HEAT KERNEL UPPER BOUNDS ON A COMPLETE NON-COMPACT MANIFOLD

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1. Introduction

Let M be a smooth connected non-compact geodesically complete Riemannian manifold, Δ denote the Laplace operator associated with the Riemannian metric, $n \geq 2$ be a dimension of M . Consider the heat equation on the manifold

$$u_t - \Delta u = 0 \tag{1.1}$$

where $u = u(x,t), x \in M, t > 0$. The heat kernel $p(x,y,t)$ is by definition the smallest positive fundamental solution to the heat equation which exists on any manifold (see [Ch] , [D]).

The purpose of the present work is to obtain uniform upper bounds of $p(x,y,t)$ which would clarify the behaviour of the heat kernel as $t \to \infty$ and $r \equiv \mathrm{dist}(x,y) \to \infty$. In the Euclidean space $\mathbf{R}^n$ the heat kernel is given by the following well known formula

$$p(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{r^2}{4t}\right) \tag{1.2}$$

so that it decreases as $\frac{\mathrm{const}}{t^{n/2}}$ as $t \to \infty$ and its behaviour for large r is determined by the Gaussian term $\exp(-\frac{r^2}{4t})$. In the n -dimensional hyperbolic space $\mathbf{H}^n_k$ (of constant curvature $-k^2 < 0$) the heat kernel is known as well. It takes the simplest form in the case of the dimension 3:

$$p(x,y,t) = \frac{1}{(4\pi t)^{3/2}} \frac{kr}{\sinh(kr)} \exp(-\frac{r^2}{4t} - k^2 t). \tag{1.3}$$

Its the most significant difference from the Euclidean heat kernel is an exponential decay in t given by the factor $\exp(-k^2 t)$.

The first question to be discussed here is which geometric properties of the manifold ensure decreasing of the heat kernel as $t \to \infty$ with a prescribed speed? More precisely, when the heat kernel satisfies the following estimate

$$p(x,y,t) \leq f(t) \tag{1.4}$$

for all $x, y \in M, t > 0$ where $f(t)$ is a monotonically decreasing function on the positive real semi-axis. This kind of a heat kernel estimate is often referred to as an on-diagonal estimate because (1.4) follows from the same inequality for $x = y$ (see the Proposition 2.1 bellow). It is well known and due to Nash [N] (see also [CKS]) that a heat kernel

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on-diagonal upper bound is deduced from a suitable isoperimetric inequality. Consider, for example, a special case when $f(t) = \mathrm{const}\, t^{-n/2}$ which takes place in the Euclidean space. Then the corresponding inequality

$$p(x, y, t) \leq \mathrm{const}\, t^{-n/2} \tag{1.5}$$

can be proved whenever we know, that for any smooth function v with a compact support the following Sobolev inequality holds

$$\int_M |\nabla v|^2 \geq c \left(\int_M |v|^{2n/(n-2)} \right)^{\frac{n-2}{n}} \tag{1.6}$$

(of course we have to assume here that $n > 2$). This inequality is close to the classical isoperimetric inequality between the volume of a bounded region and the area of its boundary

$$\mathrm{Area}\,(\partial\Omega) \geq c\mathrm{Vol}\,(\Omega)^{(n-1)/n}$$

for any bounded domain Ω with a smooth boundary $\partial\Omega$. Namely, the isoperimetric inequality above implies (1.6) (converse is not true - see [CL]).

N.Varopoulos [V85] proved that the Sobolev inequality (1.6) is not only sufficient but a necessary condition as well for the upper bound (1.5) to be valid. On the other hand, Carlen, Kusuoka and Stroock [CKS] found an alternative form of this phenomenon: the upper bound (1.5) is equivalent to the Nash inequality

$$\left(\int_M f^2 \right)^{1+2/n} \leq C \int_M |\nabla f|^2 \left(\int_M |f| \right)^{4/n} \tag{1.6$'$}$$

supposed to hold for any $f \in C_c^\infty(M)$. This theorem has the advantage that it does not require the hypothesis $n > 2$. Therefore, the upper estimate (1.5) is equivalent to either Sobolev and Nash inequality.

The Gaussian factor in heat kernel upper bounds on Riemannian manifolds appeared in the paper of Cheng, Li and Yau [ChLY] but they considered the heat kernel behaviour only in a finite time interval. For the whole range of time the on-diagonal bound (1.5) was shown by several authors to imply the following Gaussian off-diagonal correction

$$p(x, y, t) \leq \mathrm{const}\, t^{-n/2} \exp\left(-c\frac{r^2}{t} \right) \quad . \tag{1.7}$$

Apparently for the first time it was proved by Ushakov [U] in 1980. He derived (1.7) from (1.5) by means of pure analytical tools without appealing to geometry. Although he treated the case of unbounded regions in Euclidean space his proof can be carried over to abstract manifolds too. Another proof was obtained by Davies and Pang [DP] (see also the preceding work [D87]) who used a logarithmic Sobolev inequality which is equivalent to both estimates (1.5) and (1.7) (where constant c can be taken arbitrarily close to $\frac{1}{4}$). Thus, we have that each the of relations (1.5), (1.6), (1.6$'$), (1.7) is equivalent to the others. The following questions arise:

1. to obtain an isoperimetric property which would be equivalent to estimate (1.4) for a general function f without the polynomial or other restrictions on the behaviour of f at infinity;

2. to obtain the corresponding estimate with the Gaussian off-diagonal correction term. If the function f has at least a polynomial decay (which means in this context that it satisfies the condition

$$f(t) \le \mathrm{const}\, f(at)$$

for all $t > 0$ and $a \in [1,2]$), for example

$$f(t) = \mathrm{const}\begin{cases} t^{-n/2}, t < 1 \\ t^{-m/2}, t \ge 1 \end{cases}$$

then as was proved by Davies (see [D89] , [DP]) (1.4) implies

$$p(x,y,t) \le \mathrm{const}_{\varepsilon} f(t) \exp\left(-\frac{r^2}{(4+\varepsilon)t}\right)$$

(where $\varepsilon > 0$ is arbitrary). For example, this bound holds for the Riemannian product $M = K \times \mathbf{R}^m$, K being a $(n-m)$ -dimensional compact manifold.

There are some other works which treat the polynomial case. In a series of papers of A.K.Gushchin (see , for example, [G] , [GMM]) heat kernel bounds are proved for the case of unbounded region in Euclidean space with the Neumann boundary condition provided some isoperimetric inequality is valid. He obtained the exhaustive results but also for the case when the heat kernel has a polynomial decay as $t \to \infty$.

At the same time there exist many important classes of manifolds whose heat kernel decreases faster than polynomially as $t \to \infty$. For hyperbolic space and for a wide class of negatively curved manifolds the heat kernel has an exponential decay. There are examples of manifolds - covering manifolds with a deck transformation group being a polycyclic one - for which the heat kernel decreases as $t \to \infty$ subexponentially but superpolynomially (see [V91] , [A] for discrete counterparts).

In the present paper we adduce a new approach of obtaining the heat kernel upper bounds which enables us to cover all results mentioned above, and, moreover, to get the corresponding estimates for a larger variety of manifolds.

The theorem to be formulated below establishes equivalence in a rather general situation between the on-diagonal estimate (1.4), the corresponding Gaussian estimate and some inequality of Faber-Krahn type which we use in place of the Sobolev and Nash inequalities mentioned above.

Let $\Lambda(v)$ be a positive continuous monotonically decreasing function on the positive real semi-axis.

Definition 1.1 Let us say that a Λ-isoperimetric inequality is valid for a region $\Omega \subset M$, if for any sub-region $D \subset \Omega$ the first Dirichlet eigenvalue $\lambda_1(D)$ is controlled below by $\Lambda(\mathrm{Vol}\, D)$:

$$\lambda_1(D) \ge \Lambda(\mathrm{Vol}\, D). \tag{1.8}$$

It is very natural to use Λ-isoperimetric inequality for evaluating the heat kernel. Indeed, the heat kernel of a bounded region $\Omega \subset M$ decreases as $t \to \infty$ as $\mathrm{const} \cdot \exp(-\lambda_1(\Omega)t)$. If $\lambda_1(M) \equiv \lim_{\Omega \to M} \lambda_1(\Omega) > 0$ then one can hope that the heat kernel $p(x,y,t)$ behaves itself as $\mathrm{const} \cdot \exp(-\lambda_1(M)t)$ for large t . Otherwise, if $\lambda_1(M) = 0$ one may expect that the order of decay of p as $t \to \infty$ depends on the speed of convergence $\lambda_1(\Omega)$ to 0 on expanding of Ω .

Note that the value $\lambda_1(M)$ is referred to as the spectral radius of the manifold M and coincides with the bottom of the spectrum in $L^2(M)$ of $-\Delta$.

A Λ-isoperimetric inequality can be easily deduced from the following inequality between the volume and the area

$$\mathrm{Area}\,(\partial D) \geq g(\mathrm{Vol}\, D) \tag{1.8'}$$

with some function g . Namely, if $g(v)/v$ is a decreasing function then (1.8′) implies (1.8) where Λ is expressed through g by

$$\Lambda(v) = \frac{1}{4}\left(\frac{g(v)}{v}\right)^2$$

(this is a consequence of the well-known inequality of Cheeger - see the Proposition 2.4 below). Conversely, at least for a polynomial function $\Lambda(v) = \mathrm{const}\, v^{-\nu}$, $\nu > 0$ and for a manifold of non-negative Ricci curvature the Λ-isoperimetric inequality implies the isoperimetric inequality (1.8′) with the function $\mathrm{const}\, g(v)$ - see [C] .

For example, in the Euclidean space we have

$$\Lambda(v) = c_n v^{-2/n} \tag{1.9}$$

while for the hyperbolic space $\mathbf{H}^n_k$

$$\Lambda(v) = \max\{c_n v^{-2/n},\, \lambda\} \tag{1.9'}$$

where $\lambda = \frac{1}{4}(n-1)^2 k^2$.

For a class of covering manifolds Coulhon and Saloff-Coste [CS] proved the isoperimetric inequality (1.8′) which implies (1.8) with the following function Λ having an intermediate magnitude

$$\Lambda(v) = \mathrm{const}(\log v)^{-\nu} \tag{1.10}$$

for large v where $\nu 0$ is a positive constant.

Let us define a function $V(t)$ by means of the following relation

$$t = \int_0^{V(t)} \frac{dv}{v\Lambda(v)}. \tag{1.11}$$

Of course, we have to suppose that the integral here converges at 0 . This is not a strong restriction because for small values v the function $\Lambda(v)$ is expected to be as in Euclidean space. Obviously, $V(t)$ is an increasing function on $(0, +\infty)$.

The following theorem is one of the main results to be presented here.

Theorem 1.1 *Consider the following hypotheses*

1. *Λ-isoperimetric inequality holds on M , i.e. for any pre-compact region $\Omega \subset M$ we have:*

$$\lambda_1(\Omega) \geq \Lambda(|\Omega|)$$

2. *$\forall x, y \in M, t > 0$*

$$p(x,y,t) \leq \frac{C}{V(ct)} \exp\left(-\frac{r^2}{Dt}\right)$$

3. $\forall x \in M, t > 0$

$$p(x,x,t) \leq \frac{C}{V(ct)}$$

4. for any pre-compact region $\Omega \subset M$ *we have:*

$$\lambda_k(\Omega) \geq c\Lambda\left(C\frac{|\Omega|}{k}\right) \quad \forall k = 1,2,...,$$

and suppose that the function $V(t)$ *is regular in some sense (see below). We claim that*

$$1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4$$

where the constants c, C *are positive and can be different in the different items;* $D > 4$ *can be taken arbitrarily close to* 4 *.*

Let us note that the implication 2$\Longrightarrow$3 is obvious and included into the theorem for the sake of completeness. This theorem means that Λ-isoperimetric inequality is equivalent to either the on-diagonal bound and the off-diagonal Gaussian estimate up to the constant multiples. Indeed, the item 4 of the Theorem 1.1 implies for $k = 1$ the Λ-isoperimetric inequality which differs from that of the item 1 only by the multiples c, C . As a consequence we see that the isoperimetric inequality for the first eigenvalue implies the corresponding inequalities for the higher eigenvalues.

The regularity conditions mentioned in the Theorem 1.1 are the following. The implication 1$\Longrightarrow$2 is proved in the Theorem 5.1 in Section 5 under the hypothesis that for some $T \in]0, +\infty]$ the function $\frac{tV'(t)}{V(t)}$ is increasing for $t > T$ and bounded for $t \leq 2T$ by some constant. This hypothesis restricts, first, the behaviour of the function $V(t)$ as $t \to 0$ - it may be equal to t^ν, $\nu > 0$ but may not be equal to $e^{-\frac{1}{t}}$, and, second, the function $V(t)$ may not have too flat parts on its graph if it grows superpolynomially as $t \to \infty$. For example, for large t the function $V(t)$ may be equal to $(\log t)^\alpha t^\beta \exp(t^\gamma)$ with the arbitrary non-negative constants α, β, γ .

The implication 3$\Longrightarrow$4 requires that, first, the function $V(t)$ is obtained by the transformation (1.11) from a function Λ (see the Proposition 2.2 in Section 2 for the explicit conditions) and, second, its logarithmic derivative $\frac{V'(t)}{V(t)}$ has at most a polynomial decay in the sense mentioned in the discussion above. This condition holds, for example, provided $V(t) = t^\nu$, $\nu > 0$ for small t and the derivatives $V'(t)$ makes no jumps for large t , for instance, the function $(\log t)^\alpha t^\beta \exp(t^\gamma)$ satisfies it.

We see that the regularity assumptions does not restrict the rate of increase of $V(t)$ as $t \to \infty$. Note that the function $V(t)$ defined from (1.11) cannot increase faster than exponentially - the fastest growth corresponds to the case when $\Lambda(v) =$ const > 0 for large v . This conforms to the fact that the heat kernel cannot decrease in t quicker than exponentially that follows from the same property of the heat kernel in a bounded region.

Let us also note that the implication 1$\Longrightarrow$3 does not require any additional condition and this is the reason why we distinguish this part of the Theorem 1.1 as a separate Theorem 2.1.

