\alpha-\beta)/2} \]

\[ - (4\pi/t)^{1/2} a^2 \int_a^{\infty} p^{1-\alpha-\beta} dp \int_0^1 \rho^{-\alpha + \rho^{-\beta}} \]

\[ \times (e^{-p^2(1-\rho)^2/(4t)} - e^{-p^2(1+\rho)^2/(4t)}) \, d\rho, \tag{63} \]

\[ A_1 = \ldots \]

\[ A_2 = \ldots \]
\[ A_1 = -4\pi a(c_{a-1,\beta} + c_{\alpha,\beta-1})t^{(2-\alpha-\beta)/2} + (4\pi/t)^{1/2}a \int_0^\infty \rho^{2-\alpha-\beta}d\rho \]

\[ \times \int_0^1 d(\rho^{1-\alpha} + \rho^{-\alpha} + \rho^{1-\beta} + \rho^{-\beta})(e^{-\rho^2(1-\rho)^2/(4t)} - e^{-\rho^2(1+\rho)^2/(4t)})d\rho, \] 

(64)

and

\[ A_2 = 4\pi c_{a-1,\beta-1}t^{(3-\alpha-\beta)/2} - (4\pi/t)^{1/2} \int_0^\infty \rho^{3-\alpha-\beta}d\rho \]

\[ \times \int_0^1 d(\rho^{1-\alpha} + \rho^{1-\beta})(e^{-\rho^2(1-\rho)^2/(4t)} - e^{-\rho^2(1+\rho)^2/(4t)})d\rho. \] 

(65)

The terms to be evaluated in (63), (64) and (65) are all of the form

\[ (4\pi/t)^{1/2}a^{2-j} \int_0^\infty \rho^{1+j-\alpha-\beta}d\rho \int_0^1 \rho^{-\gamma}(e^{-\rho^2(1-\rho)^2/(4t)} - e^{-\rho^2(1+\rho)^2/(4t)})d\rho, \]

(66)

where \( j = 0, 1, 2 \) respectively. Following arguments similar to the proof of Lemma 9 we see that the contribution of the integral with respect to \( \rho \in [0, 1/2] \) in (66) is at most \( O(e^{-a^2/(18t)}) \). Furthermore

\[ (4\pi/t)^{1/2}a^{2-j} \int_0^\infty \rho^{1+j-\alpha-\beta}d\rho \int_1^{1/2} \rho^{-\gamma}e^{-\rho^2(1+\rho)^2/(4t)}d\rho = O(e^{-a^2/(18t)}). \]

(67)

Hence the expression under (66) equals

\[ (4\pi/t)^{1/2}a^{2-j} \int_0^\infty \rho^{1+j-\alpha-\beta}d\rho \int_1^{1/2} \rho^{-\gamma}e^{-\rho^2(1+\rho)^2/(4t)}d\rho + O(e^{-a^2/(18t)}). \]

(68)

Expanding \( \rho^{-\gamma} \) about \( \rho = 1 \) we obtain that

\[ |\rho^{-\gamma} - 1 - \gamma(1-\rho) - 2^{-1}\gamma(\gamma + 1)(1-\rho)^2 - 6^{-1}\gamma(\gamma + 1)(\gamma + 2)(1-\rho)^3| \leq C(1-\rho)^4, \quad 0 \leq \rho \leq 1/2, \]

(69)

where \( C \) depends on \( \gamma \) only. By (69) and (68) we obtain that (66) is equal to

\[ 2\pi(\alpha + \beta - j - 1)^{-1}a^{3-\alpha-\beta} + 4\pi^{1/2}\gamma(\alpha + \beta - j)^{-1}a^{2-\alpha-\beta}t^{1/2} \]

\[ + 2\pi\gamma(\gamma + 1)(\alpha + \beta - j + 1)^{-1}a^{1-\alpha-\beta}t \]

\[ + 8\pi^{1/2}3^{-1}\gamma(\gamma + 1)(\gamma + 2)(\alpha + \beta - j + 2)^{-1}a^{2-\alpha-\beta}t^{3/2} + O(a^2). \]

(70)

It remains to compute the coefficients \( b_0, b_1 \) and \( b_2 \) in Theorem 7. Altogether there are eight terms which contribute to the terms in (70):

\[ j = 0, \quad \gamma = \alpha, \quad \gamma = \beta \]

\[ j = 1, \quad \gamma = \alpha - 1, \quad \gamma = \beta - 1, \quad \gamma = \alpha, \quad \gamma = \beta \]

\[ j = 2, \quad \gamma = \alpha - 1, \quad \gamma = \beta - 1. \]

Summing these eight terms yield the expressions for \( b_0, b_1 \) and \( b_2 \) under (55). To calculate \( b_3 \) we have that the above eight \( \gamma(\gamma + 1)(\gamma + 2) \) terms in (70) cancel the contribution from (59). This completes the proof of Theorem 7.
References