Examples. 1. Let us set

$$\Lambda(v) = \text{const} \begin{cases} v^{-2/n}, & v < 1 \\ v^{-2/m}, & v \geq 1 \end{cases}, \tag{1.12}$$

then by the Theorem 1.1 we have the following estimate

$$p(x,y,t) \le \frac{\text{const}}{\min(t^{n/2},\ t^{m/2})} \exp\left(-\frac{r^2}{Dt}\right) .$$

Taking here $m = n$ we see that the heat kernel estimate (1.7) of a Euclidean type is equivalent to the Λ-isoperimetric inequality (1.9). On the other hand as was mentioned above (1.7) is equivalent to the Sobolev inequality (1.6). Hence, the Sobolev inequality (1.6) and the Λ-isoperimetric inequality (1.9) are equivalent. Of course, that can be proved directly too, see [C] .

2. If Λ is given by the formula (1.10) for large v (and is Euclidean one for small v) then by the Theorem 1.1 we have for large t

$$p(x,y,t) \le \text{const} \exp\left(-c_\nu t^{\frac{1}{\nu+1}} - \frac{r^2}{Dt}\right) \tag{1.13}$$

In particular, if $\nu = 2$ as it takes place on a covering manifold with a deck transformation group of the exponential growth (see [CS]), then the heat kernel decreases for large t at least as fast as $\exp(-ct^{\frac{1}{3}})$. This is a sharp order of the heat kernel decay as has been shown in [A] .

3. Let Λ be the function (1.9$'$) with some $\lambda > 0$ as it holds on a simply connected manifolds with a strictly negative curvature, then by the Theorem 1.1

$$p(x,y,t) \le Ct^{-n/2} \exp(-c\lambda t - \frac{r^2}{Dt}) .$$

This estimate will be improved in Section 5 for t bounded away from 0 , for example $t > 1$:

$$p(x,y,t) \le \text{const} \left(1 + \frac{r^2}{t}\right)^{1+\frac{n}{2}} \exp(-\lambda t - \frac{r^2}{4t}) \tag{1.14}$$

The coefficient λ at the exponent is sharp as can be seen in the case of the hyperbolic space. Note that the largest possible value of λ here is the spectral radius of the manifold $\lambda_1(M)$.

A part of the Theorem 1.1 which relates to behaviour of the heat kernel in time variable is proved in Section 2 below (see the Theorems 2.1 and 2.2 which treat the cases $1 \Longrightarrow 3$ and $3 \Longrightarrow 4$ respectively). We apply Λ-isoperimetric inequality in order to derive some inequality of Nash type which we use in place of the Sobolev inequality.

The main efforts are directed to get estimates containing the Gaussian factor. For this purpose we consider some weighted integral of p^2 over the entire manifold (in place of the usual integral of p^2 over an exterior of a ball - see Section 4 for details) which happens to decrease in t . This property simplifies considerably the proof and makes our approach very flexible. To estimate this integral (Theorems 4.1 and 4.2 in Section 4) we apply some mean-value type theorem which relies upon Λ-isoperimetric inequality and is of independent interest (Theorems 3.1 and 3.2 in Section 3).

Having the integral estimates it is easy to pass to pointwise heat kernel upper bounds using the semigroup property of the heat kernel (see Section 5). The Theorem 5.1 completes the remaining part $1 \Longrightarrow 2$ of the Theorem 1.1. The method of proving of the Theorem 5.1 enables us to get heat kernel estimates not only in the case when a Λ-isoperimetric inequality holds on the entire manifold but also when a Λ-isoperimetric inequality holds with different functions Λ on different parts of the manifold. We consider two kinds of such a situation.

1. Suppose that Λ-isoperimetric inequality is known to be true in any geodesic ball but, possibly, with its own function Λ . Under this hypothesis we obtain a heat kernel upper bound (see Theorem 5.2) which is applicable, for example, to an arbitrary manifold and yields the estimate (1.14) above where the const has to depend on x, y . Another example of applications of the Theorem 5.2 is the heat kernel estimate on a manifold of a non-negative Ricci curvature

$$p(x,y,t) \leq \frac{\text{const}}{\operatorname{Vol} B^x_{\sqrt{t}}} \exp\left(-\frac{r^2}{(4+\varepsilon)t}\right) \quad , \ \varepsilon > 0$$

(where B^x_R denotes the geodesic ball with the centre $x \in M$ and of the radius R) obtained first by P.Li and S.-T.Yau [LY] . A similar result of B.Davies [D88] for a manifold of a bounded below Ricci curvature is covered by our approach too.

2. On the other hand we are able to get an extra information concerning the heat kernel behaviour whenever in addition to a global Λ-isoperimetric inequality there is a stronger isoperimetric inequality in a neighbourhood of infinity. The Theorem 5.3 appears to be an example of such a statement and yields on a Cartan-Hadamard manifold the following estimate

$$p(x,y,t) \leq \frac{\text{const}}{t^{n/2}} \exp\left(-\text{const}\left(\frac{r^2}{4t} - \lambda_1(M)t - rk(r/2)\right)\right)$$

where $-k^2(R)$ denotes the supremum of the sectional curvature in the exterior of the ball B^y_R . What is new here is the third term at the exponent

$$\exp\Big(-\text{const}\, rk(r/2)\Big)$$

which associates with the term

$$\frac{kr}{\sinh(kr)}$$

in the case of the heat kernel (1.3) of the hyperbolic space. It can cause the heat kernel to decrease as $r \to \infty$ faster than predicted by the Gaussian term provided the curvature $-k^2(R)$ outside the ball of the radius R approaches to $-\infty$ fast enough as $R \to \infty$.

The results of this paper were partially announced in [G87b] , [G87c] , [G87a] , [G88] and [G91a] .

Notations

$\text{const}_{a,b,...}$ - a positive constant, depending on $a, b, ...$;
$\operatorname{dist}(x,y)$ - a geodesic distance between the points $x, y \in M$;
B^x_R - an open geodesic ball of the radius R centered at the point $x \in M$;
$\operatorname{meas}_k A$ - k -dimensional Riemannian measure of set A;

$$|A| \equiv \operatorname{Vol} A \equiv \begin{cases} \operatorname{meas}_n A, & \text{if } A \subset M \\ \operatorname{meas}_{n+1} A, & \text{if } A \subset M \times \mathbf{R} \end{cases}$$

$\lambda_k(\Omega)$ - the k -th eigenvalue of the Dirichlet boundary value problem in Ω .

2. Decay of the heat kernel in time

The heat kernel on a manifold can be constructed by means of the following process. For any relatively compact subset $\Omega \subset M$ with a smooth boundary one can define the heat kernel $p_\Omega(x,y,t)$ as Green function of mixed problem the for heat equation in $\Omega \times (0,\infty)$ with vanishing Dirichlet boundary values. Let us denote by $\varphi_k(x)$ the k -th eigenfunction of the Dirichlet boundary value problem in $\Omega,\ k = 1, 2, ...$, so that sequence $\{\varphi_k\}$ is an orthonormal basis in space $L^2(\Omega)$, then the following eigenfunction expansion holds

$$p_\Omega(x,y,t) = \sum_{k=1}^{\infty} \exp\Big(-\lambda_k(\Omega)t\Big)\ \varphi_k(x)\varphi_k(y) \tag{2.1}$$

A proof of this expansion as well as a justification of other properties of p_Ω to be used in this Section the reader can find in [D] and [Ch] .

Maximum principle implies that p_Ω is non-negative and monotone in Ω : it increases on expansion of Ω . At the same time the integral of the heat kernel remains bounded:

$$\int_\Omega p_\Omega(x,y,t)dy \le 1$$

which implies that p_Ω has a finite limit as $\Omega \to M$ where $\Omega \to M$ denotes an exhaustion of M by a sequence of relatively compact domains Ω . The limit

$$p(x,y,t) = \lim_{\Omega \to M} p_\Omega(x,y,t)$$

is obviously the smallest positive fundamental solution to the heat equation i.e. the heat kernel on M . The function $p(x,y,t)$ inherits from $p_\Omega(x,y,t)$ the following properties:

1. $p(x,y,t) > 0$ for all $x, y \in M, t > 0$ and satisfies the heat equation with respect to x, t for any fixed y ;
2. $p(x,y,t) \to \delta_y(x)$ in sense of distributions as $t \to 0$;
3. symmetry: $p(x,y,t) = p(y,x,t)$
4. boundedness of the entire heat flow

$$\int_M p(x,y,t)dy \le 1 \tag{2.2}$$

5. semi-group property

$$\int_M p(x,y,t)p(y,z,s)dy = p(x,z,t+s) \tag{2.3}$$

6 if $v \in L^p(M), 1 \le p < \infty$ it follows that the following operator

$$T_t v(x) = \int_M p(x,y,t)v(y)dy \tag{2.4}$$

defines a contraction semi-group in $L^p(M)$ and the function $u(x,t) = T_t v(x)$ is a solution to the Cauchy problem for the heat equation (1.1) with an initial function $v(x)$ (the latter means that

$$\lim_{t \to 0} \|u(x,t) - v(x)\|_{L^p(M)} = 0$$

see [S]).

It is standard now that to obtain pointwise upper bounds of the heat kernel one proves first a suitable integral estimate and then applies semi-group property (2.3). In the simplest setting this idea is illustrated by the following proposition.

Proposition 2.1 *The following inequalities are equivalent for any fixed* $t > 0$ *:*

1. $p(x, y, t) \le f(t) \quad \forall x, y \in M$
2. $p(x, x, t) \le f(t) \quad \forall x \in M$
3. $\int_M p^2(x, y, t)dy \le f(2t) \quad \forall x \in M$

This proposition is not a new one, nonetheless we shall prove it for convenience of the reader.

$1 \Longrightarrow 2$. Evident.

$2 \Longrightarrow 3$. According to semi-group property (2.3) we have for $x = z$

$$\int_M p^2(x, y, t)dy = p(x, x, 2t) \le f(2t)$$

This argument implies, in particular, that function $p(x, y, t)$ as a function of y lies in $L^2(M)$.

$3 \Longrightarrow 1$. According to the symmetry property of the heat kernel and by the Cauchy-Schwarz inequality we have

$$p(x, y, t) = \int_M p(x, \xi, \frac{t}{2})p(\xi, y, \frac{t}{2})d\xi$$

$$\le \left(\int_M p^2(x, \xi, \frac{t}{2})d\xi\right)^{\frac{1}{2}} \left(\int_M p^2(y, \xi, \frac{t}{2})d\xi\right)^{\frac{1}{2}} \le (f(t)f(t))^{\frac{1}{2}} = f(t)$$

□

Let us note that , in addition, each of conditions 1-3 is equivalent to each of the following hypotheses

4. $\|T_t v\|_\infty \le f(t)\,\|v\|_{L^1(M)} \quad \forall v \in L^1(M);$
5. $\|T_t v\|_\infty \le f(2t)\,\|v\|_{L^2(M)} \quad \forall v \in L^2(M);$

Next we shall assume that Λ-isoperimetric inequality holds on the manifold under consideration, function $\Lambda(v)$ being positive, continuous and monotonically decreasing on $(0, +\infty)$. In addition, we suppose that function Λ satisfies the condition

$$\int_{0+} \frac{dv}{v\Lambda(v)} < \infty \tag{2.5}$$

This relation holds, for example, when $\Lambda(v) = \mathrm{const}\, v^{-\nu}$ for small values of v where $\nu > 0$, as it takes place in Euclidean space.

Let us denote the set of such functions $\Lambda(v)$ by $\mathcal{L}$. For any $\Lambda \in \mathcal{L}$ we define a function $V(t)$ as follows

$$t = \int_0^{V(t)} \frac{dv}{v\Lambda(v)} \tag{2.6}$$

Equivalently, $V(t)$ is a positive solution to the Cauchy problem

$$V'(t) = V\Lambda(V), \quad V(0) = 0 \tag{2.7}$$

Since $\int_0^\infty \frac{dv}{v\Lambda(v)} = \infty$ due to decreasing of function Λ it follows that function $V(t)$ is defined on the entire interval $(0, +\infty)$.

A class of functions $V(t)$ obtained by (2.6) (or (2.7)) will be denoted by $\mathcal{V}$. More explicit description of this class will be given below. The mapping from $\mathcal{L}$ onto $\mathcal{V}$ given by (2.6) will be referred to as $V-$ *transformation* .

Theorem 2.1 *If Λ-isoperimetric inequality holds on manifold M with a function $\Lambda \in \mathcal{L}$ then for all $x, y \in M, t > 0$ the following heat kernel estimate is valid:*

$$p(x, y, t) \leq \frac{2}{\delta V((1-\delta)t)} \tag{2.8}$$

where $\delta > 0$ is arbitrary.

Note that this theorem contains the part $1 \Longrightarrow 3$ of the Theorem 1.1.

Proof. Let Ω be an open pre-compact subset of M with a smooth boundary. It suffices to proof that for all $y \in \Omega,\ t > 0,\ \delta \in (0, 1)$

$$\int_\Omega p_\Omega^2(x, y, t)dx \leq \frac{2}{\delta V((2-2\delta)t)} \tag{2.8$'$}$$

Indeed, the integral in (2.8$'$) is equal to $p_\Omega(y, y, 2t)$. Passing to the limit as $\Omega \to M$ we get from (2.8$'$) a similar inequality for $p(y, y, 2t)$ which implies, in its turn, pointwise estimate (2.8) as it follows from the Proposition 2.1.

The proof of (2.8$'$) relies upon the following lemma.

Lemma 2.1 *For any non-negative function $v(x) \in C_c^\infty(\Omega)$ and for any $\delta > 0$ the following inequality is valid*

$$\int_\Omega |\nabla v|^2 \geq (1-\delta) B \Lambda\left(\frac{2A^2}{\delta B}\right) \tag{2.9}$$

where

$$A = \int_\Omega v, \quad B = \int_\Omega v^2$$

Proof of the lemma. The proof follows the arguments of A.K.Gushchin [G] . For any positive τ the following inequality is evidently true

$$v^2 \leq (v-\tau)_+^2 + 2\tau v\ . \tag{2.10}$$

Integrating (2.10) over Ω we get

$$\int_\Omega v^2 \leq \int_{\{v>\tau\}} (v-\tau)^2 + 2\tau \int_\Omega v \tag{2.11}$$

By the minimax property of the first eigenvalue in the region $\{v > \tau\}$ and according to the Λ-isoperimetric inequality we have

$$\frac{\int_\Omega |\nabla v|^2}{\int_{\{v>\tau\}} (v-\tau)^2} \geq \lambda_1(\{v > \tau\}) \geq \Lambda\Big(\mathrm{Vol}\,\{v > \tau\}\Big) \quad .$$

Since

$$\mathrm{Vol}\,\{v > \tau\} \leq \frac{1}{\tau}\int_\Omega v = \frac{A}{\tau}$$

it follows that

$$\int_{\{v>\tau\}} (v-\tau)^2 \leq \frac{\int_\Omega |\nabla v|^2}{\Lambda\big(\frac{A}{\tau}\big)} \; .$$

Substituting it into (2.11) we get

$$B \leq \frac{\int_\Omega |\nabla v|^2}{\Lambda\big(\frac{A}{\tau}\big)} + 2\tau A$$

and

$$\int_\Omega |\nabla v|^2 \geq (B - 2\tau A)\Lambda\big(\frac{A}{\tau}\big) \; .$$

Taking here $\tau = \frac{\delta B}{2A}$ we obtain (2.9). $\square$

To proceed with the proof of the Theorem 2.1 let us fix some $y \in M$ and introduce the notations

$$u(x,t) = p_\Omega(x,y,t), \quad I(t) = \int_\Omega u(x,t)^2 dx \; .$$

Taking into account that $\int_\Omega u(x,t)dx \leq 1$ and applying the Lemma 2.1 we obtain

$$\int_\Omega |\nabla u|^2 \geq (1-\delta) I(t) \Lambda\Bigg(\frac{2}{\delta I(t)}\Bigg) \tag{2.12}$$

Note that although the function $u(\cdot,t)$ is not in $C_c^\infty(\Omega)$ as required by the Lemma 2.1 this Lemma is nonetheless applicable because u vanishes on $\partial\Omega$ and, thereby, can be approximated in $W^{1,2}(\Omega)$ by an element of $C_c^\infty(\Omega)$.

On the other hand differentiating $I(t)$ with respect to t we get

$$I'(t) = 2\int_\Omega u_t u = 2\int_\Omega u\Delta u = -2\int_\Omega |\nabla u|^2 \quad .$$

Substituting it into (2.12) we obtain a differential inequality:

$$I'(t) \leq -2(1-\delta) I(t) \Lambda\Bigg(\frac{2}{\delta I(t)}\Bigg) \quad . \tag{2.13}$$

Integrating this inequality we have

$$\int_{I(t_0)}^{I(t)} \frac{dI}{I\Lambda\Big(\frac{2}{\delta I}\Big)} \leq -2(1-\delta)\int_{t_0}^{t} dt = -2(1-\delta)(t-t_0)$$

where $t > t_0 > 0$. Changing a variable $v = \frac{2}{\delta I}$ we get

$$\int_0^{\frac{2}{\delta I(t)}} \frac{dv}{v\Lambda(v)} \geq \int_{\frac{2}{\delta I(t_0)}}^{\frac{2}{\delta I(t)}} \frac{dv}{v\Lambda(v)} \geq 2(1-\delta)(t-t_0) \ . \tag{2.14}$$

Letting here $t_0 \to 0$ and applying the definition of the function $V(t)$ we finally obtain

$$\frac{2}{\delta I(t)} \geq V(2(1-\delta)t), \quad I(t) \leq \frac{2}{\delta V(2(1-\delta)t)}$$

which is equivalent to $(2.8')$ $\Box$

Now we are going to show that a Λ-isoperimetric inequality is also a necessary condition for the heat kernel upper bounds of kind $p(x,y,t) \leq f(t)$. Let us start with the following observation.

Proposition 2.2 *Class $\mathcal{V}$ contains all positive functions $V(t) \in C^1(0,+\infty)$ such that*
1. $V'(t) > 0$
2. $V(0) = 0, \quad V(\infty) = \infty$
3. $\frac{V'(t)}{V(t)}$ *is monotonically decreasing.*

There are no other functions in $\mathcal{V}$.

Indeed, the fact that any function from $\mathcal{V}$ satisfies 1-3 is a simple consequence of the definition (2.7). For example, the property 3 follows from the monotone decreasing of Λ . Inversely, let $V(t)$ be a function satisfying 1-3, then we can define $\Lambda(v)$ by means of the relation

$$\Lambda(V(t)) = \frac{V'(t)}{V(t)} \tag{2.15}$$

which is nothing but transformed (2.7). Since the function $V(t)$ is a bijection by1-3 it follows that (2.15) determines a unique function $\Lambda \in \mathcal{L}$. $\Box$

Thus, the V -transformation has an inverse one define by (2.15) which will be referred to as a Λ -transformation.

Definition 2.1 A positive function $f(t)$ defined on $(0,+\infty)$ is said to be of *a polynomial decay* if for some positive constant α the following inequality holds for all $t > 0, \ a \in [1,2]$

$$f(at) \geq \alpha f(t) \tag{2.16}$$

Note that any monotone increasing function as well as a decreasing function $f(t) = t^{-N}$ satisfies this definition. It is easy to see that the condition (2.16) holds whenever we have the following differential inequality

$$f'(t) \geq -N\frac{f(t)}{t} \tag{2.16'}$$

where $\alpha = 2^{-N}$.

The next theorem is converse in some sense to the Theorem 2.1.

Theorem 2.2 *Let $V(t) \in \mathcal{V}$ and suppose that*

$$\textit{the function } \Big(\log V(t)\Big)' \textit{ is of polynomial decay} \tag{2.17}$$

Suppose that for all $x \in M$ and $t > 0$ we have the estimate:

$$p(x,x,t) \le \frac{1}{V(t)}, \tag{2.18}$$

then for any precompact open set $\Omega \subset M$

$$\lambda_k(\Omega) \ge \mathrm{const}_\alpha \Lambda\left(\frac{|\Omega|}{k}\right), \quad k = 1,2,3,... \tag{2.19}$$

where function the Λ is the Λ -transformation of V .

Remark. The condition (2.17) does not restrict the rate of increase of $V(t)$ as $t \to \infty$. In any case such standard functions as

$$V(t) = \exp(t^\nu), \quad t^\nu, \quad (\log t)^\nu, \quad (\log\log t)^\nu, \quad etc.$$

where $\nu > 0$, satisfy (2.17). Let us notice also that the Theorem 2.2 coincides with the part $3 \Longrightarrow 4$ of the main Theorem 1.1.

Corollary 2.1 *Suppose that under the hypotheses of the Theorem 2.1 the following estimate holds in place of (2.18):*

$$p(x,x,t) \le \frac{C}{V(ct)}, \tag{2.18$'$}$$

then we have instead of (2.19)

$$\lambda_k(\Omega) \ge \mathrm{const}_\alpha c\Lambda\left(C\frac{|\Omega|}{k}\right), \quad k = 1,2,3,... \tag{2.19$'$}$$

Indeed, this is a simple consequence of the following proposition which, in turn, follows obviously from the definition of the V -transformation.

Proposition 2.3 *Suppose that a function $V(t)$ is the V -transformation of $\Lambda \in \mathcal{L}$, then the function $b\Lambda(av)$ has the V -transformation $a^{-1}V(bt)$, where $a,\ b$ are arbitrary positive numbers.*

Corollary 2.2 *Let a Λ-isoperimetric inequality hold on the manifold where $\Lambda \in \mathcal{L}$ and its V -transformation $V(t)$ satisfies the condition (2.17), then for any pre-compact open set $\Omega \subset M$*

$$\lambda_k(\Omega) \ge \mathrm{const}_{\alpha,\gamma} \Lambda\left(\gamma\frac{|\Omega|}{k}\right), \quad k = 1,2,3.... \tag{2.20}$$

where $\gamma > 2$ can be chosen arbitrarily.

Indeed, by the Theorem 2.1 we have

$$p(x,x,t) \le \frac{1}{\frac{\delta}{2}V((1-\delta)t)}$$

Since the Λ -transformation of the function $\frac{\delta}{2}V((1-\delta)t)$ is the function $(1-\delta)\Lambda(\frac{2}{\delta}v)$ (see the Proposition 2.3) it follows from the Theorem 2.2 that

$$\lambda_k(\Omega) \geq \mathrm{const}_\alpha (1-\delta)\Lambda\bigg(\frac{2}{\delta}\frac{|\Omega|}{k}\bigg)$$

which implies (2.20) with $\gamma = 2/\delta$.

It would be interesting find out to what extent the constants γ and $\mathrm{const}_{\alpha,\gamma}$ in (2.20) can be optimized.

Finally, concluding the discussion around the Theorem 2.2 let us notice that property (2.17) of function V can be derived from the following one of function Λ

$$\Lambda D^2 \Lambda \geq N^{-1}(D\Lambda)^2 \tag{2.21}$$

where $D \equiv d/d(\log v)$

Proof of Theorem 2.2. An idea behind the proof is close to that of [ChL] . Let $\{\varphi_k(x)\}$ be an orthonormal basis consisting of eigenfunctions of the Dirichlet boundary value problem in $\Omega \subset M$. According to the eigenfunction expansion (2.1) we have for $x = y$

$$p_\Omega(x,x,t) = \sum_{k=1}^{\infty} \exp\Big(-\lambda_k(\Omega)t\Big)\ \varphi_k(x)^2$$

Integrating over Ω we obtain

$$\int_\Omega p_\Omega(x,x,t)dx = \sum_{k=1}^{\infty} \exp\Big(-\lambda_k(\Omega)t\Big) \tag{2.22}$$

On the other hand as it follows from (2.18)

$$\int_\Omega p_\Omega(x,x,t)dx \leq \frac{|\Omega|}{V(t)}$$

Combining this with (2.22) and noting that for any $k \geq 1$

$$\sum_{m=1}^{\infty} \exp(-\lambda_m(\Omega)t) > k\exp(-\lambda_k(\Omega)t)$$

we get

$$k\exp(-\lambda_k(\Omega)t) \leq \frac{|\Omega|}{V(t)}$$

$$\lambda_k(\Omega) \geq \frac{1}{t}\log\frac{kV(t)}{|\Omega|} \tag{2.23}$$

This inequality is true for all $t > 0$ and we can choose t arbitrarily. Let us denote for the sake of simplicity $v = |\Omega|/k$, find τ from equation $V(\tau) = v$ and take $t = 2\tau$. For this t we get from (2.23):

$$\lambda_k(\Omega) \geq \frac{1}{2\tau}\Big(\log V(2\tau) - \log V(\tau)\Big) = \frac{1}{2}f(\theta)$$

where $f(t) \equiv \frac{d}{dt} \log V(t), \quad \theta \in (\tau, 2\tau)$ being a mean value. Since function f has a polynomial decay it follows that

$$f(\theta) \geq \alpha f(\tau)$$

Hence, $\lambda_k(\Omega) \geq \frac{\alpha}{2} f(\tau)$. Since

$$f(\tau) = \frac{V'(\tau)}{V(\tau)} = \Lambda(V(\tau)) = \Lambda(v) = \Lambda\left(\frac{|\Omega|}{k}\right)$$

it follows that

$$\lambda_k(\Omega) \geq \frac{\alpha}{2} \Lambda\left(\frac{|\Omega|}{k}\right)$$

which was to be proved. □

In the conclusion of this Section we present a sufficient condition for the transience property of the Brownian motion on M which follows from the arguments of the proof of the Theorem 2.1. The Brownian motion is transient if for some (and hence for all) pairs of distinct points $x, y \in M$

$$\int_0^\infty p(x, y, t) dt < \infty \tag{2.24}$$

which means that there exists a positive fundamental solution $E(x, y)$ of the Laplace equation which is just given by the integral $E(x, y) = \int_0^\infty p(x, y, t) dt$. Of course, one can take the integral in (2.24) from some positive t_0 rather than from 0 because for $x \neq y$ function $p(x, y, t)$ is bounded on any bounded time interval. Since $p(x, y, t) \leq \sqrt{p(x, x, t) p(y, y, t)}$ (see the proof of the Proposition 2.1) it follows that the Brownian motion is transient provided for some $t_0 > 0$ and any $y \in M$

$$\int_{t_0}^\infty p(y, y, t) dt < \infty \tag{2.24$'$}$$

As soon as we have an upper estimate for the heat kernel we can check whether (2.24′) holds. In fact we need for this purpose an upper bound only for large t . That leads us to the following theorem.

Theorem 2.3 *Suppose that for any open pre-compact region $D \subset M$ with the volume* $\operatorname{Vol} D > V_0$ *(where V_0 is some positive number) the isoperimetric inequality (1.8) holds with an arbitrary continuous decreasing function Λ , then the Brownian motion is transient provided*

$$\int^\infty \frac{dv}{v^2 \Lambda(v)} < \infty. \tag{2.25}$$

Proof. Let us extend the function $\Lambda(v)$ into the interval $(0, V_0)$ as a constant: $\Lambda(v) \equiv \Lambda(V_0)$. Since $\lambda_1(D)$ decreases on expansion D we can claim that the Λ-isoperimetric inequality is valid now for all domains D . We cannot apply directly the Theorem 2.1 because $\Lambda \notin \mathcal{L}$ but we can apply the formula (2.14) obtained in the course of the proof of the Theorem 2.1 without using $\Lambda \in \mathcal{L}$. Putting there $\delta = \frac{1}{2}$ and noting that $I(t) = p_\Omega(y, y, 2t)$ we obtain that for any region Ω and for any $y \in \Omega, \ t > t_0$

$$\int_{\frac{4}{p_\Omega(y,y,2t_0)}}^{\frac{4}{p_\Omega(y,y,2t)}} \frac{dv}{v\Lambda(v)} \geq t - t_0 \ . \tag{2.14$'$}$$

Let us introduce the function $v(t)$ from the relation

$$t - t_0 = \int_{v_0}^{v(t)} \frac{dv}{v\Lambda(v)} \tag{2.26}$$

where $v_0 = \frac{4}{p_\Omega(y,y,2t_0)}$. As it follows from ($2.14'$)

$$\frac{4}{p_\Omega(y,y,2t)} \geq v(t) \ , \ p_\Omega(y,y,2t) \leq \frac{4}{v(t)}$$

Hence, in order to estimate the integral ($2.24'$) from above it suffices to obtain a uniform in Ω bound of $\int_{t_0}^{\infty} \frac{1}{v(t)} dt$. Changing a variable in the integral we have

$$\int_{t_0}^{\infty} \frac{1}{v(t)} dt = \int_{v_0}^{\infty} \frac{1}{v} \frac{dt}{dv} dv$$

$$= \int_{v_0}^{\infty} \frac{1}{v^2\Lambda(v)} dv = \int_{\frac{4}{p_\Omega(y,y,2t_0)}}^{\infty} \frac{1}{v^2\Lambda(v)} dv$$

Thus, collecting all these relations together, we obtain

$$\int_{t_0}^{\infty} p_\Omega(y,y,2t)dt \leq 4 \int_{\frac{4}{p_\Omega(y,y,2t_0)}}^{\infty} \frac{1}{v^2\Lambda(v)} dv$$

Letting $\Omega \to M$ we see that the integral ($2.24'$) is finite, which was to be proved. $\square$

Let us compare this result with a similar one obtained earlier in [G85] . Namely, the theorem of [G85] establishes transience provided the following isoperimetric inequality holds for any bounded region $D \subset M$ with a smooth boundary:

$$\mathrm{Area}\,(\partial D) \geq g(\mathrm{Vol}\, D) \tag{2.27}$$

and the function g satisfies the inequality

$$\int^{\infty} \frac{dv}{g(v)^2} < \infty. \tag{2.28}$$

Recall first that as was mentioned already in Section 1 inequality (2.27) implies some Λ-isoperimetric inequality . Indeed, the following is true.

Proposition 2.4 *Suppose that for any bounded region D with a smooth boundary such that $\bar{D} \subset \Omega$ the inequality (2.27) holds, where Ω is an arbitrary Riemannian manifold and the function g is such that $g(v)/v$ is a decreasing one, then the Λ-isoperimetric inequality holds in Ω where Λ is given by the relation*

$$\Lambda(v) = \frac{1}{4}\left(\frac{g(v)}{v}\right)^2 \tag{2.29}$$

Indeed, let ω be some sub-region in Ω and the closure of D lie inside ω . By the hypothesis (2.27)

$$\text{Area}\,(\partial D) \geq g(|D|) = |D|\,\frac{g(|D|)}{|D|} \geq |D|\,\frac{g(|\omega|)}{|\omega|}$$

Thus, the Cheeger's isoperimetric constant (see [Chr])

$$h(\omega) \equiv \inf_{D \subset \omega} \frac{\text{Area}\,(\partial D)}{|D|}$$

satisfies the following estimate

$$h(\omega) \geq \frac{g(|\omega|)}{|\omega|}$$

By the Cheeger's theorem we have

$$\lambda_1(\omega) \geq \frac{1}{4} h(\omega)^2$$

which implies (2.29). □

We are left to notice that (2.25) is transformed to (2.28) when substituting g from (2.29).

3. Mean-value type theorem

In this Section we deal with a mean-value type theorem. We call so a theorem which establishes a relation between a value which a solution of heat equation takes at some point and an integral of the solution over some neighbourhood of this point. This theorem will enable us to obtain a dependence on distance function when estimating the heat kernel. In fact, the theorem is already proved in its the most important part in [G91] . Here we are going to simplify the final result.

Let us introduce the following notations. Let Λ be a function of class $\mathcal{L}$. Since the function $\frac{v}{\Lambda(v)}$ is strictly monotone on $(0, +\infty)$ and has a range $(0, +\infty)$ it follows that it is invertible. Let us denote the inverse function by ω and define the functions $\widetilde{V}(t),\ \widetilde{W}(r)$ for $t > 0,\ r > 0$ as follows

$$t = \int_0^{\widetilde{V}(t)} \frac{d\xi}{\omega(\xi)}\ ,\ r = \int_0^{\widetilde{W}(r)} \frac{d\xi}{\sqrt{\xi\omega(\xi)}} \tag{3.1}$$

The following theorem was proved in [G91] .

Theorem 3.1 *Suppose that the Λ-isoperimetric inequality holds in some ball B_R^z where the function Λ is such that the integrals in (3.1) converge at zero and the functions $\widetilde{V},\ \widetilde{W}$ are defined on $[0, +\infty[$. Let $\mathcal{C} \equiv B_R^z \times (0, T),\ T > 0$ and suppose that a function $u(x, t) \in C^\infty(\bar{\mathcal{C}})$ satisfies in $\mathcal{C}$ the inequality*

$$u_t - \Delta u \leq 0, \tag{3.2}$$

then

$$u(z, T)_+^2 \leq \frac{4}{\min\Big(\widetilde{V}(cT), \widetilde{W}(cR)\Big)} \int_{\mathcal{C}} u_+^2 \tag{3.3}$$

where $c>0$ is an absolute constant, for example, $c=0.0001$.

Examples. 1. If $\Lambda = av^{-2/n}$ then

$$\widetilde{V}(t)=C_1(n)a^{n/2}t^{(n+2)/2}, \quad \widetilde{W}(r)=C_2(n)a^{n/2}r^{n+2}$$

and (3.3) acquires the form

$$u(z,T)_+^2 \le \frac{\mathrm{const}_n a^{-n/2}}{\min\Big(\sqrt{T},\, R\Big)^{n+2}} \int_{\mathcal{C}} u_+^2 \tag{3.4}$$

In the Euclidean space $\mathbf{R}^n$ this inequality was proved by Moser [M] .

2. If $\Lambda=\max(av^{-2/n},\, A),\ A>0$, then

$$\widetilde{V}(t) \asymp a^{n/2}\min(t,\ A^{-1})^{\frac{(n+2)}{2}}\exp(cAt)$$

$$\widetilde{W}(r) \asymp a^{n/2}\min(r^2,\ A^{-1})^{\frac{(n+2)}{2}}\exp(c\sqrt{A}r)$$

where $\asymp$ means "is in finite ratio with" and the constants bounding the ratio of the right- and left-hand sides in these relations depend only on n .

3. Let us set

$$\Lambda(v)=\frac{b}{R^2}\left(\frac{\mathrm{Vol}\, B_R^z}{v}\right)^{\beta}$$

with some positive constants b,β . The Λ-isoperimetric inequality with this function Λ holds in any ball B_R^x on a manifold with a non-negative Ricci curvature with the constants $\beta=\frac{2}{n},\ b=\mathrm{const}_n$ (see [G91] for the proof). This Λ-isoperimetric inequality is in some sense a natural one when obtaining the heat kernel upper bound of the following kind: $p(x,y,t) \le \frac{\mathrm{const}}{\mathrm{Vol}\, B^x_{\sqrt{t}}}$ - see the Section 5 below for details. But when staying inside the ball B_R^z the Λ-isoperimetric inequality in question is essentially the same as that of the Example 1 provided we take $a=\frac{b}{R^2}\Big(\mathrm{Vol}\, B_R^z\Big)^{\beta}$. Substituting into (3.4) and noting that $T\mathrm{Vol}\, B_R^z=\mathrm{Vol}\,\mathcal{C}$, we obtain

$$u(z,T)_+^2 \le \frac{\mathrm{const}_n}{\min\Big((T/R^2)^{1/\beta},\, R^2/T\Big)} \frac{1}{\mathrm{Vol}\,\mathcal{C}} \int_{\mathcal{C}} u_+^2 \tag{3.5}$$

If $T=R^2$ then the first fraction in front of the integral in (3.5) does not depend on T and R at all and (3.5) means that the value of the function u at the top point is estimated from above by the L^2 -norm of u over the cylinder $\mathcal{C}$. This is why we call the Theorem 3.1 as a mean-value type theorem.

The purpose of this Section is to replace the functions $\widetilde{V},\widetilde{W}$ from the statement of the Theorem 3.1 by other functions which are more convenient for applications. Next we assume that the function Λ under consideration lies in the class $\mathcal{L}$ and, moreover, $\sqrt{\Lambda}$ is also in $\mathcal{L}$, then we can consider the functions $V(t)$ and $W(r)$, being the $V-$ transformations of Λ and $\sqrt{\Lambda}$ respectively. The latter means that $W(r)$ is defined by the following relation

$$r=\int_0^{W(r)}\frac{dv}{v\sqrt{\Lambda(v)}} \tag{3.6}$$

Proposition 3.1 *If Λ and $\sqrt{\Lambda}$ belong to $\mathcal{L}$ it follows that the functions $\widetilde{V}(t)$ and $\widetilde{W}(r)$ defined from (3.1) exist for all $t > 0,\ r > 0$ and satisfy the estimates*

$$\widetilde{V}(t) \geq \frac{1}{4}tV(\frac{1}{4}t) \tag{3.7}$$

$$\widetilde{W}(r) \geq \frac{1}{64}r^2W(\frac{1}{8}r) \tag{3.8}$$

Remark. The inequalities of the opposite direction are valid too:

$$\widetilde{V}(t) \leq tV(t), \quad \widetilde{W}(r) \leq \frac{1}{4}r^2W(r)$$

but we do not use them.

First we prove the following lemma.

Lemma 3.1 *Under the assumptions made above the functions $\frac{1}{V(t)}$ and $\frac{1}{\sqrt{W(r)}}$ are convex.*

Proof. Consider first the former of these functions. It suffices to prove that t as a function of the argument $\frac{1}{V}$ is convex. Let us consider the derivative

$$\frac{dt}{d\frac{1}{V}} = -\frac{V^2}{V'} = -\frac{V}{\Lambda(V)}$$

where we have applied that $V' = V\Lambda(V)$. When $1/V$ is increasing V is decreasing, $\Lambda(V)$ increasing, $\frac{V}{\Lambda(V)}$ decreasing. Hence, $\frac{dt}{d\frac{1}{V}}$ is increasing and t as a function of $\frac{1}{V}$ is convex.

Similarly one can prove that r as a function of $\frac{2}{\sqrt{W}}$ is convex because

$$\frac{dr}{d\frac{2}{\sqrt{W}}} = -\sqrt{\frac{W}{\Lambda(W)}}$$

Proof of (3.7). By definition the function ω satisfies the relation

$$\xi = \frac{\omega(\xi)}{\Lambda(\omega(\xi))} \tag{3.9}$$

According to (3.1) we have

$$t = \int_0^{\widetilde{V}(t)} \frac{d\xi}{\omega(\xi)} = \int_0^{\omega(\widetilde{V})} \frac{1}{\omega}\frac{d\xi}{d\omega}d\omega = \int_0^{\omega(\widetilde{V})} \frac{1}{\omega}\frac{\Lambda d\omega - \omega d\Lambda}{\Lambda^2}$$

$$= \int_0^{\omega(\widetilde{V})} \frac{1}{\omega}\frac{d\omega}{\Lambda(\omega)} - \int_\infty^{\Lambda(\omega(\widetilde{V}))} \frac{d\Lambda}{\Lambda^2}\ ,$$

$$t = \int_0^{\omega(\widetilde{V})} \frac{d\omega}{\omega\Lambda(\omega)} + \frac{1}{\Lambda(\omega(\widetilde{V}))} \tag{3.10}$$

This implies, in particular, that the integral in the definition (3.1) of the function $\widetilde{V}$ converges at 0 . Moreover, since $\Lambda(\omega)$ is a monotone decreasing function it follows that $\int_0^\infty \frac{d\omega}{\omega\Lambda(\omega)} = \infty$ which means that the function $\widetilde{V}(t)$ is defined for all positive t . Let us consider two cases. Let first the integral on the right-hand side of (3.10) be at least as large as the second summand, then

$$\frac{t}{2} \leq \int_0^{\omega(\widetilde{V})} \frac{d\omega}{\omega\Lambda(\omega)}$$

whence $\omega(\widetilde{V}(t)) \geq V(t/2)$ follows. Together with (3.9) and (2.7) this implies

$$\widetilde{V}(t) \geq \frac{V(t/2)}{\Lambda\Big(V(t/2)\Big)} = \frac{V^2(t/2)}{V'(t/2)} \tag{3.11}$$

We are left to show that for all $\tau > 0$

$$\frac{V^2(\tau)}{V'(\tau)} \geq \frac{1}{2}\tau V(\frac{1}{2}\tau) \tag{3.12}$$

(it is evident that (3.12) for $\tau = t/2$ and (3.11) imply (3.7)).

In order to prove (3.12) consider the function $v(\tau) \equiv \frac{1}{V(\tau)}$. Due to the Lemma 3.1 this function is a convex one which means, in particular, that

$$v'(\tau) \geq \frac{v(\tau) - v(\tau/2)}{\tau/2} \geq -\frac{v(\tau/2)}{\tau/2}$$

Substituting here $v = 1/V$ we obtain (3.12).

Consider now the second case when the former summand on the right-hand side of (3.12) is less than the latter, then

$$\frac{t}{2} < \frac{1}{\Lambda(\omega(\widetilde{V}))} , \quad \frac{2}{t} > \Lambda(\omega(\widetilde{V}))$$

Since the range of the function Λ covers the interval $[\Lambda(\omega(\widetilde{V})), +\infty[$ it follows that $2/t$ lands into the range of Λ . Denote by $u(t)$ the smallest number u for which $\Lambda(u) = 2/t$, then $\Lambda(u) > \Lambda(\omega(\widetilde{V}))$ and, hence, $u(t) < \omega(\widetilde{V}(t))$. Using (3.9) we get

$$\widetilde{V}(t) > \frac{u}{\Lambda(u)} = \frac{tu(t)}{2} \tag{3.13}$$

It follows from (2.6) that

$$t = \int_0^{V(t)} \frac{dv}{v\Lambda(v)} > \int_{V(t)/2}^{V(t)} \frac{dv}{v\Lambda(v)} \geq \frac{\frac{1}{2}V(t)}{V(t)\Lambda(V(t)/2)} = \frac{1}{2\Lambda(V(t)/2)} \ .$$

Replacing here t by $t/4$ we obtain

$$\Lambda\Big(\frac{1}{2}V(\frac{1}{4}t)\Big) > \frac{2}{t} \ .$$

It yields together with $2/t = \Lambda(u(t))$ and (3.13)

$$\frac{1}{2}V(\frac{1}{4}t) \leq u(t) < \frac{2}{t}\widetilde{V}(t)$$

which coincides with (3.7).

The inequality (3.8) is proved in the same way. Let us sketch briefly the main points of the proof. It follows from the definition (3.1) of the function $\widetilde{W}$ that

$$r = \int_0^{\omega(\widetilde{W})} \frac{dv}{v\sqrt{\Lambda(v)}} + \frac{2}{\sqrt{\Lambda(\omega(\widetilde{W}))}} \tag{3.14}$$

Suppose first that the integral on the right-hand side of (3.14) is greater than or equal to the second summand, then $\omega(\widetilde{W}) \geq W(r/2)$ which implies together with (3.9) and $W' = W\sqrt{\Lambda(W)}$ that

$$\widetilde{W}(r) \geq \frac{W^3(r/2)}{W'^2(r/2)} \ . \tag{3.15}$$

Next we use the facts that the function $v(r) \equiv \frac{2}{\sqrt{W(r)}}$ is by the Lemma 3.1 a convex one and for such a function

$$v'(\tau) \geq -\frac{v(\tau/2)}{\tau/2} \ .$$

Hence, we obtain

$$\frac{W^{3/2}(\tau)}{W'(\tau)} \geq \frac{\tau}{4}\sqrt{W(\frac{1}{2}\tau)}$$

Taking here $\tau = r/2$ and substituting into (3.15) we obtain (3.8).

Suppose now the integral in (3.13) is smaller than the second summand, then we have

$$\Lambda(\omega(\widetilde{W})) < \frac{16}{r^2} \ , \quad \omega(\widetilde{W}) > u(r) \quad \widetilde{W}(r) > \frac{r^2 u(r)}{16} \tag{3.16}$$

where the function $u(r)$ is defined from the relation $\Lambda(u(r)) \equiv \frac{16}{r^2}$.

On the other hand (3.6) yields

$$r > \frac{1}{2\sqrt{\Lambda(W(r)/2)}}$$

or, replacing r by $r/8$

$$\Lambda\Big(\frac{1}{2}W(\frac{1}{8}r)\Big) > \frac{16}{r^2}$$

which implies $\frac{1}{2}W(r/8) \leq u(r)$. Collecting all these inequalities together we obtain finally (3.8). □

Now the Theorem 3.1 can be reformulated as follows.

Theorem 3.2 *Suppose that the Λ-isoperimetric inequality holds in some ball $B_R^z \subset M$ with the function $\Lambda(v)$ such that $\sqrt{\Lambda} \in \mathcal{L}$. Let $\mathcal{C} \equiv B_R^z \times (0,T)$, $T > 0$ and suppose that a function $u(x,t) \in C^\infty(\bar{\mathcal{C}})$ satisfies in $\mathcal{C}$ the inequality*

$$u_t - \Delta u \leq 0,$$

then

$$u(z,T)_+^2 \le \frac{\text{const}}{\min\Big(TV(cT),\ R^2W(cR)\Big)} \int_C u_+^2 \tag{3.17}$$

where $c > 0$, const *are absolute constants.*

4. Integral estimates with Gaussian term

We start this Section with some auxiliary properties of the $\Lambda-$ transformation which are going to be used in obtaining upper bounds of the heat kernel containing the factor $\exp(-c\frac{r^2}{t})$. Let us assume that Λ is a function of class $\mathcal{L}$ and, moreover, $\sqrt{\Lambda} \in \mathcal{L}$. Let $V(t)$ and $W(r)$ be $V-$ transformations of Λ and $\sqrt{\Lambda}$ respectively. Let us denote by $\mathcal{R}(t)$ the function $W^{-1} \circ V$ i.e.

$$\mathcal{R}(t) \equiv \int_0^{V(t)} \frac{dv}{v\sqrt{\Lambda(v)}} \tag{4.1}$$

Lemma 4.1 *For all* $t_2 > t_1 > 0$ *the inequality holds*

$$\frac{(\mathcal{R}(t_2) - \mathcal{R}(t_1))^2}{t_2 - t_1} \le \log \frac{V(t_2)}{V(t_1)} \tag{4.2}$$

Proof. Let $V_i = V(t_i)$, then (4.1) implies

$$(\mathcal{R}(t_2) - \mathcal{R}(t_1))^2 = \left(\int_{V_1}^{V_2} \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \le \int_{V_1}^{V_2} \frac{dv}{v\Lambda(v)} \int_{V_1}^{V_2} \frac{dv}{v} = (t_2 - t_1) \log \frac{V_2}{V_1}$$

whence (4.2) follows.

Proposition 4.1 *Let* $T > 0$ *and* $\delta \in (0,1)$ *be given.*

(i) If the function

$$\frac{V(t)}{V(\delta t)} \tag{4.3}$$

is bounded above by a constant N *for* $t \le T$ *(where* T *may be equal to* ∞ *) then for any* $t \le T$

$$\frac{\mathcal{R}^2(t)}{t} \le \text{const}_{N,\delta}\ . \tag{4.4}$$

(ii) Suppose that the function $\frac{V(t)}{V(\delta t)}$ *is monotonically increasing for* $t \ge T$ *, then for any* $t > T$ *the inequality holds*

$$\frac{\mathcal{R}^2(t)}{t} \le \text{const}_\delta \log \frac{V(t)}{V(\delta t)} + \text{const}_{T,\delta} \tag{4.5}$$

where $\text{const}_\delta = \frac{1-\delta}{(1-\sqrt{\delta})^2}$.

Proof of (ı). Let us consider a sequence

$$t_k = t\delta^k,\ \ k = 0, 1, 2, \dots \ . \tag{4.6}$$

By the Lemma 4.1 we have

$$\mathcal{R}(t_k) - \mathcal{R}(t_{k+1}) \le \sqrt{(t_k - t_{k+1})}\left(\log \frac{V(t_k)}{V(t_{k+1})}\right)^{\frac{1}{2}} \tag{4.7}$$

or, taking into account the hypothesis (ı)

$$\mathcal{R}(t_k) - \mathcal{R}(t_{k+1}) \le \delta^{k/2}\sqrt{(1-\delta)t\log N}\ .$$

Adding these inequalities over all k we obtain

$$\mathcal{R}(t) \le \mathrm{const}_{N,\delta}\sqrt{t}$$

which coincides with (4.4).

Proof of (ıı). Consider again the sequence (4.6). Denote by K the biggest integer for which $t_K > T$ is still valid. This means in particular, that $t_K \in [T, T/\delta]$. Let us denote for the sake of brevity $V_i = V(t_i),\ r_i = R(t_i)$ and consider the sequence

$$\frac{r_k^2}{t_k}, \qquad k = 0, 1, 2, \dots K\ . \tag{4.8}$$

Suppose first that this sequence is a monotone increasing one, then we have

$$\frac{\mathcal{R}^2(t)}{t} \le \frac{r_K^2}{t_K} \le \sup_{\tau\in[T,\frac{T}{\delta}]} \frac{\mathcal{R}^2(\tau)}{\tau} = \mathrm{const}_{T,\delta} \tag{4.9}$$

which implies, of course, (4.5).

Consider now the second case when there is a term of the sequence (4.8), say, t_m such that

$$\frac{r_m^2}{t_m} > \frac{r_{m+1}^2}{t_{m+1}}\ . \tag{4.10}$$

We may assume that m is the smallest number for which (4.10) is true. It follows from (4.10) that

$$r_{m+1} < r_m\sqrt{\delta}\ .$$

Applying the Lemma 4.1 we obtain

$$\frac{(r_m - r_m\sqrt{\delta})^2}{(1-\delta)t_m} \le \log\frac{V(t_m)}{V(\delta t_m)}\ . \tag{4.11}$$

Since the function $\frac{V(t)}{V(\delta t)}$ is increasing it follows that

$$\log\frac{V(t_m)}{V(\delta t_m)} \le \log\frac{V(t)}{V(\delta t)}\ .$$

Due to the choice of m we have
$$\frac{\mathcal{R}^2(t)}{t} \leq \frac{r_m^2}{t_m} \ .$$
Substituting these relations into (4.11) we obtain finally
$$\frac{\mathcal{R}^2(t)}{t} \leq \frac{1-\delta}{(1-\sqrt{\delta})^2} \log \frac{V(t)}{V(\delta t)}$$
whence (4.5) follows. □

The following lemma reduces the question whether the function $\frac{V(t)}{V(\delta t)}$ satisfies the hypotheses of the Proposition 4.1 to a simpler one.

Lemma 4.2 *Suppose that* $\delta \in (0,1) \quad \infty \geq T' > T/\delta \geq 0$.

(i) If the function
$$\frac{d \log V(t)}{d \log t} \tag{4.12}$$
is bounded by a constant N *for* $t \in]T, T']$, *then the function* $\frac{V(t)}{V(\delta t)}$ *is bounded by the constant* δ^{-N} *on* $]T/\delta, T']$.

(ii) If the function (4.12) is monotone increasing on $]T, T']$ *it follows that the function* $\frac{V(t)}{V(\delta t)}$ *is monotone increasing on* $]T/\delta, T']$.

Proof. We have for $t \in]T, T']$
$$\log \frac{V(t)}{V(\delta t)} = \log V(t) - \log V(\delta t) = \int_{\log \delta t}^{\log t} \frac{d \log V(\tau)}{d \log \tau} d \log \tau \tag{4.13}$$
Note that the integral is taken over an interval of a constant length $-\log \delta$ lying in $]\log T, \ \log T']$. If the function to be integrated is bounded by N then the left-hand side in (4.13) does not exceed $N \log \delta$. Else if the function is increasing then the left-hand side of (4.13) is increasing too because the interval of integration is moving to right as t is getting larger. □

Now we proceed to our main estimates of some integrals of the heat kernel. The next theorem does not still make use of the foregoing properties of the $\Lambda-$ transformation - they will simplify the further applications of this theorem.

Theorem 4.1 *Let the* Λ*-isoperimetric inequality hold in some ball* B_R^z , $\sqrt{\Lambda}$ *being a function of class* $\mathcal{L}$. *Let* $V(t)$ *and* $W(r)$ *be* $V-$ *transformations of* Λ *and* $\sqrt{\Lambda}$ *respectively, then for all* $\rho \in (0, R]$ *and* $0 < \tau \leq t$ *the inequality holds*
$$\int_M p^2(x,z,t) \exp\left(\frac{d^2(x)}{2t} \right) dx \leq \frac{\text{const}}{\min\left(V(c\tau), \ \frac{\rho^2}{\tau} W(c\rho) \right)} \tag{4.14}$$
where $d(x) \equiv \operatorname{dist}(x, B_\rho^z) \equiv (\operatorname{dist}(x,z) - \rho)_+$ *and* c, const *are some absolute positive constants (c is the same as in the Theorem 3.2).*

Remark. Of course, one could replace in the statement τ by t and ρ by R - that would simplify the formulating, but, generally speaking, these are not the optimal choice of $\tau, \ \rho$.

Proof. The idea behind the proof is similar to that of [G87c] . Let $\varphi(x) \in C_c^\infty(M)$ be an arbitrary but fixed function and $\Omega \subset M$ be an arbitrary pre-compact region with a smooth boundary containing $\operatorname{supp}\varphi$ and B_R^z . Set

$$u_\Omega(x,t) \equiv \int_\Omega p_\Omega(x,y,t)\varphi(y)dy$$

and apply to this function the Theorem 3.2 in the cylinder $B_\rho^z \times (t, t-\tau)$:

$$u_\Omega(z,t)^2 \le K \int_{t-\tau}^t \int_{B_\rho^z} u^2(x,s)dxds$$

where

$$K \equiv \frac{\mathrm{const}}{\min\Big(\tau V(c\tau),\ \rho^2 W(c\rho)\Big)} .$$

Note that the function being integrated here can be multiplied by the following term

$$\exp\left(-\frac{d^2(x)}{2(t-s)}\right)$$

which is equal identically to 1 in the ball B_ρ^z and, besides, the domain of integrating can be extended to the entire region Ω . Thus, we obtain

$$u_\Omega(z,t)^2 \le K \int_{t-\tau}^t \int_\Omega u_\Omega^2(x,s) \exp\left(-\frac{d^2(x)}{2(t-s)}\right) dxds . \tag{4.15}$$

Next we need the following lemma.

Lemma 4.3 *(integral maximum principle) Let Ω be some pre-compact region in M with a smooth boundary and $w(x,s)$ be a solution to the heat equation $w_s - \Delta w = 0$ in $\Omega \times (T_0, T)$ vanishing on the boundary $\partial\Omega \times (T_0,T)$. Suppose that $\xi(x,s)$ is a Lipschitz function in $\Omega \times (T_0,T)$ such that*

$$\xi_s + \frac{1}{2}|\nabla\xi|^2 \le 0 , \tag{4.16}$$

then the following integral

$$I(s) \equiv \int_\Omega w^2(x,s)e^{\xi(x,s)}dx \tag{4.17}$$

is a decreasing function in $s \in (T_0,T)$. Moreover, if $0 \le \lambda \le \lambda_1(\Omega)$ it follows that the function

$$I(s)\exp(2\lambda s)$$

decreases in s as well.

Remark. We use the derivatives of the function $\xi(x,s)$ although it may be not differentiable. But this function is assumed to be Lipschitz and, thereby, is locally differentiable in a weak sense. The inequality (4.16) is also understood in the sense of distributions.

The proof of the first part of this Lemma is well-known (see , for example, [ChLY] , [PE]) and consists of checking that $I'(s) \le 0$. The expression on the left-hand side of (4.16) appears as a discriminant of some quadratic polynomial which is to be non-negative. For the second part including the exponential decay of $I(s)$, see [G92] .

Let us apply this Lemma to the functions u_Ω and $\xi(x,s) = -\frac{d^2(x)}{2(t-s)}$. The distance function $d(x)$ is evidently Lipschitz and $|\nabla d| \le 1$ whence the validity of (4.16) follows. We obtain that the integral over Ω in (4.15) does not exceed that for $s=0$. Therefore

$$u_\Omega(z,t)^2 \le K\tau \int_\Omega \varphi^2(x) \exp\left(-\frac{d^2(x)}{2t}\right) dx$$

Letting here $\Omega \to M$ we deduce that the same estimate is valid for the function

$$u(x,t) = \lim_{\Omega\to M} u_\Omega(x,t) = \int_M p(x,y,t)\varphi(y)dt$$

which implies

$$u(z,t)^2 \le K\tau \int_M \varphi^2(x) \exp\left(-\frac{d^2(x)}{2t}\right) dx \tag{4.18}$$

Consider now a mapping $\Phi : L^2(M) \to \mathbf{R}$ given by the rule

$$\Phi(\eta) \equiv \int_M p(x,z,t) \exp\left(\frac{d^2(x)}{4t}\right) \eta(x)dx \tag{4.19}$$

t being fixed. Let us first explain why this mapping is defined i.e. the integral converges. For a function η of the form $\varphi(x)\exp(-d^2(x)/4t)$, where $\varphi \in C_c^\infty(M)$ we have

$$\Phi(\eta) = \int_M p(x,z,t)\varphi(x)dx$$

This integral here coincides with $u(z,t)$ and by (4.18) we have

$$\Phi(\eta)^2 \le K\tau \int_M \varphi^2(x) \exp\left(-\frac{d^2(x)}{2t}\right) dx = K\tau \int_M \eta^2(x)dx$$

whence an estimate follows

$$\Phi(\eta)^2 \le K\tau \,\|\eta\|_2^2 \ .$$

Thus, the mapping Φ is bounded on the set of functions η of the form under consideration. Since this set is dense in $L^2(M)$ it follows that

$$\|\Phi\|^2 \le K\tau \tag{4.20}$$

On the other hand the definition (4.19) of Φ implies that

$$\|\Phi\|^2 = \int_M p^2(x,z,t) \exp\left(\frac{d^2(x)}{2t}\right) dx \ .$$

Combining this with (4.20) we get (4.14) which was to be proved. $\square$

Corollary 4.1 *Under hypotheses of the Theorem 4.1 the following estimate is valid for any $D>2$*

$$E_D(z,t)\equiv\int_M p^2(x,z,t)\exp\left(\frac{r^2}{Dt}\right)\,dx\le\frac{\text{const}\;\cdot\exp\left(\frac{\rho^2}{(D-2)t}\right)}{\min\left(V(c\tau),\,\frac{\rho^2}{\tau}W(c\rho)\right)}\tag{4.21}$$

where $r=\operatorname{dist}(x,z)$.

Proof. It is easy to check that for any $D>2$ a quadratic inequality holds

$$d(x)^2=(r-\rho)_+^2\ge\frac{2}{D}r^2-\frac{2}{D-2}\rho^2$$

which implies together with (4.14) inequality (4.21) . □

Remark. For $D=2$ one cannot hope to estimate E_D because E_2 can be equal to ∞ as happens in the Euclidean space.

The expression $E_D(z,t)$ is of much importance for us. As we have seen it is estimated above via an isoperimetric property of a manifold. On the other hand, it will enable us to obtain pointwise estimates of the heat kernel in the next section. The following property of E_D is also very useful.

Proposition 4.2 *On an arbitrary manifold M for any $D>2$ the quantity $E_D(x,t)$ is finite for all $x\in M$ and $t>0$; moreover, the function*

$$E_D(x,t)\exp(2\lambda t)\tag{4.22}$$

is decreasing in t provided $0\le\lambda\le\lambda_1(M)$.

Proof. For any point $x\in M$ there exists a small positive R such that the ball B_R^x is diffeomorphic to a Euclidean one and the Euclidean metric is finite proportional to the Riemannian one i.e. the ball B_R^x is quasi-isometric to the Euclidean one. We claim that in the ball B_R^x a Λ-isoperimetric inequality holds with a Euclidean function i.e. for any region $\Omega\subset B_R^x$

$$\lambda_1(\Omega)\ge\text{const}\,(\operatorname{Vol}\Omega)^{-2/n}.$$

Indeed, the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is defined as the infimum of the ratio of the integrals $\int_\Omega|\nabla\varphi|^2$ and $\int_\Omega\varphi^2$ over all test functions $\varphi\in C_c^\infty(\Omega)$. Either integral is altered under a quasi-isometric transformation at most by a constant factor and, thereby, the same is happening with their ratio. Hence, the first eigenvalue is changed at most by a constant multiple too and the relationship between it and the volume of Ω remains as that in the Euclidean space up to a constant multiple.

The fact of the presence of a local Λ-isoperimetric inequality allows us to apply the corollary 4.1 which ensures the finiteness of $E_D(x,t)$.

To prove the second assertion let us fix some $x\in M$ and consider a pre-compact region $\Omega\subset M,\ x\in\Omega$ with a smooth boundary, then by the Lemma 4.3 the function

$$\exp(2\lambda t)\int_\Omega\exp\left(\frac{r^2}{Dt}\right)\,p_\Omega^2(x,y,t)dy\tag{4.23}$$

decreases in t whenever $0 \leq \lambda \leq \lambda_1(M)$ (note that $\lambda_1(M) \leq \lambda_1(\Omega)$). Let $\Omega \to M$ then function (4.23) tends to $\exp(2\lambda t) E_D(x,t)$ whence the decreasing of the function (4.22) follows. □

Our next step is to simplify the right-hand side of (4.21) . For this purpose we have to impose some restriction on function the $V(t)$ which, however, will not affect the rate of increase of $V(t)$ as $t \to \infty$. Namely, suppose that for some $T \in]0,+\infty]$ and $N > 0$

$$\begin{cases} \frac{d\log V(t)}{d\log t} \text{ is increasing for } t > T \\ \frac{d\log V(t)}{d\log t} \leq N \text{for } t \leq 2T \end{cases} . \tag{4.24}$$

This condition needs some comments. For all reasonable applications of the Theorem 4.1 we have for small values of t that $V(t) = \text{const} \cdot t^{\nu},\ \nu > 0$. Therefore for small t the function $\frac{d\log V(t)}{d\log t}$ is bounded above. If this function remains bounded at ∞ then (4.24) is satisfied for $T = \infty$. Otherwise the function in question is unbounded and we may assume it to be increasing at a neighbourhood of ∞ (this restriction causes no troubles for applications). If this is the case then condition (4.24) is satisfied for some finite value of T . Let us note that the first case (i.e. $T = \infty$) takes place for a polynomial function $V(t)$ whereas the second one (when T is finite) holds for a function $V(t)$ of superpolynomial growth.

Theorem 4.2 *Suppose that as in the Theorem 4.1 the Λ-isoperimetric inequality holds in some ball B_R^z and that $\sqrt{\Lambda} \in \mathcal{L}$. Let $V(t)$ and $W(r)$ be as above $V-$ transformations of Λ and $\sqrt{\Lambda}$ respectively and, in addition, suppose $V(t)$ satisfies the condition (4.24) . Suppose also that for some $\tau > 0$*

$$\tau \leq t, \quad \tau \leq R^2, \quad \mathcal{R}(c\tau) \leq cR \tag{4.25}$$

where $\mathcal{R}(\cdot)$ is defined by (4.1), then for any $D > 2$

$$E_D(z,t) \leq \frac{\text{const}_{N,T}}{\delta V(\widetilde{c}\delta\tau)} \tag{4.26}$$

where $\widetilde{c} = c^2/12, \quad \delta = \min(D-2, 6/c)$ and c is the same as in the Theorem 4.1.

Proof. Let us first observe that for $D > 6/c$ the right-hand side of (4.26) does not depend on D while the left-hand side (i.e. the quantity E_D) is decreasing with respect to D . Hence, the estimate (4.26) for $D > 6/c$ will follow from that of $D = 6/c$. This is why we assume from now on that $D \leq 6/c$ and, thereby, $\delta = D - 2$. Similarly, due to the monotone decreasing of $E_D(z,t)$ in t it suffices to consider the case $t = \tau$. Let us apply the Corollary 4.1 for $\tau = t$ and for

$$\rho = \max(\sqrt{\varepsilon t},\ c^{-1}\mathcal{R}(\varepsilon ct)) \tag{4.27}$$

where $\varepsilon \leq 1$ is to be chosen later, then according to (4.27) and (4.25) $\rho \leq R$ and by (4.21) we have

$$E_D(z,t) \leq \frac{\text{const} \cdot \exp\left(\frac{\rho^2}{\delta t}\right)}{\min\left(V(ct),\ \frac{\rho^2}{t} W(c\rho)\right)} \tag{4.28}$$

Since $\rho^2/t \geq \varepsilon$ and $W(c\rho) \geq W(\mathcal{R}(\varepsilon ct) = V(\varepsilon ct)$ it follows that the denominator in (4.28) is at least as large as $\varepsilon V(\varepsilon ct)$. To estimate the numerator first note that due to (4.24) and the Lemma 4.2 (applied for $\delta = \frac{1}{2}$ - this is not the δ from the Theorem 4.2 !) the function $V(t)/V(\frac{1}{2}t)$ is bounded above by const_N in $]0, 2T]$ and increasing in $[2T, \infty[$. Therefore, by the Proposition 4.1 we have for all $t > 0$

$$\frac{\mathcal{R}^2(t)}{t} \leq 6 \log \frac{V(t)}{V(\frac{1}{2}t)} + C$$

where $C = C(N, T)$. Replacing here t by εct we obtain

$$\frac{\mathcal{R}^2(\varepsilon ct)}{\varepsilon ct} \leq 6 \log \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} + C$$

or

$$\frac{(c^{-1}\mathcal{R}(\varepsilon ct))^2}{\delta t} \leq \frac{6\varepsilon}{\delta c} \log \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} + \frac{\varepsilon}{\delta c} C \ .$$

Let us take

$$\varepsilon = \frac{c\delta}{6}$$

and note that the hypothesis $D \leq 2 + 6/c$ implies $\delta \leq 6/c$ and $\varepsilon \leq 1$. We obtain thereby

$$\frac{(c^{-1}\mathcal{R}(\varepsilon ct))^2}{\delta t} \leq \log \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} + C \ .$$

Since

$$\frac{(\sqrt{\varepsilon t})^2}{\delta t} = \frac{\varepsilon}{\delta} \leq \frac{c}{6}$$

it follows from (4.27) that

$$\frac{\rho^2}{\delta t} \leq \log \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} + C + \frac{c}{6}$$

whence we obtain

$$\exp\left(\frac{\rho^2}{\delta t}\right) \leq \mathrm{const}_{N,T} \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} \ .$$

This is a desired estimation of the numerator in (4.28) . Substituting it into (4.28) together with the obtained above estimate of the denominator we get finally

$$E_D(z, t) \leq \frac{\mathrm{const}_{N,T}}{\varepsilon V(\varepsilon ct)} \frac{V(\varepsilon ct)}{V(\frac{1}{2}\varepsilon ct)} = \frac{\mathrm{const}_{N,T}}{\delta V(\widetilde{c}\delta t)}$$

as was to be proved. □

Corollary 4.2 *Suppose that in addition to the hypotheses of the Theorem 4.2 the function* $\Lambda(v)$ *has a polynomial decay* (see definition 2.1)*, in particular*

$$\Lambda(2v) \geq \alpha \Lambda(v) \tag{4.29}$$

for some $\alpha > 0$ *and in place of the condition (4.25) we have*

$$\tau \le t, \quad \mathcal{R}(c\tau) \le c_1 R \tag{4.30}$$

where $c_1 = c_1(\alpha)$, *then the estimate (4.26) continues to hold.*

Remark. If the function $\Lambda(v)$ is obtained from the relation (2.29) of the Proposition 2.4 then it satisfies (4.29) automatically with $\alpha = \frac{1}{4}$ because the function $g(v)$ in (2.29) is increasing. That means that the hypothesis (4.29) holds in all reasonable cases whenever a Λ-isoperimetric inequality takes place.

Proof of corollary. As in the proof of the Theorem 4.2 we can take $\tau = t$. The constant c_1 is to be chosen in the course of the proof. First impose the restriction $c_1 \le c$. Then (4.30) implies $\mathcal{R}(ct) \le ct$ so we have the second half of condition (4.25) . To obtain its first half it suffices to prove that for all $t > 0$

$$\mathcal{R}(t) \ge \beta\sqrt{t} \tag{4.31}$$

where $\beta = \beta(\alpha) > 0$. Indeed, as soon as we have proved (4.31) we obtain from (4.30)

$$t \le \left(\frac{c_1}{\beta}\right)^2 R^2$$

whence the first of inequalities (4.25) follows provided $c_1 \le \beta$. Thus, we can take $c_1 = \min(c, \beta)$ and apply the Theorem 4.2.

To prove (4.31) let us use definition (4.1) of $\mathcal{R}(t)$ and definition (2.6) of $V(t)$. The inequality (4.31) is transformed to the form

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}}\right)^2 \ge \beta^2 \int_0^V \frac{dv}{v\Lambda(v)} \tag{4.32}$$

Consider a sequence of points $V_i = \frac{V}{2^i}$, $i = 0, 1, 2, \ldots$, then

$$\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}} = \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\sqrt{\Lambda(v)}} \ge$$

$$\ge \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\sqrt{\Lambda(V_{i+1})}} \ge \sum_{i=0}^{\infty} \log \frac{V_i}{V_{i+1}} \frac{1}{\sqrt{\Lambda(V_{i+1})}} \ge$$

$$\ge \sum_{i=0}^{\infty} \log 2 \sqrt{\frac{\alpha}{\Lambda(V_i)}}$$

where we have used $\Lambda(V_{i+1}) \le \frac{1}{\alpha}\Lambda(V_i)$. Applying a simple inequality

$$\left(\sum X_i\right)^2 \ge \sum X_i^2$$

we obtain

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}} \right)^2 \geq \sum_{i=0}^{\infty} \frac{\alpha \log^2 2}{\Lambda(V_i)} \ .$$

On the other hand

$$\int_0^V \frac{dv}{v\Lambda(v)} = \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\Lambda(v)} \leq$$

$$\leq \sum_{i=0}^{\infty} \int_{V_{i+1}}^{V_i} \frac{dv}{v\Lambda(V_i)} = \sum_{i=0}^{\infty} \frac{\log 2}{\Lambda(V_i)} \ .$$

Finally we get

$$\left(\int_0^V \frac{dv}{v\sqrt{\Lambda(v)}} \right)^2 \geq \alpha \log 2 \int_0^V \frac{dv}{v\Lambda(v)}$$

and $\beta = \sqrt{\alpha \log 2}$. □

5. Pointwise estimates with the Gaussian term

The following statement plays the main role in obtaining pointwise estimates of the heat kernel from the integral estimates. As in the previous Section we shall use the notation

$$E_D(x,t) = \int_M \exp\left(\frac{r^2}{Dt} \right) p^2(x,y,t)dy \tag{5.1}$$

where $r = \operatorname{dist}(x,y)$.

Proposition 5.1 *On an arbitrary manifold M the following inequality holds for all $x, y \in M,\ t > 0$ and $D > 2$*

$$p(x,y,t) \leq \exp\left(-\frac{r^2}{2Dt} \right) \sqrt{E_D(x,\frac{t}{2})E_D(y,\frac{t}{2})} \tag{5.2}$$

where $r = \operatorname{dist}(x,y)$, and, besides

$$p(x,y,t) \leq \exp\left(-\frac{r^2}{2Dt} - \lambda t \right) \exp(\lambda t_0)\sqrt{E_D(x,\frac{t_0}{2})E_D(y,\frac{t_0}{2})} \tag{5.3}$$

provided $t \geq t_0 > 0,\ \ 0 \leq \lambda \leq \lambda_1(M)$.

Proof. Let us denote by r_1, r_2 the distances from an arbitrary point $z \in M$ to x, y . Applying the semi-group property of the heat kernel and an elementary inequality $r_1^2 + r_2^2 \geq r^2/2$ we obtain

$$p(x,y,2t) = \int_M p(x,z,t)p(z,y,t)dz \leq$$

$$\leq \exp\left(-\frac{r^2}{4Dt} \right) \int_M p(x,z,t) \exp\left(\frac{r_1^2}{2Dt} \right) p(z,y,t) \exp\left(\frac{r_2^2}{2Dt} \right) dz \leq$$

$$\leq \exp\left(-\frac{r^2}{4Dt}\right)\left(\int_M p^2(x,z,t)\exp\frac{r_1^2}{Dt}dz\right)^{\frac{1}{2}}\left(\int_M p^2(y,z,t)\exp\frac{r_2^2}{Dt}dz\right)^{\frac{1}{2}}$$

whence (5.2) follows.

In order to prove (5.3) note that as it follows from the Proposition 4.2

$$\exp(2\lambda t_0)E_D(x,\frac{t_0}{2})E_D(y,\frac{t_0}{2}) \geq \exp(2\lambda t)E_D(x,\frac{t}{2})E_D(y,\frac{t}{2})$$

whence

$$\exp(-\lambda(t-t_0))\sqrt{E_D(x,\frac{t_0}{2})E_D(y,\frac{t_0}{2})} \geq \sqrt{E_D(x,\frac{t}{2})E_D(y,\frac{t}{2})}$$

which together with (5.2) imply (5.3) . □

Theorem 5.1 *Suppose that the Λ-isoperimetric inequality holds on the manifold with a function $\Lambda \in \mathcal{L}$. Assume also that its $V-$ transformation $V(t)$ satisfies the condition (4.24) with some constants T, N , then for all $x, y \in M,\ t > 0$ and any $D > 2$*

$$p(x,y,t) \leq \frac{\mathrm{const}_{D,N,T}}{V(\hat{c}t)}\exp\left(-\frac{r^2}{2Dt}\right) \tag{5.4}$$

where $\hat{c} = \hat{c}(D) = \min(D-2,\ 6/c)c^2/24$ and c is the constant from the Theorem 3.2.

Indeed, by the Proposition 5.1 we have the inequality (5.2) whence we obtain (5.4) by estimating the quantities E_D applying the Theorem 4.2 for $R = \infty$ and $\tau = t$. □

Examples. In all the following examples the function $\Lambda(v)$ is equal to const $\cdot\, v^{-2/n}$ for small values v , say, for $v < v_0$, so that $V(t) = \mathrm{const}\cdot t^{n/2}$ for $t < t_0$ and by the Theorem 5.1 for these values of t we have the estimate

$$p(x,y,t) \leq \frac{\mathrm{const}_D}{t^{n/2}}\exp\left(-\frac{r^2}{2Dt}\right) \tag{5.5}$$

Consider now different possibilities of behaviour of the function $\Lambda(v)$ for large values of the argument v . Note that the following identity is useful for checking the condition (4.24) which is satisfied in all the following examples:

$$\frac{d\log V(t)}{d\log t} = t\frac{V'(t)}{V(t)} = t\Lambda(V(t)) \tag{5.6}$$

1. Let for $v > v_0$

$$\Lambda(v) = \mathrm{const}\cdot v^{-\nu},\ \nu > 0\ ,$$

then for $t > t_0$

$$V(t) \geq \mathrm{const}\cdot t^{1/\nu}$$

and by the Theorem 5.1

$$p(x,y,t) \leq \frac{\mathrm{const}_D}{t^{1/\nu}}\exp\left(-\frac{r^2}{2Dt}\right)\ . \tag{5.7}$$

2. Suppose that for large v

$$\Lambda(v) = \mathrm{const}(\log v)^{-\nu} \ ,$$

then for large values of t we get

$$V(t) \geq \mathrm{const} \cdot \exp\Big(\mathrm{const} \cdot t^{\frac{1}{\nu+1}}\Big)$$

and the corresponding estimate

$$p(x,y,t) \leq \mathrm{const} \cdot \exp\left(-\mathrm{const} \cdot t^{\frac{1}{\nu+1}} - \frac{r^2}{2Dt}\right) \ . \tag{5.8}$$

3. Let $\Lambda(v) \equiv \lambda \equiv \mathrm{const} > 0$ for $v > v_0$ that is the manifold has a positive spectral radius. The Theorem 5.1 yields

$$p(x,y,t) \leq \mathrm{const} \cdot \exp\left(-\mathrm{const} \cdot \lambda t - \frac{r^2}{2Dt}\right) \tag{5.9}$$

In the situation with a positive spectral radius one can obtain a sharper information about the rate of decay of the heat kernel as $t \to \infty$. Namely, const in front of λt is a superfluous term. The following theorem treats this case.

Theorem 5.2 *Suppose that in any ball B_R^x of a fixed radius $R > 0$ the Λ-isoperimetric inequality holds with the function $\Lambda = \Lambda_{x,R}$ defined as follows*

$$\Lambda_{x,R}(v) = a(x,R)v^{-\nu} \tag{5.10}$$

where $a(x,R) > 0,\ \nu > 0$, then for all $x, y \in M,\ t > t_0 > 0$,

$$p(x,y,t) \leq \frac{\mathrm{const}_\nu \left(1 + \frac{r^2}{t}\right)^{1+\frac{1}{\nu}} \exp\left(-\frac{r^2}{4t} - \lambda t\right) \exp(\lambda t_0)}{\min(t_0, R^2)^{1/\nu} \left(a\left(x,R\right) a\left(y,R\right)\right)^{1/2\nu}} \tag{5.11}$$

where $\lambda = \lambda_1(M),\ r = \mathrm{dist}(x,y)$.

Remark. Note that the isoperimetric inequality (5.10) takes place on any complete manifold for $\nu = 2/n$ which follows from compactness arguments. Therefore, the estimate (5.11) holds also on any manifold and gives the precise speed of decay of the heat kernel as $t \to \infty$. If, in addition, one knows that all the coefficients $a(x,R)$ are uniformly bounded away from 0 , then the estimate (5.11) takes the following form

$$p(x,y,t) \leq \mathrm{const} \cdot \left(1 + \frac{r^2}{t}\right)^{1+\frac{1}{\nu}} \exp\left(-\frac{r^2}{4t} - \lambda t\right) \tag{5.12}$$

provided $t > t_0$.

A similar inequality (without the term $-\lambda t$) is obtained in [D92] under the hypothesis of "weak bounded geometry" which is stronger than our uniform local Λ-isoperimetric inequality .

Proof of Theorem 5.2. Computing the $V-$ transformations of $\Lambda_{x,R}$ and $\sqrt{\Lambda_{x,R}}$ one obtains

$$V(t)=(\nu a t)^{1/\nu}, \quad W(r)=(\frac{1}{4}\nu^2 a r^2)^{1/\nu} \tag{5.13}$$

whence it follows that

$$\mathcal{R}(t)=2\sqrt{t/\nu}$$

Let us note also that the function $t\Lambda(V(t))\equiv\frac{1}{\nu}$ is bounded. Applying the Theorem 4.2 for $\tau=\min(t,\ \mathrm{const}_\nu R^2)$ and $D\in(2,2+6/c)$ we get

$$E_D(x,t)\le\frac{\mathrm{const}_\nu}{\delta^{1+\frac{1}{\nu}}a(x,R)^{1/\nu}\min(t,R^2)^{1/\nu}}$$

which together with (5.3) implies

$$p(x,y,t)\le\frac{\mathrm{const}_\nu}{\delta^{1+\frac{1}{\nu}}}\exp\Bigg(-\frac{r^2}{2(2+\delta)t}-\lambda t\Bigg)\ A \tag{5.14}$$

where

$$A=\frac{\exp(\lambda t_0)}{(a(x,R)a(y,R))^{1/2\nu}\min(t_0,R^2)^{1/\nu}}$$

and $\delta=D-2<6/c$. Taking here $\delta=\min\Big(\frac{6}{c},\frac{t}{r^2}\Big)$ and noting that for this δ

$$\frac{r^2}{2t}-\frac{r^2}{(2+\delta)t}=\frac{\delta r^2}{2(2+\delta)t}\le\frac{1}{4}$$

we obtain finally (5.11) . □

The Theorem 5.2 can give a non-trivial information also in the case $\lambda_1(M)=0$. Next we suppose that the Λ-isoperimetric inequality holds in any ball $B_R^x\in M$ with the following function $\Lambda=\Lambda_{x,R}$

$$\Lambda_{x,R}(v)=\frac{b}{R^2}\Big(\mathrm{Vol}\,B_R^x\Big)^{\frac{2}{n}}v^{-2/n}\ ,\ b=\mathrm{const}>0\ . \tag{5.15}$$

Here n is the dimension of the manifold M but formally this is not necessarily: n may be any positive number. For example, this is true with $n=\dim M$ for a manifold M of a non-negative Ricci curvature (see [G91] for the proof).

Proposition 5.2 *Under the hypothesis above the heat kernel admits the following estimate for all $x,y\in M$ and $t>0$*

$$p(x,y,t)\le\mathrm{const}_{n,b}\Bigg(1+\frac{r^2}{t}\Bigg)^{\frac{3}{4}n+1}\frac{\exp\Big(-\frac{r^2}{4t}\Big)}{\mathrm{Vol}\,B_{\sqrt{t}}^x}\ . \tag{5.16}$$

Besides, for any two balls B_R^x and $B_\rho^y\subset B_R^x$ we have

$$\frac{\mathrm{Vol}\,B_R^x}{\mathrm{Vol}\,B_\rho^y}\le\mathrm{const}_{n,b}\Big(\frac{R}{\rho}\Big)^n\ . \tag{5.17}$$

Inversely, suppose that a manifold M is known to have the heat kernel satisfying the inequality

$$p(x,x,t) \le \frac{\mathrm{const}}{\mathrm{Vol}\, B^x_{\sqrt{t}}} \tag{5.18}$$

for all $x \in M$ and $t > 0$ and assume in addition that for any couple of concentric balls $B^x_\rho,\ B^x_R$ where $\rho \le R$ we have

$$\frac{\mathrm{Vol}\, B^x_R}{\mathrm{Vol}\, B^x_\rho} \le \mathrm{const}\left(\frac{R}{\rho}\right)^n , \tag{5.19}$$

then in any ball the Λ-isoperimetric inequality holds with the function $\Lambda = \Lambda_{x,R}$ defined by (5.15) .

Remark. The statement of the theorem means that the Λ-isoperimetric inequality in any ball given by (5.15) is simply equivalent to the conjuncture of the heat kernel estimate (5.16) and the volume ratio estimate (5.17) . On a manifold with a non-negative Ricci curvature a similar estimate of the heat kernel was proved first by Li and Yau [LY] . In the view of the sharp results of B.Davies (see [D88] , [DP]) the order $\frac{3}{4}n+1$ of the polynomial correction term in (5.16) seems not to be optimal. The advantage of our approach is that the result is stable under a quasi-isometric transformation of the Riemannian metric. Indeed, it is easy to see that the Λ-isoperimetric inequality in question is a quasi-isometric invariant. It can be considered as a replacement of the notion of a non-negative Ricci curvature when dealing with a manifold whose metric is not smooth enough to have a curvature at all.

Proof. Let us prove first (5.17) applying the result of Carron [C] that a Λ-isoperimetric inequality implies some lower bounds for the volume of a geodesic ball. Indeed, the Proposition 2.4 from [C] states the following:

if the Λ-isoperimetric inequality holds in a region $\Omega \subset M$ with the function $\Lambda(v) = av^{-2/n}$, then any ball $B^y_\rho \subset \Omega$ admits the following volume estimate:

$$\mathrm{Vol}\, B^y_\rho \ge \mathrm{const}_n a^{n/2} \rho^n .$$

Applying this result to the set $\Omega = B^x_R$ and taking $a = \frac{b}{R^2}\left(\mathrm{Vol}\, B^x_R\right)^{\frac{2}{n}}$ we get (5.17) .

To prove the heat kernel estimate (5.16) let us apply the Theorem 5.2 with the function $a(x,R) = \frac{b}{R^2}\left(\mathrm{Vol}\, B^x_R\right)^{\frac{2}{n}}$ and with $t_0 = t,\ R = \sqrt{t}$. We obtain:

$$p(x,y,t) \le \frac{\mathrm{const}_n\left(1+\frac{r^2}{t}\right)^{\frac{n}{2}+1}\exp\left(-\frac{r^2}{4t}\right)}{\left(\mathrm{Vol}\, B^x_{\sqrt{t}}\mathrm{Vol}\, B^y_{\sqrt{t}}\right)^{\frac{1}{2}}} \tag{5.20}$$

The expression on the right-hand side of (5.20) is going to be simplified to get rid of $\mathrm{Vol}\, B^y_{\sqrt{t}}$. It is known how to do that on a non-negatively curved manifold (see [LY]): one should estimate the ratio of the volumes $\mathrm{Vol}\, B^x_{\sqrt{t}}$ and $\mathrm{Vol}\, B^y_{\sqrt{t}}$ via the distance r between points x, y using the volume comparison theorem for such a manifold. Here we apply the inequality (5.17) in place of the comparison theorem. Namely, we have

$$\operatorname{Vol} B^x_{\sqrt{t}} \le \operatorname{Vol} B^y_{r+\sqrt{t}} \le \text{const}_{n,b} \left(\frac{r+\sqrt{t}}{\sqrt{t}} \right)^n \operatorname{Vol} B^y_{\sqrt{t}}$$

$$\le \text{const}_{n,b} \left(1 + \frac{r^2}{t} \right)^{n/2} \operatorname{Vol} B^y_{\sqrt{t}} \; . \tag{5.21}$$

Replacing the volume $\operatorname{Vol} B^y_{\sqrt{t}}$ in (5.20) by its lower bound from (5.21) we get (5.16) .

Let us turn to the proof of the converse statement. The idea is the same as in the Theorem 2.2 but first we observe that the hypothesis (5.19) implies that for any two intersecting balls B^x_R and B^y_ρ such that $\rho \le R$ the following holds:

$$\text{const} \left(\frac{R}{\rho} \right)^{n_1} \le \frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_\rho} \le \text{const} \left(\frac{R}{\rho} \right)^{n} \tag{5.22}$$

where $n_1 > 0$ depends on n and on the const in (5.19) .

This is proved in [G91] , see the Theorem 1.1 there. We only mention that the right inequality in (5.22) follows evidently from (5.19) whereas the left one exploits (5.19) as well as the non-compactness of the manifold under consideration.

Hence, we can claim that for any two intersecting ball B^x_R and B^y_ρ the following relation holds without any restriction on the radii R, ρ :

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_\rho} \le \text{const}\, f\left(\frac{R}{\rho}\right) \tag{5.23}$$

where

$$f(\xi) = \begin{cases} \xi^{n_1}, \ \xi < 1 \\ \xi^n, \ \xi \ge 1 \end{cases} . \tag{5.24}$$

Indeed, if $R \ge \rho$ then right inequality (5.22) is applicable. Otherwise we apply the left inequality (5.22) exchanging R and ρ .

To prove the Λ-isoperimetric inequality in a given ball B^x_R with the function (5.15) let us apply the eigenfunction expansion as was done in the course of the proof of the Theorem 2.2 and obtain for any region Ω lying in a ball B^x_R and for any value of time $t > 0$ the estimate:

$$\exp(-\lambda_1(\Omega)t) \le \int_\Omega p(y,y,t)dy \le \text{const} \int_\Omega \frac{dy}{\operatorname{Vol} B^y_{\sqrt{t}}} \; .$$

We have according to (5.23) that

$$\frac{\operatorname{Vol} B^x_R}{\operatorname{Vol} B^y_{\sqrt{t}}} \le \text{const}\, f\left(\frac{R}{\sqrt{t}}\right)$$

which implies

$$\exp(-\lambda_1(\Omega)t) \le \text{const}\, f\left(\frac{R}{\sqrt{t}}\right) \frac{\operatorname{Vol} \Omega}{\operatorname{Vol} B^x_R} \; .$$

Now we choose the time t in order to get at most e^{-1} on the right-hand side above:

$$\operatorname{const}\cdot ef\Big(\frac{R}{\sqrt{t}}\Big)=\frac{\operatorname{Vol}B_R^x}{\operatorname{Vol}\Omega} \tag{5.25}$$

whence it follows that

$$\lambda_1(\Omega)\geq\frac{1}{t}\ . \tag{5.26}$$

We can evaluate t using (5.25) and the following property of the function f which reflects its polynomial nature and follows obviously from the definition (5.24) : for any $\gamma>0$ there exists a positive constant c_γ such that for all $\xi>0$

$$\gamma f(\xi)\leq f(c_\gamma\xi)\ .$$

Therefore, taking here $\gamma=\operatorname{const}\cdot e$ we have

$$\gamma f\Big(\frac{R}{\sqrt{t}}\Big)\leq f\Big(c_\gamma\frac{R}{\sqrt{t}}\Big)$$

which together with (5.25) yields

$$f\Big(c_\gamma\frac{R}{\sqrt{t}}\Big)\geq\frac{\operatorname{Vol}B_R^x}{\operatorname{Vol}\Omega}\ .$$

Let us note that the ratio on the right-hand side here is at least as much as 1 whence it follows that

$$c_\gamma\frac{R}{\sqrt{t}}\geq\left(\frac{\operatorname{Vol}B_R^x}{\operatorname{Vol}\Omega}\right)^{1/n}\ .$$

Evaluating t from this inequality and substituting into (5.26) we get finally the desired Λ-isoperimetric inequality . □

A similar heat kernel estimate can be obtained on a manifold of a Ricci curvature bounded from below by some constant $-K$. For such a manifold it was proved in [V89] that there is some small positive constant $\rho=\rho(K)$ such that in any ball B_ρ^x of radius ρ Λ-isoperimetric inequality is satisfied with the function

$$\Lambda_x(v)=\operatorname{const}_K\Big(\operatorname{Vol}B_\rho^x\Big)^{2/n}v^{-2/n}\ .$$

Applying the Theorem 5.2 for $R=\rho,\ t_0=\rho^2$ we obtain

$$p(x,y,t)\leq\operatorname{const}_{K,n,\lambda}\frac{\Big(1+\frac{r^2}{t}\Big)^{1+\frac{n}{2}}\exp\Big(-\frac{r^2}{4t}-\lambda t\Big)}{\Big(\operatorname{Vol}B_\rho^x\operatorname{Vol}B_\rho^y\Big)^{\frac{1}{2}}} \tag{5.27}$$

provided $t>\rho^2,\quad \lambda=\lambda_1(M)$.

In all the foregoing examples the heat kernel bounds contain two main multiples: a multiple which is responsible for behaviour of heat kernel in time and the term $\exp\Big(-c\frac{r^2}{t}\Big)$ which, in fact, controls a decay of the heat kernel when $r\to\infty$ although the Gaussian factor appears on a non-geometric ground. The rest of this Section is devoted to a situation when one more factor emerges depending on r . This term is expected on a quickly expanding manifold for the following reason. As it was mentioned in Section 2, we have always

$$\int_M p(x,y,t)dy \le 1 \tag{5.28}$$

If the volume of a ball of the radius R is growing up faster than $\exp(CR^2)$ as $R \to \infty$ then the heat kernel $p(x,y,t)$ has to decrease, generally speaking, faster than $\exp(-Cr^2)$ so that the integral (5.28) is balanced. This is why one more factor is expected with a quick decay as $r \to \infty$

Let us fix some point $z \in M$ and consider the function

$$\lambda(R) \equiv \lambda_1(M \setminus \overline{B^z_R}) \tag{5.29}$$

where the exterior of the ball is regarded as a submanifold and $R \in [0, +\infty)$. Obviously, $\lambda(R)$ is an increasing function of R and $\lambda(0)$ is nothing but the spectral radius $\lambda_1(M)$.

For example, if M is a Cartan-Hadamard manifold (i.e. its sectional curvature is non-positive and it is simply connected) then

$$\lambda(R) \ge \frac{1}{4}(n-1)^2 k(R)^2 \tag{5.30}$$

where $-k^2(R)$ is equal to *sup* of sectional curvature in exterior of ball B^z_R .

Theorem 5.3 *Suppose that the Λ-isoperimetric inequality holds on the manifold with the function*

$$\Lambda(v) = av^{-2/n} \tag{5.31}$$

with some $a > 0$, then for all $x \in M$ such that $r \equiv \operatorname{dist}(x,z) > \sqrt{t}$ the inequality is valid

$$p(x,z,t) \le \frac{\mathrm{const}_{a,n,\gamma}}{t^{n/2}} \exp\left(-\gamma\frac{r^2}{4t} - \gamma\lambda(0)t - \bar{c}r\sqrt{\lambda(\gamma r)}\right) \tag{5.32}$$

where $0 < \gamma < 1$ is arbitrary, $\bar{c} = \bar{c}(\gamma) > 0$.

Remark. For small values of r and even for all $x, z \in M,\ t > 0$ we have by the Theorem 5.2 the estimate (5.11) for $R = \infty,\ a(x,R) \equiv a$ or by the Theorem 5.1 the estimate (5.5) .

The Theorem 5.3 is applicable for a Cartan-Hadamard manifold because the Euclidean isoperimetric inequality holds for such a manifold. The third term $\exp\bigl(-\bar{c}r\sqrt{\lambda(\gamma r)}\bigr)$ acquires on a Cartan-Hadamard manifold the form

$$\exp\left(-\bar{c}\frac{1}{2}(n-1)k(\gamma r)r\right) \quad . \tag{5.33}$$

Let us compare the estimate (5.32) in this setting with the exact formula (1.3) of the heat kernel of $\mathbf{H}^3_k$. The multiple (5.33) corresponds to the term

$$\frac{kr}{\sinh(kr)}$$

whose decay is similar to that of (5.33) up to the constant $\bar{c}$.

Of course, if the curvature $-k^2(R)$ is growing (to the negative side) fast enough then the term (5.33) can play the main role in the heat kernel behaviour as $r \to \infty$.

Proof of theorem. In the exterior of any ball B^z_R the Λ-isoperimetric inequality holds with the function

$$\Lambda_R(v) = \max(av^{-2/n},\ \lambda(R)) \ . \tag{5.34}$$

Let us denote by $V_R(t)$ and $W_R(r)$ $V-$ transformations of Λ and $\sqrt{\Lambda}$ respectively. One can compute that

$$V_R(t) \geq \mathrm{const}_{a,n} t^{n/2} \exp\Big(\frac{1}{2}\lambda(R)t\Big) \tag{5.35}$$

and

$$W_R(t) \geq \mathrm{const}_{a,n} r^n \exp\Big(\frac{1}{2}\sqrt{\lambda(R)}r\Big) \tag{5.36}$$

Of course, $\frac{1}{2}$ in the exponent is too rough but all the same we will apply further the Theorem 3.2 which does not give a sharp constant at the corresponding place.

Let us fix some ball B^x_ρ of radius $\rho < r = \mathrm{dist}(x,z)$ and put $R = r - \rho$. Since the ball B^x_ρ lies in the exterior of B^z_R it follows that the Λ-isoperimetric inequality holds in B^x_ρ with the function $\Lambda = \Lambda_R$. Applying the Theorem 3.2 to the function $p(\cdot\,, z, \cdot)$ in the cylinder $B^x_\rho \times (t/2, t)$ we get

$$p(x,z,t)^2 \leq \frac{\mathrm{const}}{\min\Big(\frac{t}{2}V_R(c\frac{t}{2}),\ \rho^2 W_R(c\rho)\Big)} \int_{\frac{t}{2}}^t \int_{B^x_\rho} p^2(y,z,s)dyds \tag{5.37}$$

On the other hand by the Theorem 4.2 we have for $D > 2$

$$\int_{B^x_\rho} p^2(y,z,s)dy \leq \int_{M\setminus B^z_R} p^2(y,z,s)dy$$

$$\leq \exp\Big(-\frac{R^2}{Ds}\Big) \int_M p^2(y,z,s)\exp\Big(\frac{r^2}{Ds}\Big)\ dy \leq \frac{\mathrm{const}_{D,a,n}}{s^{n/2}} \exp\Big(-\frac{R^2}{Ds}\Big)$$

Substituting into (5.37) and noting that $t/2 \leq s \leq t$ we obtain

$$p^2(x,z,t) \leq \frac{\mathrm{const}_{D,a,n}\exp\Big(-\frac{R^2}{Dt}\Big)}{t^{n/2}\min\Big(V_R(c\frac{t}{2}),\ W_R(c\rho)\frac{\rho^2}{t}\Big)}$$

or, estimating V_R and W_R from (5.35) and (5.36) respectively,

$$p^2(x,z,t) \leq \frac{\mathrm{const}_{D,a,n}}{t^n \min\Big(1,\ \frac{\rho^2}{t}\Big)^{\frac{n}{2}+1}} \exp\Big(-\frac{R^2}{Dt} - \frac{1}{4}c\min(\lambda(R)t,\ \rho\sqrt{\lambda(R)})\Big) \tag{5.38}$$

Now we apply the following elementary inequality

$$\kappa^2X^2 - 2\kappa XY + Y\min(X,Y) \geq 0$$

which is valid for all $X, Y \geq 0$ and $0 < \kappa < \frac{1}{2}$. Taking here $X = \rho\sqrt{\lambda(R)},\ Y = \lambda(R)t$ we have

$$\min\Big(\rho\sqrt{\lambda(R)},\ \lambda(R)t\Big) \geq 2\kappa\rho\sqrt{\lambda(R)} - \kappa^2\frac{\rho^2}{t} \ .$$

Therefore (5.38) implies

$$p^2(x,z,t) \leq \frac{\mathrm{const}_{D,a,n}}{t^n \min\left(1,\ \frac{\rho^2}{t}\right)^{\frac{n}{2}+1}} \exp\left(-\frac{R^2}{Dt} - \frac{1}{2}c\kappa\rho\sqrt{\lambda(R)} + c\kappa^2\frac{\rho^2}{4t}\right) \ .$$

Let us set $R = \gamma r$ and, correspondingly, $\rho = (1-\gamma)r$ where $0 < \gamma < 1$. Since by the hypothesis $r > \sqrt{t}$ it follows that $\frac{\rho^2}{t} \geq \mathrm{const}_\gamma$. Taking κ small enough and increasing a bit D we get rid of the summand $c\kappa^2\frac{\rho^2}{4t}$:

$$p^2(x,z,t) \leq \frac{\mathrm{const}_{D,a,n,\gamma}}{t^n} \exp\left(-\frac{\gamma^2 r^2}{Dt} - \frac{1}{2}\gamma\kappa c\sqrt{\lambda(\gamma r)}r \ .\right) \tag{5.39}$$

Finally we apply estimate (5.11) which yields for $t_0 = (1-\gamma)t$

$$p(x,z,t) \leq \frac{\mathrm{const}_{D,a,n,\gamma}}{t^{n/2}} \exp\left(-\frac{r^2}{2Dt} - \gamma\lambda(0)t\right) \tag{5.40}$$

Multiplying (5.40) and (5.39) to powers γ and $\frac{1-\gamma}{2}$ respectively and taking D being close enough to 2 we obtain (5.32) (one has to replace γ^2 by γ in the final expression). □

Corollary 5.1 *If the Λ-isoperimetric inequality holds on a manifold with the function $\Lambda(v) = av^{-2/n}$, $a > 0$ then for all $z \in M$*

$$\limsup_{r\to\infty} \frac{1}{r} \sup_{t>0} \log p(x,z,t) \leq -\mathrm{const}\left(\sqrt{\lambda_1(M)} + \sqrt{\lambda_{ess}(M)}\right) \tag{5.41}$$

where $r = \mathrm{dist}(x,z)$, $\lambda_{ess}(M)$ is the bottom of the essential spectrum of $-\Delta$ in $L^2(M)$, $\mathrm{const} > 0$ is an absolute constant.

First note that for $r > 1$ we can give up the factor $t^{n/2}$ in (5.32) . Indeed, if $t > 1$ this is evident, otherwise this term is majorized by $\exp(\mathrm{const}\,\frac{r^2}{t})$. Thus, taking $\gamma = \frac{1}{2}$ in (5.32) we have for $r > 1$

$$\log p(x,z,t) \leq -\mathrm{const}\left(\frac{r^2}{t} + \lambda(0)t + r\sqrt{\lambda(r/2)}\right) + \mathrm{const}_{a,n} \ .$$

Since

$$\frac{r^2}{t} + \lambda(0)t \geq 2r\sqrt{\lambda(0)}$$

it follows that

$$\sup_{t>0} \log p(x,z,t) \leq -\mathrm{const}\left(r\sqrt{\lambda(0)} + r\sqrt{\lambda(r/2)}\right) + \mathrm{const}_{a,n} \ .$$

Dividing this inequality by r and passing to limit as $r \to \infty$ we get (5.41) because $\lambda(0) = \lambda_1(M)$ and

$$\lim_{r\to\infty} \lambda(r) = \lambda_{ess}(M)$$

(see [DL]) □

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